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The Hopf Bifurcations in the Permanent Magnet Synchronous Motors*

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Abstract Based on the focus quantities and other techniques, the stability properties of equilibria and the limit cycles arising from Hopf bifurcations are investigated for two models of permanent magnet synchronous motors. The first model is of surface-magnet type and can have at most two unstable small limit cycles, which are symmetric with respect to x-axis. The other model is of interior-magnet type and can have at most four small limit cycles in two symmetric nests.

Keywords Focus quantity, Limit cycle, Hopf bifurcation.

MSC(2010) 34C05, 34C07.

1. The mathematical model of PMSM

A motor is an electrical machine that converts electrical energy into mechanical energy. The permanent magnet synchronous motors (PMSM) are widely used in industry and electric vehicle applications. It has many advantages, such as high efficiency, high-power density and low-cost maintenance, see [2, 7] and references therein. There are two major types of PMSM: one with permanent magnets mounted on the rotor surface, called the surface-magnet type; and one with permanent magnets buried inside the rotor, called the interior-magnet type.

In [2], using the d-q frame, the PMSM model is written as

$$\begin{cases} L_{ds}\dfrac{i_{ds}}{dt'} = -u_{ds} - R_s i_{ds} + n_p\omega_r L_{qs} i_{qs}, \\ L_{qs}\dfrac{di_{qs}}{dt'} = -u_{qs} - R_s i_{qs} - n_p\omega_r L_{ds} i_{ds} + n_p\psi_a\omega_r, \\ J\dfrac{d\omega_r}{dt'} = T_m - \dfrac{3}{2}n_p\psi_a i_{qs} + \dfrac{3}{2}n_p(L_{ds}-L_{qs}) + i_{ds}i_{qs} - B_m\omega_r. \end{cases} \tag{1.1}$$

The variables and parameters are listed in Table 1.

For brevity, a symmetric load of resistance R is used, so that u_{ds} and u_{qs} can be expressed in terms of $i_{ds}R$ and $i_{qs}R$, respectively. Additionally, the net driving torque is considered to be proportional to i_{qs}, i.e., $T_m - T_{pm} = i_{qs}\mu$, where $T_{pm} = \dfrac{3}{2}n_p\psi_a i_{qs}$ is the PM torque and μ is a positive constant.

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Table 1. Variables and parameters of PMSM

Names	Descriptions	Units
i_{ds}, i_{qs}	Stator currents	A
u_{ds}, u_{qs}	Stator voltages	V
L_{ds}, L_{qs}	Stator inductances	H
R_s	Stator resistance	Ω
ψ_a	PM flux	Wb
n_p	Number of pole pairs	\
ω_r	Mechanical rotor speed	rad/s
T_m	Mechanical driving torque	N·m
J	Rotor inertia	kg·m^2
B_m	Viscosity friction coefficient	N·m·s

System (1.1) can be further simplified by transforming t' to τt, and i_{ds} to bkx, i_{qs} to ky, and ω_r to $\dfrac{z}{\tau n_p}$, where $b = \dfrac{L_{qs}}{L_{ds}}$, $\tau = \dfrac{L_{qs}}{R_s + R}$, and k is a positive constant. Therefore, system (1.1) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} = -bx + yz, \\ \dfrac{dy}{dt} = -y - xz + cz, \\ \dfrac{dz}{dt} = a(\gamma k y - z) + \eta k^2 xy, \end{cases} \tag{1.2}$$

where $a = \dfrac{B_m \tau}{J}$, $c = \dfrac{\psi_a}{kL_{qs}}, \eta = \dfrac{3n_p^2(L_{ds} - L_{qs})b\tau^2}{2J}$, and $\gamma = \dfrac{n_p \tau \mu}{B_m}$.

Magnetic saliency describes the relationship between the rotor's flux (d-axis) inductance and the torque-producing (q-axis) inductance. Since surface-magnet PMSM exhibits no saliency (i.e. $L_{ds} = L_{qs}$), we have $\eta = 0$. In order to avoid the trivial case when $\gamma = 0$, γ is assumed nonzero and k is defined as $k = \dfrac{1}{\gamma}$. Thus, system (1.2) can be written as

$$\begin{cases} \dfrac{dx}{dt} = -x + yz, \\ \dfrac{dy}{dt} = -y - xz + cz, \\ \dfrac{dz}{dt} = a(y - z). \end{cases} \tag{1.3}$$

For more details, see [2].

