Journal of Nonlinear Modeling and Analysis
Volume 3, Number 2, June 2021, 301–319
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.301

Modeling the Transmission of West Nile Virus with *Wolbachia* in a Heterogeneous Environment*

Mengli Liu[1], Min Zhu[1,†] and Xiaofei Song[1]

Abstract *Wolbachia* are maternally transmitted endosymbiotic bacteria. To investigate the effect of *Wolbachia* on the spreading and vanishing of West Nile virus, we construct a reaction-diffusion model associated with the *Wolbachia* parameter in a heterogeneous environment, which has nonlinear infectious disease parameters. Based on the spectral radius of next infection operator and the related eigenvalue problem, we present a corresponding explicit expression describing the basic reproduction number. Furthermore, utilizing this number, we not only give out the stability of disease-free equilibrium, but also analyze the uniqueness and globally asymptotic behavior of endemic equilibrium. Our theoretical results and numerical simulations indicate that only if *Wolbachia* reach a certain magnitude in mosquitoes, it can be effective in the control of West Nile virus.

Keywords West Nile virus model, *Wolbachia*, Basic reproduction number, Stability.

MSC(2010) 35K20, 35B40, 92D30.

1. Introduction and Model formulation

With the development of economy, the excessive exploitation and unreasonable use of the nature resources aggravate the environmental problems, and the habitats of animals are suffering more and more destruction [32]. Some viruses, which originally transmitted only from animal to animal, are now found to be showing signs of human-to-animal or human-to-human transmission. In this paper, we will focus on West Nile virus(WNv) whose transmission process accords with the aforementioned animal-to-human mechanism.

The WNv, a member of the Flavivirus, usually transmitted between birds and mosquitoes [17]. A wide range of vertebrate including mankind are likely to be accidental hosts of this virus [2]. When birds are bitten by an infected mosquito, the birds' titers in the body continue to rise for three to five days [9], and then the bird transmit the virus to the mosquitoes that bite them [26]. Although mosquitoes may avoid the disadvantageous effects of WNv, birds (especially corvids) have high mortality risk caused by this virus. In 1937, WNv was isolated and identified in the blood of an Ugandan woman [4] for the first time. To begin with, the virus

[†]the corresponding author.
Email address: min_zhuly@163.com(M. Zhu)
[1]School of Mathematics and Statistics, Anhui Normal University, Wuhu, Anhui 241000, China
[*]The authors were supported by National Natural Science Foundation China (Grant No. 11801009) and Initial Scientific Research Fund of Doctor in Anhui Normal University (Grant No. 751993).

was prevalent throughout Israel, France, South Africa, Algeria, Romania, Czech republic, Congo, Russia, etc. Subsequently, in August 1999, it broke out in North America and spread rapidly across much of the country [2, 4, 8, 16]. Except for causing onset and death of large numbers of birds and other wildlife, WNv has also influenced some human [3]. In 2011, WNv was isolated from mosquito samples in Xinjiang Uygur Autonomous Region of China [12]. A large number of serological studies have proved that there once existed diseases caused by WNv infection in this area. WNv has constantly threatened public health constantly, and impeded the development of economy seriously around the world over the last few decades. Therefore, it is of great practical significance to study the transmission mechanism of this virus and take effective measures to control it.

The mathematical model plays a crucial role in terms of preventing and controlling diseases, which can not only help us understand the transmission mechanism of infectious diseases, but also have an impact on predicting, estimating and guiding its development tendency. In order to control and eliminate WNv, we need to pay close attention to how it sustains between organisms and how to keep the morbidity and mortality within certain levels. Faced with these problems, Wonham *et al.* [31] developed an SIR model of WNv cross-infection involving mosquitoes and birds, as well as provided a simple way to determine the control levels of mosquitoes. In literature [9], the authors incorporated the factor of vertical transmission to a WNv model, and found that if the vertical propagation coefficient was high enough, the endemic proportion of infected birds would rise. In [24], researchers were more interested in identifying predictors of WNv incidence in mosquitoes based on a Bayesian space-time model. According to a single-season model to WNv, authors proved that the strategies to decrease mosquitoes and personal protection may prevent WNv effectively in [6]. Besides, Lin and Zhu *et al.* [13, 20] established a reaction-diffusion system with free boundary to explore the transmission mechanism of time and space based on a WNv model between mosquitoes and birds. To study the transmission rates and traveling waves of WNv, Lewis *et al.* [16, 17] initiated a survey of spatial-temporal transmission of disease and considered the spatially-independent and spatially-dependent WNv models with the following forms respectively,

$$\begin{cases} \frac{dI_b}{dt} = \alpha_b\beta_b\frac{N_b - I_b}{N_b}I_m - \gamma_b I_b, \\ \frac{dI_m}{dt} = \alpha_m\beta_b\frac{A_m - I_m}{N_b}I_b - d_m I_m, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \frac{\partial I_b}{\partial t} = d_1\Delta I_b + \alpha_b\beta_b\frac{N_b - I_b}{N_b}I_m - \gamma_b I_b, & t \in (0, +\infty), x \in \Omega, \\ \frac{\partial I_m}{\partial t} = d_2\Delta I_m + \alpha_m\beta_b\frac{A_m - I_m}{N_b}I_b - d_m I_m, & t \in (0, +\infty), x \in \Omega, \\ I_b(0, x) = I_{b,0}(x),\ I_m(0, x) = I_{m,0}(x), & x \in \overline{\Omega}. \end{cases} \tag{1.2}$$

The parameters and variables mentioned above are defined in Table 1.

Meanwhile, Lewis *et al.* exhibited the corresponding basic reproduction number to problem (1.1) as follows:

$$\Re_0 = \sqrt{\frac{\frac{A_m}{N_b}\alpha_m\alpha_b\beta_b^2}{d_m\gamma_b}}. \tag{1.3}$$

Table 1. Parameters and variables in models (1.1) and (1.2)

Parameters or Variables	Definition	Value	Reference
I_m	numbers of infected mosquitoes	–	–
I_b	population of infected birds	–	–
d_1	diffusion coefficient of birds	–	–
d_2	diffusion coefficient for mosquitoes	–	–
N_b	total population of birds	–	–
A_m	total numbers for adult mosquitoes	A_m/N_b=20	[16]
d_m	mortality for adult mosquitoes	0.029	[16]
β_b	mosquitoes biting on birds	0.3	[16]
γ_b	recovery rate of birds which infected WNv	0.01	[16]
α_m	WNv spreading rate per bite to vector(mosquito)	0.16	[31]
α_b	WNv spreading rate per bite to host(bird)	0.88	[31]

As we know, mathematically, most of WNv models are spatially-independent [1,5,31], in which the recovery and contact rates are usually constants. Considering the space diffusion and spatial heterogeneity [13, 16–18] have gradually become important factors affecting the persistence and spread of infectious diseases, in [25], Allen *et al.* not only introduced spatial diffusion into the classical SIS model, but also adopted the spatially-dependent recovery rate and contact transmission rate so as to explore the influence of random diffusion and environmental difference on infectious diseases.

However, in the real society, the spread of WNv is not merely related to space, but also directly affected by the density of mosquito population. In order to prevent and control mosquito-borne diseases for the past few years, in biology, scientists have tried to use *Wolbachia* to alter the reproductive mechanism of mosquitoes. Studies find that *Wolbachia* are a group of intracellular bacteria that infect the reproductive tissues of mosquitoes, alter reproduction in their mosquito hosts in various ways [30] and to some extent affect the lifespan of mosquitoes [23]. Considering the utility of *Wolbachia* on disease transmission [14, 33, 36], we will use the parameter p to represent the effect of *Wolbachia* on the disease and adopt the death rate of adult mosquitoes involving p. Meanwhile, it is worth observing that the death rate of infected birds is influenced by *Wolbachia* and the number of infected birds simultaneously. Take the above two factors into consideration, we use $\mu_b(x,p,I_b)$ to represent the mortality of infected birds in WNv model. What's more, some researches show that the flight distance of mosquito is much less then birds', which means that the diffusion coefficients $d_2 \ll d_1$ in model (1.2) [29]. Based on the fact, we can suppose that $d_2 = 0$. Combining these factors with models (1.1) and (1.2), we establish the following WNv model

$$\begin{cases} \frac{\partial I_b}{\partial t} - d_1\Delta I_b = \alpha_b(x)\beta_b(x)\frac{N_b-I_b}{N_b}I_m - \gamma_b(x)I_b - \mu_b(x,p,I_b)I_b, & x\in\Omega, t>0, \\ \frac{\partial I_m}{\partial t} = \alpha_m(x)\beta_b(x)\frac{A_m-I_m}{N_b}I_b - d_m(p)I_m, & x\in\Omega, t>0, \\ I_b(x,t) = I_m(x,t) = 0, & x\in\partial\Omega, t>0, \\ 0 \le I_b(x,0) = I_{b,0}(x) \le N_b,\ 0 \le I_m(x,0) = I_{m,0}(x) \le A_m, & x\in\overline{\Omega}, \end{cases} \tag{1.4}$$

where $\gamma_b(x), \alpha_b(x), \alpha_m(x)$ and $\beta_b(x)$ for $x \in \overline{\Omega}$ are sufficiently smooth and positive. Besides, $\mu_b(x, p, I_b)$ is a sufficiently smooth non-negative function, which increases with respect to I_b and decreases with respect to p, while $d_m(p)$ increases with respect to p. The initial functions $I_{b,0}(x), I_{m,0}(x) \in \mathcal{C}^2(\Omega)$ are both nontrivial. At the same time, the boundary condition $I_b(x,t) = I_m(x,t) = 0$ means that there is no infection on the boundary $\partial\Omega$.

In this paper, we are devoted to exploring how the parameter p caused by the *Wolbachia* affect the dynamic of transmission on model (1.4). The framework of this paper is arranged as follows: To begin with, in Section 2, the basic reproduction number to model (1.4) is presented and the corresponding analytical findings are discussed. What's more, in Section 3, we are concerned with the stability of equilibria. At last, in Section 4, in order to understand the conclusion of this paper more clearly, we will show some numerical simulations, and then end up giving some epidemiological explanations.

2. The basic reproduction number

In general, the basic reproduction number is a threshold parameter, and is often defined as the average number of secondary infections that one infected person would produce in the entire duration of the infectious period [10]. For ordinary differential epidemic models, this number is usually obtained by the next generation matrix method [11, 25, 34]. However, in reaction-diffusion equation models, we can use the spectral radius of the next infection operators to express this number. Meanwhile, Allen *et al.* [25] applied the variational method to solve most scalar eigenvalue problems.

It is obvious that model (1.4) has a disease-free equilibrium $(I_b, I_m) = (0, 0)$. Linearizing model (1.4) at this equilibrium, we obtain the following system

$$\begin{cases} \frac{\partial I_b}{\partial t} = d_1 \Delta I_b + \alpha_b(x)\beta_b(x) I_m - \gamma_b(x) I_b - \mu_b(x, p, 0) I_b, & x \in \Omega, t > 0, \\ \frac{\partial I_m}{\partial t} = \alpha_m(x)\beta_b(x)\frac{A_m}{N_b} I_b - d_m(p) I_m, & x \in \Omega, t > 0, \\ I_b(x,t) = I_m(x,t) = 0, & x \in \partial\Omega, t > 0, \\ 0 \le I_b(x,0) = I_{b,0}(x) \le N_b, \ 0 \le I_m(x,0) = I_{m,0}(x) \le A_m, \ x \in \overline{\Omega}. \end{cases} \tag{2.1}$$

Evidently, the first two equations in the above problem can be rewritten as

$$\begin{cases} \frac{\partial \omega}{\partial t} = d\Delta\omega + u(x)\omega - v(x)\omega, \ x \in \Omega, t > 0, \\ \omega = 0, \qquad\qquad\qquad\qquad\qquad x \in \partial\Omega, t > 0, \end{cases}$$

in which

$$\omega = \begin{pmatrix} I_b \\ I_m \end{pmatrix}, \quad u(x) = \begin{pmatrix} 0 & \alpha_b(x)\beta_b(x) \\ \alpha_m(x)\beta_b(x)\frac{A_m}{N_b} & 0 \end{pmatrix},$$

$$d = \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad v(x) = \begin{pmatrix} \gamma_b(x) + \mu_b(x, p, 0) & 0 \\ 0 & d_m(p) \end{pmatrix}. \tag{2.2}$$

Based on the analysis of infectious disease model in [34], the self-evolution of individuals in the infection compartment is determined by system

$$\begin{cases} \frac{\partial \omega}{\partial t} = d\Delta\omega - v(x)\omega, & x \in \Omega, t > 0, \\ \omega = 0, & x \in \partial\Omega, t > 0. \end{cases}$$

Let W be a solution semigroup formed by the operator $d\Delta - v(x)$ on an ordered Banach space X_1 which is defined as $C(\overline{\Omega}, \mathbb{R}^2)$ and associated with $X_1^+ := C(\overline{\Omega}, \mathbb{R}_+^2)$. At the same time, L can be defined

$$L(\phi)(x) := \int_0^\infty u(x)[W(t)\phi](x)dt.$$

Consequently, L can map ϕ(the initial distribution function of infected individual), in an infectious period, into the distribution functions which have infectious individual, as well as L is a continuous and positive operator. According to the method of next infection operator stated in [34], one defines

$$R_0^p := r(L)$$

as the basic reproduction number to problem (1.4), which is precisely the spectral radius of the operator L. In addition, for the sake of later research, we show some properties of R_0^p.

Lemma 2.1. The following statements are true.

(i) $R_0^p = \frac{1}{\mu_0}$, where μ_0 is the unique principal eigenvalue of the following elliptic eigenvalue problem

$$\begin{cases} -d_1\Delta\phi = \mu\alpha_b(x)\beta_b(x)\psi - \gamma_b(x)\phi - \mu_b(x,p,0)\phi, & x \in \Omega, \\ 0 = \mu\alpha_m(x)\beta_b(x)\frac{A_m}{N_b}\phi - d_m(p)\psi, & x \in \Omega, \\ \phi(x) = \psi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

In other wards, R_0^p meets with

$$\begin{cases} -d_1\Delta\phi = \frac{1}{R_0^p}\alpha_b(x)\beta_b(x)\psi - \gamma_b(x)\phi - \mu_b(x,p,0)\phi, & x \in \Omega, \\ 0 = \frac{1}{R_0^p}\alpha_m(x)\beta_b(x)\frac{A_m}{N_b}\phi - d_m(p)\psi, & x \in \Omega, \\ \phi(x) = \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

in which $(\phi(x), \psi(x))$ is the eigenfunction-pair and satisfies $\phi(x) > 0$,$\psi(x) > 0$ in Ω.

(ii) $1 - R_0^p$ has the same sign as λ_1, which accords with the eigenvalue problem

$$\begin{cases} -d_1\Delta\phi = \alpha_b(x)\beta_b(x)\psi - \gamma_b(x)\phi - \mu_b(x,p,0)\phi + \lambda_1\phi, & x \in \Omega, \\ 0 = \alpha_m(x)\beta_b(x)\frac{A_m}{N_b}\phi - d_m(p)\psi + \lambda_1\psi, & x \in \Omega, \\ \phi(x) = \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

and λ_1 is the corresponding principal eigenvalue.

(iii) The basic reproduction number R_0^p of problem (1.4) is expressed by

$$(R_0^p)^2 = \sup_{\phi\in H_0^1(\Omega),\phi\neq 0} \frac{\displaystyle\int_\Omega \frac{\alpha_b(x)\alpha_m(x)\beta_b^2(x)A_m}{d_m(p)N_b}\phi^2 dx}{\displaystyle\int_\Omega (d_1|\nabla\phi|^2 + \gamma_b(x)\phi^2 + \mu_b(x,p,0)\phi^2)dx}. \tag{2.6}$$

(iv) Assume that $\alpha_b(x) = \alpha_b, \beta_b(x) = \beta_b, \gamma_b(x) = \gamma_b, \alpha_m(x) = \alpha_m, d_m(p) = d_m, \mu_b(x,p,I_b) = \mu_b$ are all positive constants, we can define $R_0^p = R_0^*$ and

$$R_0^* = \sqrt{\frac{\frac{\alpha_b\alpha_m\beta_b^2A_m}{d_mN_b}}{d_1\lambda^* + \gamma_b + \mu_b}}, \tag{2.7}$$

here, (λ^*, ψ^*) is the principal eigen-pair of eigenvalue problems $-\Delta\psi = \lambda\psi(x \in \Omega)$ with null-boundary condition.

Proof. We will give proofs of the above four lemmas in turn.