The foregoing system is a special case of

$$\begin{cases} \dfrac{dx}{dt} =& -bx + yz, \\ \dfrac{dy}{dt} =& -y - xz + cz, \\ \dfrac{dz}{dt} =& a(y-z), \end{cases} \tag{1.4}$$

where $a, b, c > 0$ and the parameter b is not specified.

The interior-magnet PMSM exhibits significant saliency ($L_{qs} \neq L_{ds}$). Hence, it offers additional salient power. In order to derive the explicit solution for the sizing of PMs, γ is assumed zero and k is defined as $k = \sqrt{\dfrac{1}{\eta}}$. Note that $\gamma = 0$ is the case when $T_m = T_{pm}$. So, system (1.2) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} =& -bx + yz, \\ \dfrac{dy}{dt} =& -y - xz + cz, \\ \dfrac{dz}{dt} =& xy - az. \end{cases} \tag{1.5}$$

For more details, see [2].

Both systems (1.4) and (1.5) are symmetric with respect to the x-axis in the sense of coordinate transformation $(x, y, z) \to (x, -y, -z)$. Despite the simplicity, these systems have rich dynamical behaviors, ranging from equilibria to periodic and even chaotic oscillations, depending on the parameter values [2, 3].

For systems (1.4) and (1.5), although in [2, 3] the generic Hopf bifurcations are analyzed numerically, the focus quantities that characterize the nature of the bifurcations were not obtained. Moreover, for these systems, there are no analytical results about the Hopf bifurcations and degenerate Hopf bifurcations in the current literature.

The rest of this paper is organized as follows. In Section 2, we review the definition of focus quantities and related computation methods. By using the method of focus quantities, we study the Hopf bifurcations of systems (1.4) and (1.5). In Section 3 it is proved that only subcritical Hopf bifurcation occurs at each of the two symmetric equilibria in system (1.4), i.e. the system can have at most two unstable small limit cycles. In the last section, it proves that system (1.5) can have at most four small limit cycles in two symmetric nests.

2. The focus quantities of three dimensional systems

Consider a family of three dimensional analytic differential systems

$$\begin{cases} \dfrac{dx}{dt} =& P(x, y, z, \mu), \\ \dfrac{dy}{dt} =& Q(x, y, z, \mu), \\ \dfrac{dz}{dt} =& R(x, y, z, \mu), \end{cases} \tag{2.1}$$

parameterized by a parameter μ. Assume that for each $\mu \in J$, the system has a unique equilibrium $p(\mu) := (x(\mu), y(\mu), z(\mu))$, where J is an interval in $\mathbb{R}$.

Let

$$A_\mu := \left(\frac{\partial(P,Q,R)}{\partial(x,y,z)}\right)(p(\mu))$$

be the Jacobian matrix of system (2.1) evaluated at the equilibrium. Assume that the characteristic polynomial of A_μ is

$$g_p(\lambda) = \lambda^3 + b_1(\mu)\lambda^2 + b_2(\mu)\lambda + b_3(\mu),$$

where $\Delta := b_1 b_2 - b_3 = 0, b_2 > 0, b_3 > 0, \dfrac{d\Delta}{d\mu} \neq 0$ for some $\mu = \mu_0$. Then according to the criterion of Hopf bifurcation [10], the matrix A_{μ_0} has three eigenvalues $\lambda_{1,2} = \pm\sqrt{b_2(\mu_0)}\mathtt{i}$ and $\lambda_3 = -b_1(\mu_0) < 0$. The Jordan canonical form of this matrix can be obtained by some similarity transformation $S^{-1}A_{\mu_0}S = J_p$, where

$$J_p = \begin{pmatrix} \sqrt{b_2(\mu_0)}\mathtt{i} & 0 & 0 \\ 0 & -\sqrt{b_2(\mu_0)}\mathtt{i} & 0 \\ 0 & 0 & -b_1(\mu_0) \end{pmatrix}.$$

and $S = (s_{i,j})_{3\times 3}$.

Introducing the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = S \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x(\mu_0) \\ y(\mu_0) \\ z(\mu_0) \end{pmatrix},$$

into system (2.1) with $\mu = \mu_0$, we obtain

$$\begin{cases} \dfrac{dx_1}{dt} = \sqrt{b_2(\mu_0)}\mathtt{i}\, x_1 + \displaystyle\sum_{j+k+s=2}^{\infty} B_{jks} {x_1}^j {y_1}^k z_1^s = B(x_1, y_1, z_1), \\ \dfrac{dy_1}{dt} = -\sqrt{b_2(\mu_0)}\mathtt{i}\, y_1 + \displaystyle\sum_{j+k+s=2}^{\infty} C_{jks} {x_1}^j {y_1}^k z_1^s = C(x_1, y_1, z_1), \\ \dfrac{dz_1}{dt} = -\, b_1(\mu_0)\, z_1 + \displaystyle\sum_{j+k+s=2}^{\infty} D_{jks} {x_1}^j {y_1}^k z_1^s = D(x_1, y_1, z_1). \end{cases} \tag{2.2}$$