(i) Considering the following eigenvalue problem

$$\begin{cases} d_1\Delta\phi + \mu\alpha_b(x)\beta_b(x)\psi - \gamma_b(x)\phi - \mu_b(x,p,0)\phi = \lambda\phi, & x\in\Omega, \\ \mu\alpha_m(x)\beta_b(x)\frac{A_m}{N_b}\phi - d_m(p)\psi = \lambda\psi, & x\in\Omega, \\ \phi(x) = \psi(x) = 0, & x\in\partial\Omega. \end{cases} \tag{2.8}$$

In order to clarify the existence of the principal eigenvalue $\lambda = \lambda(\mu)$ for any given $\mu > 0$, we define the linear operator as follows,

$$L_\lambda\phi = d_1\Delta\phi - (\gamma_b(x)+\mu_b(x,p,0))\phi + \frac{\mu\alpha_b(x)\beta_b(x)\cdot\mu\alpha_m(x)\beta_b(x)\frac{A_m}{N_b}}{(\lambda + d_m(p))}\phi, \quad \forall\lambda > -d_m(p).$$

Since $\alpha_m(x),\ \alpha_b(x),\ \beta_b(x)$ are all positive for $x \in \overline{\Omega}$, we set

$$K := \mu^2\frac{A_m}{N_b}\min_{x\in\overline{\Omega}}(\alpha_b(x)\beta_b(x)\alpha_m(x)\beta_b(x)),$$

and let λ_0^* be the principle eigenvalue of the following elliptic eigenvalue problem

$$\begin{cases} d_1\Delta\phi - (\gamma_b(x) + \mu_b(x,p,0))\phi = \lambda\phi, & x\in\Omega, \\ \phi(x) = 0, & x\in\partial\Omega, \end{cases}$$

with positive eigenfunction $\phi_1(x)$. Set

$$\lambda_0 := \frac{\lambda_0^* - d_m(p) + \sqrt{(\lambda_0^* + d_m(p))^2 + 4K}}{2},$$

we have $\lambda_0 > -d_m(p)$ due to $K > 0$. It then follows that

$$L_{\lambda_0}\phi_1(x) \geq d_1\Delta\phi_1(x) - (\gamma_b(x) + \mu_b(x,p,0))\phi_1(x) + \frac{K}{(\lambda_0 + d_m(p))}\phi_1(x)$$

$$= (\lambda_0^* + \frac{K}{\lambda_0 + d_m(p)})\phi_1(x)$$
$$= \lambda_0 \phi_1(x), \quad \forall x \in \Omega.$$

Combining Theorem 11.3.2 and Remark 11.3.1 in [34], problem (i) has an eigenvalue with geometric multiplicity one and a nonnegative eigenfunction. According to (2.8) and its associated parabolic system, we see that this eigenfunction is positive. Therefore, μ_0 is the unique solution of $\lambda(\mu) = 0$. Using the same idea of Theorem 11.3.4 in [34], we obtain $R_0^p = \frac{1}{\mu_0}$.

(ii) Based on the statement (i), we can clearly see that the principal eigenvalue λ_1 exists when $\mu = 1$ in (2.8). Therefore, we omit the rest proof of (ii) here because it is a straightforward consequence of Theorem 11.3.3 in [34].

(iii) In what follows, we give the proof of assertion (iii). From the second equation of (2.4), one gets

$$\psi = \frac{1}{R_0^p} \frac{\alpha_m(x)\beta_b(x)}{N_b} \frac{A_m}{d_m(p)} \phi, \quad x \in \Omega.$$

Putting ψ into first equation in (2.4) leads to

$$-d_1 \Delta \phi = \frac{1}{(R_0^p)^2} \alpha_b(x)\beta_b(x) \frac{\alpha_m(x)\beta_b(x)}{N_b} \frac{A_m}{d_m(p)} \phi - \gamma_b(x)\phi - \mu_b(x, p, 0)\phi, \quad x \in \Omega.$$

We multiply ϕ simultaneously in above equation and then integrate on Ω, which can result in

$$\int_\Omega -d_1 \phi \Delta \phi dx = \frac{1}{(R_0^p)^2} \int_\Omega \alpha_b(x)\beta_b(x) \frac{\alpha_m(x)\beta_b(x)}{N_b} \frac{A_m}{d_m(p)} \phi^2 dx - \int_\Omega \gamma_b(x)\phi^2 dx - \int_\Omega \mu_b(x, p, 0)\phi^2 dx.$$

Combining the null-boundary conditions, we acquire the explicit expression (2.6) of R_0^p according to the variational method [7].

(iv) We are in a position to prove (iv). Setting

$$C^* = \frac{\frac{\alpha_b \alpha_m \beta_b^2 A_m}{d_m N_b}}{d_1 \lambda^* + \gamma_b + \mu_b},$$

we can yield that (ϕ^*, ψ^*) is one positive solution of system (2.4) associated with $R_0^* = \sqrt{C^*}$ through a simple calculation, where

$$(\phi^*, \psi^*) = (C^* \psi^*, \psi^*) = \left(\sqrt{\frac{\alpha_b d_m}{\alpha_m \frac{A_m}{N_b}(d_1 \lambda^* + \gamma_b + \mu_b)}} \psi^*, \psi^* \right).$$

Owing to the uniqueness of the principal eigen-pair, we naturally have the expression equation (2.7).

□

Remark 2.1. Under the hypothesis in Lemma 2.1(iv), the basic reproduction number defined as R_0^* in (2.7), is strictly less than $\mathfrak{R}_0$ which is corresponding one to the ordinary differential equation (1.1) and given by (1.3).

Remark 2.2. $R_0^p(d_m(p_1)) \geq R_0^p(d_m(p_2))$ provided that $p_1 \leq p_2$ when other parameters are fixed.

Remark 2.3. $R_0^p(\mu_b(x,p_1,I_b)) \leq R_0^p(\mu_b(x,p_2,I_b))$ provided that $p_1 \leq p_2$ encountered if other parameters are fixed.

3. The stability of the model

We aim at analyzing the equilibrium stability of problem (1.4), and begin this section with the following definition and lemma in view of [27].

Definition 3.1. (Upper-lower solutions) Assume $(\tilde{I}_b, \tilde{I}_m)(x,t)$ and $(\hat{I}_b, \hat{I}_m)(x,t)$ are two pairs of nonnegative functions in $\mathcal{C}^{1,2}(\Omega \times [0,+\infty)) \cap \mathcal{C}(\overline{\Omega} \times [0,+\infty))$ with $(\tilde{I}_b, \tilde{I}_m) \geq (\hat{I}_b, \hat{I}_m)$ in $\overline{\Omega} \times [0,+\infty)$. If

$$\begin{cases} \frac{\partial \tilde{I}_b}{\partial t} - d_1 \Delta \tilde{I}_b \geq \alpha_b(x)\beta_b(x)\frac{N_b - \tilde{I}_b}{N_b}\tilde{I}_m - \gamma_b(x)\tilde{I}_b - \mu_b(x,p,\tilde{I}_b)\tilde{I}_b, & x \in \Omega, t > 0, \\ \frac{\partial \tilde{I}_m}{\partial t} \geq \alpha_m(x)\beta_b(x)\frac{A_m - \tilde{I}_m}{N_b}\tilde{I}_b - d_m(p)\tilde{I}_m, & x \in \Omega, t > 0, \\ \frac{\partial \hat{I}_b}{\partial t} - d_1 \Delta \hat{I}_b \leq \alpha_b(x)\beta_b(x)\frac{N_b - \hat{I}_b}{N_b}\hat{I}_m - \gamma_b(x)\hat{I}_b - \mu_b(x,p,\hat{I}_b)\hat{I}_b, & x \in \Omega, t > 0, \\ \frac{\partial \hat{I}_m}{\partial t} \leq \alpha_m(x)\beta_b(x)\frac{A_m - \hat{I}_m}{N_b}\hat{I}_b - d_m(p)\hat{I}_m, & x \in \Omega, t > 0, \\ (\hat{I}_b, \hat{I}_m)(x,t) \leq (0,0) \leq (\tilde{I}_b, \tilde{I}_m)(x,t), & x \in \partial\Omega, t > 0, \\ (0,0) \leq (\hat{I}_{b,0}, \hat{I}_{m,0}) \leq (\tilde{I}_{b,0}, \tilde{I}_{m,0}) \leq (N_b, A_m), & x \in \overline{\Omega}, \end{cases} \tag{3.1}$$

holds, then $(\tilde{I}_b, \tilde{I}_m)(x,t)$ and $(\hat{I}_b, \hat{I}_m)(x,t)$ are the ordered upper-lower solutions of (1.4).