For system (2.2), according to [23], we can successively derive the following formal series

$$F(x_1, y_1, z_1) = x_1 y_1 + \sum_{s=3}^{\infty} \sum_{k=0}^{s} \sum_{j=0}^{s-k} M_{s,k,j} x_1^{s-k-j} y_1^k z_1^j, \tag{2.3}$$

such that

$$\frac{dF}{dt}\Big|_{(2.2)} = \frac{\partial F}{\partial x_1}B + \frac{\partial F}{\partial y_1}C + \frac{\partial F}{\partial z_1}D = \sum_{n=1}^{\infty} W_n (x_1 y_1)^{n+1}, \tag{2.4}$$

where $M_{s,k,j}$ can be determined uniquely if we set $M_{2k,k,0} = 0$.

Definition 2.1. In (2.4), we call W_n the nth focus quantities of the original system $(2.1)|_{\mu=\mu_0}$ at $p(\mu_0)$.

Since $\lambda_3 < 0$, the stability of $p(\mu_0)$ is determined by the first non-vanishing focus quantity W_n. For $W_n < 0$, $p(\mu_0)$ is asymptotically stable, and for $W_n > 0$, the point is unstable.

Consider a family of quadratic systems in the form of (2.2), i.e.

$$\begin{cases} \dfrac{dx_1}{dt} = \alpha_1 x_1 + \alpha_2 {x_1}^2 + \alpha_3 x_1 y_1 + \alpha_4 {y_1}^2 + \alpha_5 x_1 z_1 + \alpha_6 y_1 z_1 + \alpha_7 {z_1}^2, \\ \dfrac{dy_1}{dt} = -\alpha_1 y_1 + \beta_2 {x_1}^2 + \beta_3 x_1 y_1 + \beta_4 {y_1}^2 + \beta_5 x_1 z_1 + \beta_6 y_1 z_1 + \beta_7 {z_1}^2, \\ \dfrac{dz_1}{dt} = \delta_1 z_1 + \delta_2 {x_1}^2 + \delta_3 x_1 y_1 + \delta_4 {y_1}^2 + \delta_5 x_1 z_1 + \delta_6 y_1 z_1 + \delta_7 {z_1}^2, \end{cases} \tag{2.5}$$

where $\alpha_1 = \omega \mathtt{i}$ with $\omega \in \mathbb{R}^+$, $\delta_1 \in \mathbb{R}^-$ and the other coefficients $\alpha_k, \beta_k, \delta_k$ are complex coefficients. Based on the algorithm developed in [18], we get the formula for the first focus quantity. The formula can facilitate the study of Hopf bifurcation in many practical problems (up to a affine transformation).

Proposition 2.1. *For system* (2.5)*, the first focus quantity of the origin is*

$$W_1 = \frac{W_{1,1}}{-4\,{\alpha_1}^3 \delta_1 + \alpha_1 {\delta_1}^3}, \tag{2.6}$$

where

$$\begin{aligned} W_{1,1} = & -4\,\delta_3 \left(\alpha_5 + \beta_6\right) {\alpha_1}^3 - 2\,\delta_1 \left(2\,\alpha_2 \alpha_3 - \delta_2 \alpha_6 - 2\,\beta_3 \beta_4 + \delta_4 \beta_5\right) {\alpha_1}^2 \\ & + {\delta_1}^2 \left(\alpha_5 \delta_3 + \delta_2 \alpha_6 + \delta_4 \beta_5 + \beta_6 \delta_3\right) \alpha_1 + {\delta_1}^3 \left(\alpha_2 \alpha_3 - \beta_3 \beta_4\right). \end{aligned}$$

In fact, we can easily get the second and third focus quantities, but we omit these lengthy expressions for brevity.

It proves in [17] that the focus quantities of three dimensional system have a structure analogous to that in two dimensional case. An algorithm based on this idea is formulated in that paper for three dimensional case. Some other methods for computing focus quantities can be found in [6, 8, 19, 20, 23, 27, 28].

For the three-dimensional system (2.1), it is shown in [19] that the focus quantities at $p(\mu_0)$ are the same with those of the restriction system on the center manifold of $p(\mu_0)$. This implies that, by using the information of focus quantities, we can obtain the local dynamics of system (2.1) for μ near μ_0.