Lemma 3.1. (Comparison principle) If (3.1) holds, any solution $(I_b(x,t), I_m(x,t))$ to problem (1.4) has the following inequality relationship

$$(\hat{I}_b(x,t), \hat{I}_m(x,t)) \leq (I_b(x,t), I_m(x,t)) \leq (\tilde{I}_b(x,t), \tilde{I}_m(x,t)), \quad x \in \overline{\Omega}, t \geq 0.$$

3.1. Stability analysis of endemic equilibrium

The upper-lower solution will be used to determine the behavior of endemic equilibrium (I_b^*, I_m^*) to system (1.4) under the hypothesis $R_0^p > 1$ in this subsection. First of all, we exhibit the steady state of system (1.4) as follows:

$$\begin{cases} -d_1 \Delta I_b = \alpha_b(x)\beta_b(x)\frac{N_b - I_b}{N_b}I_m - \gamma_b(x)I_b - \mu_b(x,p,I_b)I_b, & x \in \Omega, \\ 0 = \alpha_m(x)\beta_b(x)\frac{A_m - I_m}{N_b}I_b - d_m(p)I_m, & x \in \Omega, \\ I_b(x) = I_m(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

In Definition 3.1 and Lemma 3.1, if time t is not taken into account, we can give a similar upper-lower definition and a corresponding comparison principle for elliptic problem (3.2), whose details we omit here. Besides, we give out a denotation for the discussion in the sequel

$$\begin{aligned} &\langle (m_1,n_1),(m_2,n_2) \rangle \\ &= \{(m,n) : m_1 \leq m(x,t) \leq m_2, n_1 \leq n(x,t) \leq n_2, (x,t) \in \overline{\Omega} \times [0,+\infty)\}. \end{aligned}$$

Theorem 3.2. When $R_0^p > 1$, the endemic equilibrium $(I_b^*(x), I_m^*(x))$ is unique and globally asymptotically stable to problem (1.4).

Proof. In order to prove the existence of solutions of problem (3.2), we first set

$$(\hat{I}_b(x), \hat{I}_m(x)) = (\delta\phi(x), \delta\psi(x)),$$

where δ is positive, sufficiently small and is determined by subsequent calculations, $\phi(x)$ and $\psi(x)$ are the corresponding eigenfunction pairs of system (2.5). Recalling that $\mu_b(x, p, I_b)$ increases with respect to I_b and combining Lemma 2.1(ii), one directly calculates

$$\begin{aligned} & -d_1\Delta\hat{I}_b - \alpha_b(x)\beta_b(x)\tfrac{N_b-\hat{I}_b}{N_b}\hat{I}_m + \gamma_b(x)\hat{I}_b + \mu_b(x,p,\hat{I}_b)\hat{I}_b \\ = \ & \delta[-d_1\Delta\phi - \alpha_b(x)\beta_b(x)(1-\tfrac{\delta\phi}{N_b})\psi + \gamma_b(x)\phi + \mu_b(x,p,\delta\phi)\phi] \\ = \ & \delta[\lambda_1\phi + \mu_b(x,p,\delta\phi)\phi - \mu_b(x,p,0)\phi + \tfrac{\alpha_b(x)\beta_b(x)}{N_b}\delta\phi\psi], \\ := \ & \delta\phi A(\delta), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} & -\alpha_m(x)\beta_b(x)\tfrac{A_m-\hat{I}_m}{N_b}\hat{I}_b + d_m(p)\hat{I}_m \\ & = \delta[-\alpha_m(x)\beta_b(x)\tfrac{A_m-\delta\psi}{N_b}\phi + d_m(p)\psi] \\ & = \delta\psi[\lambda_1 + \tfrac{\alpha_m(x)\beta_b(x)}{N_b}\delta\phi]. \end{aligned} \tag{3.4}$$

From the continuity of $\mu_b(x, p, I_b)$ with respect to I_b, it follows that $\lim_{\delta\to 0^+} A(\delta) = 0$. Thus, both (3.3) and (3.4) are negative for sufficiently small δ owing to $\lambda_1 < 0$. Therefore, the pair $(\hat{I}_b, \hat{I}_m) = (\delta\phi(x), \delta\psi(x))$ is the lower solution to problem (3.2). On the other hand, it is easy to verify that the pair $(\tilde{I}_b, \tilde{I}_m) = (N_b, A_m)$ is the upper solution of (3.2).

Subsequently, we select the Lipschitz constants K_1 and K_2 as follows:

$$\begin{aligned} K_1 &= \tfrac{\max_{x\in\overline{\Omega}}(\alpha_b(x)\beta_b(x))A_m}{N_b} + \max_{x\in\overline{\Omega}}(\gamma_b(x)) + \max_{x\in\overline{\Omega}}(\mu_b(x,p,N_b)), \\ K_2 &= \max_{x\in\overline{\Omega}}(\alpha_m(x)\beta_b(x)) + d_m(p), \end{aligned}$$

and conduct the following iteration process

$$\begin{cases} -d_1\Delta\overline{I}_b^{(k)} + K_1\overline{I}_b^{(k)} = K_1\overline{I}_b^{(k-1)} + \alpha_b(x)\beta_b(x)\tfrac{N_b-\overline{I}_b^{(k-1)}}{N_b}\overline{I}_m^{(k-1)} \\ \qquad\qquad - (\gamma_b(x) + \mu_b(x,p,\overline{I}_b^{(k-1)}))\overline{I}_b^{(k-1)}, & x\in\Omega, \\ K_2\overline{I}_m^{(k)} = K_2\overline{I}_m^{(k-1)} + \alpha_m(x)\beta_b(x)\tfrac{A_m-\overline{I}_m^{(k-1)}}{N_b}\overline{I}_b^{(k-1)} \\ \qquad\qquad - d_m(p)\overline{I}_m^{(k-1)}, & x\in\Omega, \\ -d_1\Delta\underline{I}_b^{(k)} + K_1\underline{I}_b^{(k)} = K_1\underline{I}_b^{(k-1)} + \alpha_b(x)\beta_b(x)\tfrac{N_b-\underline{I}_b^{(k-1)}}{N_b}\underline{I}_m^{(k-1)} \\ \qquad\qquad - (\gamma_b(x) + \mu_b(x,p,\underline{I}_b^{(k-1)}))\underline{I}_b^{(k-1)}, & x\in\Omega, \\ K_2\underline{I}_m^{(k)} = K_2\underline{I}_m^{(k-1)} + \alpha_m(x)\beta_b(x)\tfrac{A_m-\underline{I}_m^{(k-1)}}{N_b}\underline{I}_b^{(k-1)} \\ \qquad\qquad - d_m(p)\underline{I}_m^{(k-1)}, & x\in\Omega, \\ \underline{I}_b^{(k)}(x) = \underline{I}_m^{(k)}(x) = \overline{I}_b^{(k)}(x) = \overline{I}_m^{(k)}(x) = 0, & x\in\partial\Omega. \end{cases} \tag{3.5}$$

which follows the initial values

$$(\overline{I}_b^{(0)}, \overline{I}_m^{(0)}) = (\tilde{I}_b, \tilde{I}_m) = (N_b, A_m), \quad (\underline{I}_b^{(0)}, \underline{I}_m^{(0)}) = (\hat{I}_b, \hat{I}_m) = (\delta\phi, \delta\psi).$$

With the aid of comparison principal, we realize the iteration sequences $(\overline{I}_b^{(k)}, \overline{I}_m^{(k)})$ and $(\underline{I}_b^{(k)}, \underline{I}_m^{(k)})$ are well-defined, as well as they admit the following relation

$$\begin{aligned}(\underline{I}_b^{(0)}, \underline{I}_m^{(0)}) \le (\underline{I}_b^{(1)}, \underline{I}_m^{(1)}) \le \cdots \le (\underline{I}_b^{(k)}, \underline{I}_m^{(k)}) \le \cdots \\ \le (\overline{I}_b^{(k)}, \overline{I}_m^{(k)}) \le \cdots \le (\overline{I}_b^{(1)}, \overline{I}_m^{(1)}) \le (\overline{I}_b^{(0)}, \overline{I}_m^{(0)}).\end{aligned}$$

According to elliptic estimation and *Sobolev* embedding theorem, we are thus lead to the result that the sequences $(\overline{I}_b^{(k)}, \overline{I}_m^{(k)}), (\underline{I}_b^{(k)}, \underline{I}_m^{(k)}) \in \mathcal{C}^2(\overline{\Omega})$. Hence, the limits

$$\lim_{k\to\infty} \overline{I}_b^{(k)} = \overline{I}_b, \lim_{k\to\infty} \overline{I}_m^{(k)} = \overline{I}_m, \lim_{k\to\infty} \underline{I}_b^{(k)} = \underline{I}_b, \lim_{k\to\infty} \underline{I}_m^{(k)} = \underline{I}_m$$

exist and satisfy

$$\begin{cases} -d_1\Delta\overline{I}_b = \alpha_b(x)\beta_b(x)\frac{N_b-\overline{I}_b}{N_b}\overline{I}_m - \gamma_b(x)\overline{I}_b - \mu_b(x,p,\overline{I}_b)\overline{I}_b, & x \in \Omega, \\ 0 = \alpha_m(x)\beta_b(x)\frac{A_m-\overline{I}_m}{N_b}\overline{I}_b - d_m(p)\overline{I}_m, & x \in \Omega, \\ -d_1\Delta\underline{I}_b = \alpha_b(x)\beta_b(x)\frac{N_b-\underline{I}_b}{N_b}\underline{I}_m - \gamma_b(x)\underline{I}_b - \mu_b(x,p,\underline{I}_b)\underline{I}_b, & x \in \Omega, \\ 0 = \alpha_m(x)\beta_b(x)\frac{A_m-\underline{I}_m}{N_b}\underline{I}_b - d_m(p)\underline{I}_m, & x \in \Omega, \\ \underline{I}_b(x) = \underline{I}_m(x) = \overline{I}_b(x) = \overline{I}_m(x), & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Therefore, from what have discussed above, it is easy to see that both $(\underline{I}_b, \underline{I}_m)$ and $(\overline{I}_b, \overline{I}_m)$ are true solutions of (3.2). Due to Theorem 2.2 in [27], we call $(\underline{I}_b, \underline{I}_m)$ and $(\overline{I}_b, \overline{I}_m)$ are the minimum and maximum solutions in set $\langle(\hat{I}_b, \hat{I}_m), (\tilde{I}_b, \tilde{I}_m)\rangle$, respectively. Meanwhile, for any solution $(I_b(x), I_m(x))$ to problem (3.2), one has

$$\underline{I}_b \le I_b \le \overline{I}_b, \quad \underline{I}_m \le I_m \le \overline{I}_m \quad uniformly\ for \quad x \in \overline{\Omega}.$$

Next, we can demonstrate that $(\overline{I}_b(x), \overline{I}_m(x)) \equiv (\underline{I}_b(x), \underline{I}_m(x)) := (I_b^*(x), I_m^*(x))$ for (3.2), in other words, the solution to problem (3.2) is unique. Actually, one can get the following equality by the second equation of problem (3.2)

$$I_m = \frac{\alpha_m(x)\beta_b(x)\frac{A_m}{N_b}}{\frac{\alpha_m(x)\beta_b(x)}{N_b}I_b + d_m(p)} I_b,$$

and insert it to the first equation which results in

$$\begin{aligned} -d_1\Delta I_b &= \left[\alpha_b(x)\beta_b(x)\frac{N_b - I_b}{N_b}\frac{\alpha_m(x)\beta_b(x)\frac{A_m}{N_b}}{\frac{\alpha_m(x)\beta_b(x)}{N_b}I_b + d_m(p)} - \gamma_b(x) - \mu_b(x,p,I_b)\right] I_b \\ &:= g(I_b)I_b. \end{aligned} \tag{3.7}$$

We find that $g(I_b)$ decreases with respect to I_b. Therefore, the equation (3.7) admits a solution denoted by I_b^* and this solution is positive and unique, which means the endemic equilibrium $(I_b^*(x), I_m^*(x))$ is unique.

Afterwards, analogous to (3.3) and (3.4), one can demonstrate that $(\tilde{I}_b(x,t), \tilde{I}_m(x,t)) = (N_b, A_m)$ and $(\hat{I}_b(x,t), \hat{I}_m(x,t)) = (\delta\phi, \delta\psi)$ are also the upper and lower solutions to parabolic system (1.4). Combining Lemma 3.1, when $R_0^p > 1$, $(I_b^*(x), I_m^*(x))$ is unique and globally asymptotically stable, which completes the proof. □

Recall model (1.2) associated with $d_2 \neq 0$, its steady state system is as follows

$$\begin{cases} 0 = d_1 \Delta I_b + \alpha_b \beta_b \frac{N_b - I_b}{N_b} I_m - \gamma_b I_b, & x \in \Omega, \\ 0 = d_2 \Delta I_m + \alpha_m \beta_b \frac{A_m - I_m}{N_b} I_b - d_m I_m, & x \in \Omega, \\ I_m(0,x) = I_{m,0}(x), I_b(0,x) = I_{b,0}(x), & x \in \overline{\Omega}. \end{cases} \tag{3.8}$$

With the same routine as the first half part of proof to Theorem 3.2, or the method of Theorem 3.4 in [36], one can also obtain the maximum solution and minimum solution of problem (3.8), denoted by $(\bar{i}_b(x), \bar{i}_m(x))$ and $(\underline{i}_b(x), \underline{i}_m(x))$, respectively. By means of upper-lower solution method and its corresponding eigenvalue problem of (3.8), whose eigen-pair is still denoted by $(\delta\phi, \delta\psi)$, we possess the following stability.

Remark 3.1. For model (1.2), if the initial value $(I_b(0,x), I_m(0,x)) = (I_{b,0}(x), I_{m,0}(x))$ meets with the condition that $(\delta\phi, \delta\psi) \le (I_{b,0}(x), I_{m,0}(x)) \le (N_b, A_m)$, where δ is small enough, then the set $\langle(\underline{i}_b(x), \underline{i}_m(x)), (\bar{i}_b(x), \bar{i}_m(x))\rangle$ is the attractive domain to problem (1.2). That is to say, any solution $(i_b(x,t), i_m(x,t))$ of problem (1.2) satisfies $(\underline{i}_b, \underline{i}_m) \le (i_b, i_m) \le (\bar{i}_b, \bar{i}_m)$ and

$$\begin{aligned} &\underline{i}_b(x) \le \liminf_{t\to\infty} i_b(x,t) \le \limsup_{t\to\infty} i_b(x,t) \le \bar{i}_b(x) \ uniformly\ for\ x \in \overline{\Omega}, \\ &\underline{i}_m(x) \le \liminf_{t\to\infty} i_m(x,t) \le \limsup_{t\to\infty} i_m(x,t) \le \bar{i}_m(x) \ uniformly\ for\ x \in \overline{\Omega}. \end{aligned} \tag{3.9}$$

Remark 3.2. If $\underline{i}_b = \bar{i}_b$ or $\underline{i}_m = \bar{i}_m$, then we have $(\underline{i}_b, \bar{i}_m) = (\underline{i}_b, \bar{i}_m) := (i_b^*, i_m^*)$, that is, (i_b, i_m) is unique for problem (3.8) and is globally asymptotically stable for system (1.2).

3.2. Stability analysis of disease-free equilibrium

The stability of disease-free equilibrium $(0,0)$ will be studied in this subsection for system (1.4). First, we give out the following theorem.

Theorem 3.3. If $R_0^p < 1$, for problem (1.4), the disease-free equilibrium $(0,0)$ is locally stable. while if $R_0^p > 1$, the equilibrium $(0,0)$ is unstable.

Proof. Let $(I_b, I_m) = (e^{-\mu t}\eta_1(x), e^{-\mu t}\eta_2(x))$ be a solution of problem (2.1) which obtained by linearizing at the point $(0,0)$. Inserting it into (2.1), one gets the following linear eigenvalue problem

$$\begin{cases} -d_1 \Delta \eta_1 = \alpha_b(x)\beta_b(x)\eta_2 - \gamma_b(x)\eta_1 + \mu_b(x,p,0)\eta_1 + \mu\eta_1, & x \in \Omega, \\ 0 = \alpha_m(x)\beta_b(x)\frac{A_m}{N_b}\eta_1 - d_m(p)\eta_2 + \mu\eta_2, & x \in \Omega, \\ \eta_1(x) = \eta_2(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

We feel like proving that if $R_0^p < 1$, then the disease-free equilibrium $(0,0)$ is locally stable, that is to say, for any solution $(\mu; \eta_1(x), \eta_2(x))$ to problem (3.10)(in which at least one of $\eta_1^*(x)$ and $\eta_2^*(x)$ is not zero identically in Ω), $Re(\mu) > 0$ always holds. Next, our remainder proof is by contraction.

Assume that $(\mu^*; \eta_1^*(x), \eta_2^*(x))$ is a solution of problem (3.10) associated with $\eta_1^*(x) \geq 0$, $\eta_2^*(x) \geq 0$ for $x \in \Omega$, and at least one of $\eta_1^*(x)$ and $\eta_2^*(x)$ is not equal to zero, in addition $Re(\mu^*) \leq 0$. In fact, $\eta_1^*(x) \not\equiv 0$ in any subinterval within Ω. Because if $\eta_1^*(x) \equiv 0$, inserting it to the first equation of (3.10) will lead to $\eta_2^*(x) \equiv 0$, which contradicts the hypothesis. Therefore, we obtain $\eta_1^*(x) \not\equiv 0$ in an arbitrary subinterval within Ω. Combining strong maximum principle, one can get $\eta_1^*(x) > 0 (x \in \Omega)$. Using the similar manner as above, we also have $\eta_2^*(x) > 0 (x \in \Omega)$.

Obviously, $(\lambda_1; \phi, \psi)$ in Lemma 2.1(ii) precisely meets with the problem (3.10) and the above requirements. Following from uniqueness, we have $\lambda_1 = \mu^*$ and $Re(\lambda_1) = Re(\mu^*) \leq 0$, so $R_0^p \geq 1$ according to Lemma 2.1(ii), which is contradict to the premise. Consequently, we could find that $Re(\mu) > 0$ is true and $(0,0)$ is locally stable if $R_0^p < 1$.