In general, the focus quantities of a three dimensional system are very difficult to be obtained. However, these quantities are not only important in theoretical study, but also useful in applications, see [1, 9, 11–16, 22, 24–26, 29] and references therein.

3. Hopf bifurcation of system (1.4)

In this section, we mainly consider system (1.4) with $a, b, c > 0$, since system (1.3) is only a special case. It has an equilibrium at the origin $O = (0, 0, 0)$, which exists for any parameter values. According to Routh-Hurwitz criterion, O is asymptotically stable for $0 < c < 1$ and is unstable for $c > 1$. For $c = 1$, the origin is non-hyperbolic and its stability is stated in the following result.

Proposition 3.1. *For system* (1.4) *with* $c = 1$*, the origin is asymptotically stable.*

Proof. In this case the eigenvalues of system (1.4) at the origin are $\lambda_1 = -b < 0, \lambda_2 = -(a+1) < 0, \lambda_3 = 0$. Hence the origin is a non-hyperbolic equilibrium.

By introducing the transformation $x = x_1, y = \dfrac{az_1 - y_1}{a}, z = y_1 + z_1$, system (1.4) is transformed into

$$\begin{cases} \dfrac{dx_1}{dt} = -\,bx_1 - \dfrac{{y_1}^2}{a} + \dfrac{(a-1)\,z_1y_1}{a} + {z_1}^2, \\ \dfrac{dy_1}{dt} = (-a-1)\,y_1 + \dfrac{ay_1x_1}{a+1} + \dfrac{az_1x_1}{a+1}, \\ \dfrac{dz_1}{dt} = -\,\dfrac{ay_1x_1}{a+1} - \dfrac{az_1x_1}{a+1}. \end{cases} \tag{3.1}$$

We seek the center manifold of system (3.1) emanating from the origin in the form of

$$x_1 = k_1z_1^2 + O(z_1^3) \quad \text{and} \quad y_1 = k_2z_1^2 + O(z_1^3), \tag{3.2}$$

Substituting (3.2) into the last equation of (3.1) leads to the following equation, describing the dynamics on the center manifold:

$$\frac{dz_1}{dt} = -\frac{ak_1}{a+1}{z_1}^3 + O(z_1^4), \tag{3.3}$$

Furthermore, substituting (3.2) and (3.3) into the first two equations of (3.1) yields

$$(bk_1 - 1)z_1^2 + O(z_1^3) = 0, \tag{3.4}$$
$$(a+1)k_2z_1^2 + O(z_1^3) = 0. \tag{3.5}$$

Equating the coefficients of z_1^2 to zero in (3.4) and (3.5), we obtain $k_1 = \dfrac{1}{b}, k_2 = 0$. Then, substituting these values into (3.3) yields the equation describing the dynamics on the center manifold

$$\frac{dz_1}{dt} = -\frac{a}{b(a+1)}{z_1}^3 + O(z_1^4), \tag{3.6}$$

so the origin of system (1.4) is an asymptotically stable node. This completes the proof. □

For $c > 1$, besides the origin the system has the symmetric equilibria $E_{1,2}(c) = (c-1, \pm\sqrt{b\,(c-1)}, \pm\sqrt{b\,(c-1)})$. According to the criterion stated in the previous section (see also [10]), there exist two Hopf bifurcation points $(E_1(c_0), c_0)$ and $(E_2(c_0), c_0)$, where $c_0 = \dfrac{(a+b+3)\,a}{a-b-1}$ and

$$E_1(c_0) = \left(\frac{(a+1)\,(a+1+b)}{a-b-1}, \sqrt{\frac{b\,(a+1)\,(a+1+b)}{a-b-1}}, \sqrt{\frac{b\,(a+1)\,(a+1+b)}{a-b-1}}\right),$$
$$E_2(c_0) = \left(\frac{(a+1)\,(a+1+b)}{a-b-1}, -\sqrt{\frac{b\,(a+1)\,(a+1+b)}{a-b-1}}, -\sqrt{\frac{b\,(a+1)\,(a+1+b)}{a-b-1}}\right).$$

In order to study the stability of $E_{1,2}(c_0)$, it only needs to study $E_1(c_0)$ due to symmetry. The Jacobian matrix of system $(1.4)|_{c=c_0}$ at $E_1(c_0)$ has a pair of imaginary eigenvalues $\lambda_{1,2} = \pm\sqrt{\dfrac{2ab(a+1)}{a-b-1}}\,\mathtt{i}$ and a negative eigenvalue $\lambda_3 = -(a+b+1)$, where $a, b > 0$ and $a > b+1$.