On the other hand, for any solution to problem (3.10), if $R_0^p > 1$, we have $(\mu; \eta_1(x), \eta_2(x)) = (\lambda_1; \phi, \psi)$ with $\lambda_1 < 0$ in view of Lemma 2.1(ii), which means $Re(\lambda_1) < 0$. Thus, the disease-free equilibrium to system (1.4) is unstable. □

After analyzing the local stability of the disease-free equilibrium point mentioned above, we also have its global property under the circumstance of $R_0^p < 1$.

Theorem 3.4. Suppose $R_0^p < 1$, for system (1.4), the disease-free equilibrium $(0,0)$ is attractive and $\langle (0,0), (M\phi, M\psi) \rangle$ is the attraction domain, in which

$$M = \min\left(\frac{N_b}{\max_{x \in \overline{\Omega}} \phi(x)}, \frac{A_m}{\max_{x \in \overline{\Omega}} \psi(x)}\right),$$

as well as (ϕ, ψ) is the eigenfunction pair of (2.4). If the initial function pair satisfies

$$I_b(x,0) \leq M\phi, \quad I_m(x,0) \leq M\psi \quad for \quad x \in \overline{\Omega},$$

then any solution of system (1.4) meets with

$$\lim_{t\to\infty} I_m(x,t) = 0, \quad \lim_{t\to\infty} I_b(x,t) = 0 \quad \text{uniformly for} \quad x \in \overline{\Omega}. \tag{3.11}$$

Proof. Assume $R_0^p < 1$, it follows that $\lambda_1 > 0$. Combine Lemma 2.1(iv) and set $(\overline{I}_b, \overline{I}_m) = (Me^{-\frac{\lambda_1}{2}t}\phi, Me^{-\frac{\lambda_1}{2}t}\psi)$, then calculate

$$\begin{aligned}
&\tfrac{\partial \bar{I}_b}{\partial t} - d_1 \Delta \bar{I}_b - \alpha_b(x)\beta_b(x)\bar{I}_m + \gamma_b(x)\bar{I}_b + \mu_b(x,p,\bar{I}_b)\bar{I}_b \\
&= Me^{-\frac{\lambda_1}{2}t}[-\tfrac{\lambda_1}{2}\phi - d_1\Delta\phi - \alpha_b(x)\beta_b(x)\psi + \gamma_b(x)\phi + \mu_b(x,p,Me^{-\frac{\lambda_1}{2}t}\phi)\phi] \\
&= Me^{-\frac{\lambda_1}{2}t}[\tfrac{\lambda_1}{2}\phi + \mu_b(x,p,Me^{-\frac{\lambda_1}{2}t}\phi)\phi - \mu_b(x,p,0)\phi] \\
&\geq \tfrac{1}{2}M\lambda_1 e^{-\frac{\lambda_1}{2}t}\phi \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{\bar{I}_m}{\partial t} - \alpha_m(x)\beta_b(x)\tfrac{A_m}{N_b}\bar{I}_b + d_m(p)\bar{I}_m \\
&= Me^{-\frac{\lambda_1}{2}t}[-\tfrac{\lambda_1}{2}\psi - \alpha_m(x)\beta_b(x)\tfrac{A_m}{N_b}\phi + d_m(p)\psi] \\
&= \tfrac{1}{2}M\lambda_1 e^{-\frac{\lambda_1}{2}t}\psi \geq 0.
\end{aligned}$$

In view of $M = \min(\frac{N_b}{\max_{x\in\overline{\Omega}}\phi(x)}, \frac{A_m}{\max_{x\in\overline{\Omega}}\psi(x)})$, we can get the initial condition satisfies $(\bar{I}_b, \bar{I}_m) = (M\phi, M\psi) \leq (N_b, A_m)$. Therefore, the $(\bar{I}_b, \bar{I}_m)(x,t)$ is the upper solution for system (1.4) based on Definition 3.1. Combined comparison principle, for any $(x,t) \in \overline{\Omega} \times [0, +\infty)$, one obtains $(I_b(x,t), I_m(x,t)) \leq (\bar{I}_b(x,t), \bar{I}_m(x,t))$. Hence, we conclude that the set $\langle(0,0),(M\phi, M\psi)\rangle$ is the attraction domain of problem (1.4) associated with the condition $I_b(x,0) \leq M\phi, I_m(x,0) \leq M\psi$.

Apparently, $\lim_{t\to\infty}\bar{I}_b = 0$ and $\lim_{t\to\infty}\bar{I}_m = 0$ uniformly for $x \in \overline{\Omega}$, so (3.11) also holds. □

Theorem 3.5. Let $E = \{(I_b, I_m) \in R_+^2, I_b(x,t) \leq N_b, I_m(x,t) \leq A_m, (x,t) \in \overline{\Omega} \times [0, +\infty)\}$, if the following inequality is true

$$\sqrt{\frac{\frac{A_m}{N_b}\max\limits_{x\in\overline{\Omega}}(\alpha_b(x)\beta_b(x)) \cdot \max\limits_{x\in\overline{\Omega}}(\alpha_m(x)\beta_b(x))}{d_m(p)\cdot \min\limits_{x\in\overline{\Omega}}(\gamma_b(x) + \mu_b(x,p,0))}} < 1, \tag{3.12}$$

then for model (1.4), the disease-free equilibrium $(0,0)$ is globally asymptotically stable on the invariant set E.

Proof. First, inequality (3.12) indicates that there exists some $\varepsilon_0 > 0$ satisfy

$$\max_{x\in\overline{\Omega}}\alpha_b(x)\beta_b(x)\max_{x\in\overline{\Omega}}\alpha_m(x)\beta_b(x)\frac{A_m}{N_b} < (1-\varepsilon_0)^2\min_{x\in\overline{\Omega}}(\gamma_b(x)+\mu_b(x,p,0))d_m(p). \tag{3.13}$$

Next, we define the following Lyapunov function in which $\theta > 0$ is to be determined later

$$V(t) = \frac{1}{2}\int_\Omega I_b^2 dx + \frac{\theta}{2}\int_\Omega I_m^2 dx.$$

Noticing that the null boundary Dirichlet condition in system (1.4), one has

$$\begin{aligned}
\tfrac{dV(t)}{dt} &= \int_\Omega I_b \cdot \frac{\partial I_b}{\partial t}dx + \theta\int_\Omega I_m \cdot \frac{\partial I_m}{\partial t}dx \\
&= \int_\Omega I_b(d_1\Delta I_b + \alpha_b(x)\beta_b(x)\frac{N_b - I_b}{N_b}I_m - \gamma_b(x)I_b - \mu_b(x,p,I_b)I_b)dx \\
&\quad +\theta\int_\Omega I_m(\alpha_m(x)\beta_b(x)\frac{A_m - I_m}{N_b}I_b - d_m(p)I_m)dx \\
&= -d_1\int_\Omega |\nabla I_b|^2 dx + \int_\Omega \alpha_b(x)\beta_b(x)I_bI_m dx - \int_\Omega \alpha_b(x)\beta_b(x)\frac{I_b^2 I_m}{N_b}dx \\
&\quad -\int_\Omega \gamma_b(x)I_b^2 dx - \int_\Omega \mu_b(x,p,I_b)I_b^2 dx + \theta\int_\Omega \alpha_m(x)\beta_b(x)\frac{A_m}{N_b}I_bI_m dx \\
&\quad -\theta\int_\Omega \alpha_m(x)\beta_b(x)\frac{I_bI_m^2}{N_b}dx - \theta\int_\Omega d_m(p)I_m^2 dx.
\end{aligned} \tag{3.14}$$

Actually, we can verify that the inequality (3.13) guarantee the following inequality to hold

$$\begin{aligned}
&\int_\Omega \alpha_b(x)\beta_b(x)I_bI_m dx + \theta\int_\Omega \alpha_m(x)\beta_b(x)\frac{A_m}{N_b}I_bI_m dx \\
&\leq (1-\varepsilon_0)\int_\Omega (\gamma_b(x) + \mu_b(x,p,0))I_b^2 dx + (1-\varepsilon_0)\theta\int_\Omega d_m(p)I_m^2 dx,
\end{aligned} \tag{3.15}$$

associated with

$$\theta = \frac{M_1 - M_2}{(\max\limits_{x\in\overline{\Omega}} \alpha_m(x)\beta_b(x)\frac{A_m}{N_b})^2},$$

where

$$M_1 = 2(1-\varepsilon_0)^2 \min_{x\in\overline{\Omega}} (\gamma_b(x) + \mu_b(x,p,0)) \min_{x\in\overline{\Omega}} d_m(p),$$
$$M_2 = \max_{x\in\overline{\Omega}} \alpha_b(x)\beta_b(x) \max_{x\in\overline{\Omega}} \alpha_m(x)\beta_b(x)\frac{A_m}{N_b},$$

and (3.13) also make sure $\theta > 0$. Therefore, according to (3.14) and (3.15), we get

$$\begin{aligned}\tfrac{dV(t)}{dt} \le& -\int_\Omega d_1|\nabla I_b|^2 dx - \int_\Omega \alpha_b(x)\beta_b(x)\frac{I_b^2 I_m}{N_b}dx - \theta\int_\Omega \alpha_m(x)\beta_b(x)\frac{I_b I_m^2}{N_b}dx\\ &- \varepsilon_0\int_\Omega (\gamma_b(x) + \mu_b(x,p,0))I_b^2 dx - \varepsilon_0\theta\int_\Omega d_m(p)I_m^2 dx,\\ \le& -\varepsilon_0\int_\Omega (\gamma_b(x) + \mu_b(x,p,0))I_b^2 dx - \varepsilon_0\theta\int_\Omega d_m(p)I_m^2 dx < 0.\end{aligned}$$