Theorem 3.1. *As c is varied to pass through c_0, a subcritical Hopf bifurcation occurs at $(E_1(c_0), c_0)$, leading to an unstable limit cycle which exists for $c < c_0$, with each c near c_0. The non-hyperbolic equilibrium $E_1(c_0)$ is unstable.*

Proof.

For c near c_0, the characteristic equation of system (1.4) at $E_1(c)$ is

$$g(\lambda, c) = \lambda^3 + (a+1+b)\,\lambda^2 + b\,(a+c)\,\lambda + 2\,ab\,(c-1) = 0. \tag{3.7}$$

Using the implicit function theorem, we can compute the derivative of the complex eigenvalue $\lambda(c)$ with respect to c for the equilibrium: i.e.

$$\frac{d\lambda}{dc} = -\frac{\partial g}{\partial c}\Big/\frac{\partial g}{\partial \lambda} = -\frac{b\,(2\,a+\lambda)}{3\,\lambda^2 + (2\,a+2\,b+2)\,\lambda + b\,(a+c)}. \tag{3.8}$$

Substituting $c = c_0$ and $\lambda_{1,2} = \pm\sqrt{\dfrac{2ab(a+1)}{a-b-1}}\,\mathtt{i}$ into (3.8), we obtain

$$\left.\frac{d\Re(\lambda)}{dc}\right|_{c=c_0,\lambda=\lambda_{1,2}} = \frac{ba\,(a-b-1)}{a^3 + 3\,a^2 b - ab^2 - b^3 + a^2 - 3\,b^2 - a - 3\,b - 1}, \tag{3.9}$$

which is positive because $a, b > 0, a > b+1$ and the denominator of (3.9) has the following Taylor expansion at $a = b+1$:

$$2\,b^3 + 6\,b^2 + 4\,b + \left(8\,b^2 + 14\,b + 4\right)(a-b-1) + (4+6\,b)\,(a-b-1)^2 + (a-b-1)^3. \tag{3.10}$$

This implies that the transversality condition holds.

Computing with the aid of Maple, we get the first focus quantity of $E_1(c_0)$ for system $(1.4)|_{c=c_0}$,

$$W_1 = \frac{2b(a-b-1)W_{1,1}}{(a+b+1)W_{1,2}W_{1,3}}, \tag{3.11}$$

where

$$\begin{aligned} W_{1,1} &= 8\,b^4 + 48\,b^3 + 104\,b^2 + 96\,b + 32 + \left(20\,b^3 + 116\,b^2 + 208\,b + 112\right)(a-b-1) \\ &\quad + \left(20\,b^2 + 116\,b + 124\right)(a-b-1)^2 + (18\,b + 56)\,(a-b-1)^3 + 9\,(a-b-1)^4, \\ W_{1,3} &= W_{1,2} + 6ab(a+1), \end{aligned}$$

and $W_{1,2}$ is the same as (3.10). This implies that the Hopf bifurcation is subcritical and the non-hyperbolic equilibrium $E_1(c_0)$ is unstable.

Since the two non-degeneracy conditions of Hopf bifurcation [8] are satisfied, the proof is completed. $\square$

Due to the symmetry, we have:

Corollary 3.1. *System (1.4) can have at most two small limit cycles, which are unstable and symmetric with respect to the x-axis.*

Corollary 3.2. *System (1.3) can have at most two small limit cycles, which are unstable and symmetric with respect to the x-axis.*

4. Degenerate Hopf bifurcation of system (1.5)

In this section, we consider system (1.5) with $a, b, c > 0$. The system always has a hyperbolic equilibrium point at the origin $O = (0,0,0)$. Since the Jacobian matrix of the system at the origin has three negative eigenvalues $\lambda_1 = -a, \lambda_2 = -b, \lambda_3 = -1$, it follows that the equilibrium O is an asymptotically stable node.

If $a > \frac{c^2}{4}$, the origin is the only equilibrium of system (1.5). If $a = \frac{c^2}{4}$, the system has two non-trivial equilibria: $M_1 = (\frac{c}{2}, \frac{c}{2}\sqrt{b}, \sqrt{b})$ and $M_2 = (\frac{c}{2}, -\frac{c}{2}\sqrt{b}, -\sqrt{b})$. If $a < \frac{c^2}{4}$, then the system has four non-trivial equilibria:

$$N_{1,2}(c) = (\frac{c+\sqrt{c^2-4a}}{2}, \pm\sqrt{ab}, \frac{x_{eq}y_{eq}}{a}), N_{3,4}(c) = (\frac{c-\sqrt{c^2-4a}}{2}, \pm\sqrt{ab}, \frac{x_{eq}y_{eq}}{a}).$$

The equilibria $N_{3,4}(c)$ are locally unstable [3]. Hence, Hopf bifurcations cannot occur from $N_{3,4}(c)$. However, Hopf bifurcations can occur from $N_{1,2}(c)$ for $c = c_0$, where $c_0 = 4\sqrt{\frac{a}{(3\,a+b+1)(a-b-1)}}a$, see [3].

Proposition 4.1. *If $a = \frac{c^2}{4}$, $0 < b \leq \frac{(c^2+4b+4)^2}{128}$ and $c > 0$, then the equilibria M_1 and M_2 of system (1.5) are saddle-nodes.*

Proof. Due to the symmetry, we only consider the equilibrium M_1. With the invertible affine transformation

$$\begin{cases} x = \dfrac{\sqrt{b}\,(c-2)\,(c+2)\,x_1}{2c(\mu_1+b)} + \dfrac{\sqrt{b}\,(c-2)\,(c+2)\,y_1}{2c(\mu_2+b)} + \dfrac{cz_1}{2\sqrt{b}} + \dfrac{c}{2}, \\ y = -2\,\dfrac{x_1}{c} - 2\,\dfrac{y_1}{c} + \dfrac{1}{2}\,c\sqrt{b}, \\ z = x_1 + y_1 + z_1 + \sqrt{b}, \end{cases} \tag{4.1}$$

system (1.5) for $a = \frac{c^2}{4}$ is transformed into

$$\begin{cases} \dfrac{dx_1}{dt} = \mu_1 x_1 + O(|x_1, y_1, z_1|^2), \\ \dfrac{dy_1}{dt} = \mu_2 y_1 + O(|x_1, y_1, z_1|^2), \\ \dfrac{dz_1}{dt} = -\dfrac{c^2 {z_1}^2}{4\sqrt{b}} + O(|x_1, y_1, z_1|^2), \end{cases} \tag{4.2}$$

where the remainder $O(|x_1, y_1, z_1|^2)$ of the third equation has no z_1^2 term and

$$\mu_{1,2} = -\frac{1}{8}\left[c^2 + 4b + 4 \mp \sqrt{(c^2+4b+4)^2 - 128b}\right] < 0 \tag{4.3}$$

since $0 < b \leq \frac{(c^2+4b+4)^2}{128}$ and $c > 0$. Thus the equilibrium M_1 is a saddle-node. $\square$

From now on, we consider the Hopf bifurcations in system (1.5) for $a < \frac{c^2}{4}$. Due to the symmetry, we only need to consider the Hopf critical point $(N_1(c_0), c_0)$.

The Jacobian matrix evaluated at this point contains a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm 2\sqrt{\dfrac{ab}{a-b-1}}\,\mathtt{i}$ and a negative eigenvalue $\lambda_3 = -(a+b+1)$, where $a, b > 0$ and $a > b+1$. Let $\lambda(c), \overline{\lambda(c)}$ be the pair of complex conjugate eigenvalues of the Jacobian matrix at $N_1(c)$, when $|c - c_0| \geq 0$ and is small enough. It can be verified that

$$\left.\frac{d\Re(\lambda)}{dc}\right|_{c=c_0, \lambda=\lambda_{1,2}} = \frac{(3\,a+b+1)\,\sqrt{a\,(3\,a+b+1)\,(a-b-1)b}}{(a+1+b)\,((a-b-1)^3+4a(b+1)(a-b-1)+4ab)} > 0, \tag{4.4}$$

since $a, b > 0$ and $a > b+1$.