Theorem 4.2 in [19] expands the remaining proof, which we do not repeat here. By using the Lyapunov function, we prove that when (3.12) holds, the disease-free equilibrium $(0,0)$ of system (1.4) is globally asymptotically stable on set E. □

Remark 3.3. Assume that $\alpha_b(x) = \alpha_b, \beta_b(x) = \beta_b, \gamma_b(x) = \gamma_b, \alpha_m(x) = \alpha_m, d_m(p) = d_m$ are all constants and $\mu_b(x,p,0) = 0$, then the condition (3.12) equals to $\mathfrak{R}_0 < 1$, in which $\mathfrak{R}_0$ is the corresponding basic reproduction number to an ordinary differential equation (1.2). Theorem 3.5 shows that when $\mathfrak{R}_0 < 1$, the disease-free equilibrium $(0,0)$ is globally asymptotically stable to system (1.4).

4. Numerical simulation and epidemiological analysis

In the last part of this paper, we are devote to illustrating some theoretical results by virtue of some numerical simulations. Based on some values on Table 1, we choose the interval and some parameters as follows:

$$\begin{aligned}&\Omega = (0,\pi), \quad A_m = 10, \quad N_b = 0.5, \quad d_1 = 1,\\ &\alpha_m(x) = 0.16 + 0.2\sin x, \quad \alpha_b(x) = 0.88 + 0.5\sin x,\\ &\gamma_b(x) = 0.01 + 0.005\cos 2x, \quad \beta_b(x) = 0.1 + 0.02\sin 3x,\\ &d_m(p) = 0.009 + \tfrac{2}{\pi}\arctan p,\\ &\mu_b(x,p,I_b) = 0.02 + 0.02\sin 3x + \tfrac{1}{2e^p} + 0.01 I_b.\end{aligned}$$

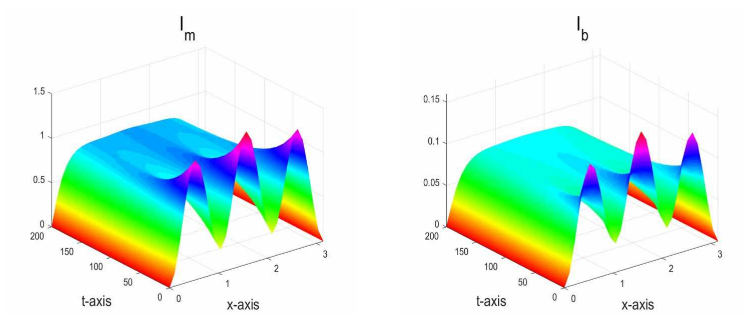
Combining the initial conditions

$$\begin{aligned}&I_{b,0}(x) = 0.06 + 0.04\sin x + 0.06\sin(6x - \tfrac{\pi}{2}),\\ &I_{m,0}(x) = 0.6 + 0.3\sin x + 0.6\sin(6x + \tfrac{3\pi}{2}),\end{aligned}$$

we could observe the asymptotic behaviors of the solution to system (1.4) through changing *Wolbachia* parameter p.

Example 4.1. Set $p = 0.02$, it is easy to calculate from (2.6)

$$
\begin{aligned}
(R_0^p)^2 &:= \sup_{\phi\in H_0^1(\Omega),\phi\neq 0} \frac{\int_\Omega \frac{\alpha_b(x)\alpha_m(x)\beta_b^2(x)A_m}{d_m(p)N_b}\phi^2 dx}{\int_\Omega (d_1|\nabla\phi|^2+\gamma_b(x)\phi^2+\mu_b(x,p,0)\phi^2)dx}.\\
&\geq \frac{\int_0^\pi \frac{(0.88+0.5)\times(0.16+0.2)\times(0.1-0.02)^2\times 10}{(0.009+\frac{2}{\pi}\times\arctan 0.02)\times 0.5}\sin^2 x dx}{\int_0^\pi 1\times\cos^2 x dx+\int_0^\pi (0.01-0.005)\sin^2 x dx+\int_0^\pi (0.02-0.02+\frac{1}{2e^{0.02}}+0.01\times 0)\sin^2 x dx}\\
&\approx \frac{2.926}{1.495} > 1.
\end{aligned}
$$



(a) (b)

Figure 1. For $p = 0.02$, graphs (a) and (b) illustrate that the solution (I_b, I_m) of problem (1.4) goes to a positive equilibrium, which means the WNv is spreading for a smaller p.

From Theorem 3.2, we know if $R_0^p > 1$, the endemic equilibrium $(I_b^*(x), I_m^*(x))$ of system (3.2) is unique and globally asymptotically stable. Following with Figure 1, one can observe that the solution $(I_b(x,t), I_m(x,t))$ of problem (1.4) stabilizes to an endemic equilibrium. That is to say, the WNv will spread if not enough *Wolbachia* are implanted.

Example 4.2. Set $p = 0.25$ and compare with Example 4.1, the parameter p becomes larger. In this way, we can calculate from (2.6),

$$
\begin{aligned}
(R_0^p)^2 &:= \sup_{\phi\in H_0^1(\Omega),\phi\neq 0} \frac{\int_\Omega \frac{\alpha_b(x)\alpha_m(x)\beta_b^2(x)A_m}{d_m(p)N_b}\phi^2 dx}{\int_\Omega (d_1|\nabla\phi|^2+\gamma_b(x)\phi^2+\mu_b(x,p,0)\phi^2)dx}\\
&\leq \frac{\int_0^\pi \frac{(0.88+0.5)\times(0.16+0.2)\times(0.1-0.02)^2\times 10}{(0.009+\frac{2}{\pi}\times\arctan 0.25)\times 0.5}\sin^2 x dx}{\int_0^\pi (0.01-0.005)\sin^2 x dx+\int_0^\pi (0.02-0.02+\frac{1}{2e^{0.25}}+0.01\times 0)\sin^2 x dx} \approx \frac{0.386}{0.394} < 1.
\end{aligned}
$$

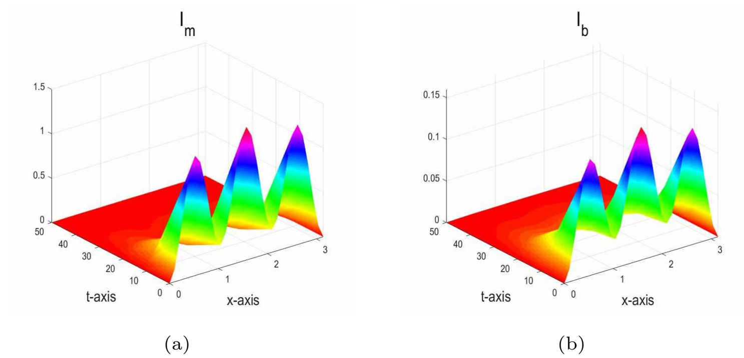


Figure 2. For $p = 0.25$, the two graphs illustrate that the solution (I_b, I_m) of problem (1.4) tends to $(0, 0)$ as time goes on, that is, the WNv will be vanish for a bigger p.

It follows from Theorem 3.4 that the disease free equilibrium $(0, 0)$ of problem (3.2) is locally stable. Figure 2 shows that the densities of infectious mosquitoes $I_m(x, t)$ and infectious birds $I_b(x, t)$ tend to zeros as time goes by, which embodies the theoretical finding of Theorem 3.5 and reveals that the WNv will vanish eventually.