Computing the first two focus quantities with Maple, we get

$$W_1 = \frac{W_{1,1}}{W_{1,2}}, \quad W_2 = \frac{W_{2,1}}{W_{2,2}},$$

where

$$\begin{aligned}
W_{1,1} &= -2b(a-b-1)W_{1,1,1},\\
W_{1,1,1} &= 3\,a^6 + a^5b - 6\,a^4b^2 - 2\,a^3b^3 + 3\,a^2b^4 + ab^5 - 26\,a^5 + 79\,a^4b - 64\,a^3b^2\\
&\quad -18\,a^2b^3 + 26\,ab^4 + 3\,b^5 - 83\,a^4 - 2\,a^3b - 128\,a^2b^2 + 6\,ab^3 + 15\,b^4 - 60\,a^3\\
&\quad -94\,a^2b - 40\,ab^2 + 30\,b^3 + 13\,a^2 + ab + 30\,b^2 + 22\,a + 15\,b + 3,\\
W_{1,2} &= (1+b)\,(3\,a+b+1)\,(a+1+b)\left((a-b-1)^3 + 4a(b+1)(a-b-1) + 4ab\right)\\
&\quad \left((a-b-1)^3 + 4a(b+1)(a-b-1) + 16ab\right),\\
W_{2,1} &= (a-b-1)^2W_{2,1,1},\\
W_{2,1,1} &= -3258059904\,a^{20} + 6516120064\,b^2a^{18} - 3258060192\,a^{16}b^4 + 32\,b^{20}\\
&\quad -259065479424\,a^{19} - 100999950848\,ba^{18} + 677775960576\,b^2a^{17}\\
&\quad -3257928384\,a^{16}b^3 - 314452601088\,a^{15}b^4 - 832\,b^{19} - 8827912263168\,a^{18}\\
&\quad -8859403367424\,ba^{17} + 31857705778176\,a^{16}b^2\\
&\quad -140936027136\,a^{15}b^3 - 13873067300352\,a^{14}b^4 - 154143851609856\,a^{17}\\
&\quad -349289871464960\,ba^{16} + 879169105794048\,b^2a^{15} - 66062310912\,a^{14}b^3\\
&\quad -361536724218112\,a^{13}b^4 + 187392\,b^{17} - 1040509658472320\,a^{16}\\
&\quad -7915666241953792\,ba^{15} + 15397825461539840\,b^2a^{14} + 116815879533568\,a^{13}b^3\\
&\quad -5981184425961600\,a^{12}b^4 - 6079488\,b^{16} + 10098713518490624\,a^{15}\\
&\quad -108400413021706240\,ba^{14} + 171175266743790592\,b^2a^{13}\\
&\quad +3096862736013056\,a^{12}b^3 - 62293822308234496\,a^{11}b^4 + 60936192\,b^{15}\\
&\quad +260677668234749952\,a^{14} - 840808229478297600\,ba^{13}\\
&\quad +1114360079077395456\,b^2a^{12} + 32614476187620352\,a^{11}b^3\\
&\quad -367783918681271296\,a^{10}b^4 + 782671872\,b^{14} + 1897839617667890176\,a^{13}\\
&\quad -2580371207904204800\,ba^{12} + 3050419049627468800\,b^2a^{11}\\
&\quad +739564445948492 80\,a^{10}b^3 - 737656140429204736\,a^9b^4 - 29340524544\,b^{13}\\
&\quad +2942907584602755840\,a^{12} + 7357680362764673024\,a^{11}b\\
&\quad -5824866116792907776\,b^2a^{10} - 1045943484665699328\,a^9b^3\\
&\quad +3316924444472137536\,a^8b^4 + 28976578560\,b^{12} - 21767415433929340416\,a^{11}
\end{aligned}$$

$$
\begin{aligned}
&+49375461605891691520\, ba^{10} - 414340396156664431104\, a^9b^2\\
&-4425427223618612352\, a^8b^3 + 125081357329225945 60\, a^7b^4\\
&+13380533125120\, b^{11} - 31902597217969533952\, a^{10}\\
&-97054545373928093696\, ba^9 + 353015238692089825 28\, a^8b^2\\
&+17926874674999192576\, a^7b^3 - 31130277921786756608\, a^6b^4\\
&-226191046344704\, b^{10} + 198371445570450798080\, a^9\\
&-258562668994865806336\, a^8b + 106138327729522342912\, a^7b^2\\
&+31344093485365255680\, a^6b^3 - 7762118542521885440\, a^5b^4\\
&-2540022509076480\, b^9 - 483401896782444273408\, a^8\\
&+208263141166818611 2000\, a^7b - 102654549984535515 6480\, a^6b^2\\
&-287688652923368793088\, a^5b^3 + 45966222908404887 1808\, a^4b^4\\
&+136898713821904896\, b^8 - 716104288171369012224\, a^7\\
&-394296625796263032 2176\, a^6b + 141510142573087000 6784\, a^5b^2\\
&+71711182256781941 1200\, a^4b^3 - 1244841800604884883200\, a^3b^4\\
&-650186986547576832\, b^7 + 550734924642626320 9984\, a^6\\
&-642230286170717493 2480\, a^5b + 489404485568212408 8320\, a^4b^2\\
&+29912765313299823 7184\, a^3b^3 - 381165531245698461696\, a^2b^4\\
&-583616028916045578 24\, b^6 - 4450292220526566220800\, a^5\\
&+429784995032298356 18304\, a^4b - 21438155204997433054208\, b^2a^3\\
&-854408837337529738 1888\, a^2b^3 + 1170686384031181179 7248\, ab^4\\
&+1142623384981179727872\, b^5 - 36087595186601715482240\, a^4\\
&+572139784847679418 3680\, ba^3 - 5823842014965174276 9664\, a^2b^2\\
&+2210481623555226504192\, ab^3 + 655593335063213771 9136\, b^4\\
&-3014511048023596665 6768\, a^3 - 45290330914201925628416\, a^2b\\
&-2045407398040993479 3216\, ab^2 + 1366087439358496251 9360\, b^3\\
&+4785166393320218221056\, a^2 - 7121374702383019100 16\, ab\\
&+13931113348894410014720\, b^2 + 10245554293415047590144\, a\\
&+7045717327768708187648\, b + 1419740623489575597696,\\
W_{2,2} ={}& 6\,a\,(1+b)^2\left((a-b-1)^3+4a(b+1)(a-b-1)+36ab\right)(3\,a+b+1)^2\\
&\left((a-b-1)^3+4a(b+1)(a-b-1)+16ab\right)^2(a+1+b)^3\\
&\left((a-b-1)^3+4a(b+1)(a-b-1)+4ab\right)^3,
\end{aligned}
$$

and $W_{2,1,1}$ is reduced w.r.t. $W_{1,1,1}$. It is easy to see that $W_{1,2}$ and $W_{2,2}$ are positive.

By using the procedure (RootFinding[Isolate]) built in Maple, we find that the two polynomial equations $W_1 = W_2 = 0$ have no solutions satisfying $a > b+1, a > 0, b > 0$. Therefore, there is no need to calculate W_3. However, there exist some positive solutions of the semi-algebraic system $W_1 = 0, W_2 \neq 0$ for a, b. Thus, system (1.5) can have at most two small limit cycles in some neighborhood of $N_1(c)$. Due to the symmetry, at most four small limit cycles can be found for the system.

Lemma 4.1. *If*

$$c = c_0 = \frac{16\sqrt{14}}{7}, \quad a = 8, \quad b = 3, \tag{4.5}$$

then the equilibrium $N_1(c_0)$ is a stable weak focus of Order 2 for the flow of system (1.5) *restricted to the center manifold.*

Proof. When $a = 8, b = 3$, the critical value of Hopf bifurcation for system (1.5) is $c = \frac{16\sqrt{14}}{7}$. If (4.5) holds, then it is easy to see that

$$W_1 = 0, \qquad W_2 = -\frac{2285}{1053696} < 0,$$

and thus the conclusion follows. □

For the case $4 < a < 8, b = 3$, we have $W_1 > 0$. Thus, as c crosses the critical value $c_0 = \frac{16\sqrt{14}}{7}$, a supercritical Hopf bifurcation occurs, resulting in an unstable limit cycle around the equilibrium $N_1(c)$ for $c < c_0$ and near c_0. For the case $a > 8, b = 3$, we have $W_1 < 0$. Thus as c crosses the critical value $c_0 = \frac{16\sqrt{14}}{7}$, a supercritical Hopf bifurcation occurs, resulting in a stable limit cycle around the equilibrium $N_1(c)$ for $c > c_0$ and near c_0.

Lemma 4.2. *By varying (c, a) in a neighborhood of $(\frac{16\sqrt{14}}{7}, 8)$, system* (1.5) *with $b = 3$ can yield two limit cycles around the equilibrium $N_1(c)$. The outermost limit cycle is stable, while the small one is unstable.*

Proof. By Lemma 4.1 and the transversal condition (4.4), it is suffice to prove the result by noting that

$$\left.\frac{\partial W_1}{\partial a}\right|_{a=8,b=3} = -\frac{431}{94080} < 0.$$

Thus, the system has a transversal Hopf point of codimension two. According to the theory of degenerate Hopf bifurcation (Bautin bifurcation), see [4, 5, 21], for $b = 3$ and $(c, a) : 0 < \frac{16\sqrt{14}}{7} - c \ll 8 - a \ll 1$, there are two limit cycles around the equilibrium $N_1(c)$ with the innermost cycle unstable and the outermost stable. □

Based on the above lemmas and due to symmetry, we get the following result.

Theorem 4.1. *System* (1.5) *can present at most four small limit cycles in two symmetric nests, and this bound is sharp.*

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