As a mosquito-borne disease, WNv not only causes the death of birds, but also influences the physical and mental health of human beings to some extent. Since there is no effective therapeutic drugs for WNv, and the main treatment is to intervene some related symptoms caused by WNv directly, it is crucial important to reduce contact transmission of infected mosquitoes through utilizing scientific methods. It has been scientifically proven that implanting *Wolbachia* can destroy the reproductive mechanisms of mosquitoes [22], so it may be effective to some extent to use this method to further reduce the transmission risk of mosquito-borne infectious diseases. In this paper, based on models (1.1) and (1.2) proposed by Lewis, we introduce the parameter p to represent the effect of *Wolbachia* on WNv. Subsequently, the impact of spatial heterogeneity and *Wolbachia* on the transmission of WNv was also studied with the help of the reaction-diffusion model (1.4).

In this paper, we have investigated the asymptotic behavior of the equilibrium by the basic reproduction number R_0^p. In comparison with $\mathfrak{R}_0$, which is the threshold of ordinary differential model (1.1), R_0^p provided by (2.6) not only reflects the difference of spatial environment, but also is affected by the parameter p, which matches the characteristics that R_0^p is influenced by many factors simultaneously in the real environment.

Centering on the explicit expression (2.6) of R_0^p and the monotonicity of $d_m(p)$, $\mu_b(x, p, 0)$ with respect to p ($d_m(p)$ increases with respect to p, and $\mu_b(x, p, 0)$ decreases with respect to p), we can clearly see that two cases need to be considered if we want to guarantee $R_0^p < 1$. On the one hand, p becomes smaller such that $d_m(p)$ is smaller and $\mu_b(x, p, 0)$ is larger. Meanwhile, from a numerical point of view on (2.6), $\mu_b(x, p, 0)$ has a slightly more influence than $d_m(p)$ on R_0^p. However, p has an impact on $\mu_b(x, p, 0)$ indirectly and the fact that implanting fewer *Wolbachia* results in a lower mortality rate among adult mosquitoes will not achieve the expectation effect to control WNv, which means this kind of operator is unreasonable. On the other hand, when p becomes larger such that $d_m(p)$ is larger and $\mu_b(x, p, 0)$ is smaller, in the meantime, $d_m(p)$ has a slightly more numerical effect than $\mu_b(x, p, 0)$ on R_0^p, which signifies that the mortality rate of adult mosquitoes is higher, the disease induced death rate of birds is lower and $R_0^p < 1$. It will be in line with our expectations. Therefore, implanting more *Wolbachia* would play an active role on the control of WNv. However, in reality, we ought to not only consider the economic cost, but also protect the ecological balance. Hence, if p is too large, that is, too much *Wolbachia* is implanted, the economic cost will be too high and the ecological balance may be damaged. Above all, we deeply understand that *Wolbachia* is a double-edged sword for WNv, and its implantation needs to reach a certain value. However, it cannot be increased indefinitely. Only by keeping the *Wolbachia* within a certain range can we control WNv and maintain the ecological balance to the full extent. As we know, WNv spreads between mosquitoes and host animals (especially birds), and then does a great harm to mankind. Some scientific researches suggest that Rabies [21], Ebola [15], SARS [28] and even 2019-nCoV [35] which was breaking out by the time we finished the paper are also transmitted from some hosts to human beings. The results of this paper also provide further insight into the transmission mechanism of infection to mankind through some hosts, and give out some reference value for the prevention and control of infectious diseases.

References

[1] A. Abdelrazec, S. Lenhart and H. P. Zhu, *Transmission dynamics of West Nile virus in mosquitoes and corvids and non-corvids*, Journal of Mathematical Biology, 2014, 68(6), 1553–1582.

[2] L. R. H. Ahlers and A. G. Goodman, *The immune responses of the animal hosts of West Nile virus: a comparison of insects, birds, and mammals*, Frontiers in Cellular and Infection Microbiology, 2018, 8, 96.

[3] D. B. Alex and D. E. Gregory, *Comparative pathology of West Nile virus in humans and non-human animals*, Pathogens, 2020, 9(1).

[4] D. S. Asnis, C. Rick, A. A. Teixeira et al., *The West Nile virus outbreak of 1999 in New York: the flushing hospital experience*, Clinical Infectious Diseases, 2000, 30(3), 413–418.

[5] K. W. Blayneh, A. B. Gumel, S. Lenhart et al., *Backward bifurcation and optimal control in transmission dynamics of West Nile virus*, Bulletin of Mathematical Biology, 2010, 72(4), 1006–1028.

[6] C. Bowman, A. B. Gumel, P. V. D. Driessche et al., *A mathematical model for assessing control strategies against West Nile virus*, Bulletin of Mathematical Biology, 2005, 67(5), 1107–1133.

[7] R. S. Cantrell and C. Cosner, *Spatial ecology via reaction-diffusion equation*, John Wiley & Sons, Hoboken, 2003.

[8] V. Capasso and L. Maddalena, *Convergence to equilibrium states for a reaction-diffusion system modelling the spatial spread of a class of bacterial and viral diseases*, Journal of Mathematical Biology, 1981, 13(2), 173–184.

[9] G. Cruz-Pacheco, L. Esteva, J. A. Montaño-Hirose and C. Vargas, *Modelling the dynamics of West Nile virus*, Bulletin of Mathematical Biology, 2005, 67(6), 1157–1172.

[10] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, *On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations*, Journal of Mathematical Biology, 1990, 28(4), 365–382.

[11] P. van den Dreessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Mathematical Biosciences, 2002, 180(1), 29–48.

[12] S. H. Fu, L. Cao, Z. Lyu et al., *Comparison of the biologic characteristics of West Nile virus and Japanese encephalitis virus isolated in China (Chinese)*, Chinese Journal of Virology, 2019, 35(03), 364–371.

[13] J. Ge, K. I. Kim, Z. G. Lin et al., *A SIS reaction-diffusion-advection model in a low-risk and high-risk domain*, Journal of Differential Equations, 2015, 259(10), 5486–5509.

[14] M. G. Huang, J. S. Yu, L. C. Hu et al., *Qualitative analysis for a Wolbachia infection model with diffusion*, Science China Mathematics, 2016, 59(7), 1249–1266.

[15] E. M. Leroy, B. Kumulungui, X. Pourrut et al., *Fruit bats as reservoirs of Ebola virus*, Nature, 2005, 438(7068), 575–576.

[16] M. A. Lewis, J. Renclawowicz and P. V. D. Driessche, *Traveling waves and spread rates for a West Nile virus model*, Bulletin of Mathematical Biology, 2006, 68(1), 3-23.

[17] M. A. Lewis, J. Renclawowicz, P. V. D. Driessche et al., *A comparison of continuous and discrete-time West Nile virus models*, Bulletin of Mathematical Biology, 2006, 68(3), 491–509.

[18] H. Li, R. Peng and F. B. Wang, *Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model*, Jounal of Differential Equations, 2017, 262(2), 885–913.

[19] Z. G. Lin and M. Pedersen, *Stability in a diffusive food-chain model with michaelis-menten functional response*, Nonlinear Analysis, 2004, 57(3), 421–433.

[20] Z. G. Lin and H. P. Zhu, *Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary*, Journal of Mathematical Biology, 2017, 75(6), 1–29.

[21] Y. Liu, S. F. Zhang, X. F. Wu et al., *Ferret badger rabies origin and its revisited importance as potential source of rabies transmission in Southeast China*, BMC Infectious Diseases, 2010, 10(1), 234–241.

[22] J. W. Mains, C. L. Brelsfoard, R. I. Rose et al., *Female adult aedes albopictus suppression by Wolbachia-infected male mosquitoes*, Scientific Reports, 2016, 6, 33846.

[23] C. J. Mcmeniman, R. V. Lane, B. N. Cass et al., *Stable introduction of a life-shortening Wolbachia infection into the mosquito aedes aegypti*, Science, 2009, 323(5910), 141–144.

[24] M. H. Myer and J. M. Johnston, *Spatiotemporal bayesian modeling of West Nile virus: identifying risk of infection in mosquitoes with local-scale predictors*, Science of the Total Environment, 2019, 650, 2818–2829.

[25] A. L. Nevai, Y. Lou, B. M. Bolker et al., *Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model*, Discrete and Continuous Dynamical Systems Series A, 2008, 21(1), 1–20.

[26] A. M. Norberto and M. Y. Hyun, *Spatial spreading of West Nile virus described by traveling waves*, Journal of Theoretical Biology, 2009, 258(3), 403–417.

[27] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[28] K. K. W. To, I. F. N. Hung, J. F. W. Chan and K. Y. Yuen, *From SARS coronavirus to novel animal and human coronaviruses*, Journal of Thoracic Disease, 2013, 5(2), 103–108.

[29] P. F. M. Verdonschot and A. A. Besse-Lototskaya, *Flight distance of mosquitoes (Culicidae): A metadata analysis to support the management of barrier zones around rewetted and newly constructed wetlands*, Limnologica, 2014, 45, 69–79.

[30] J. H. Werren, *Biology of Wolbachia*, Annual Review of Entomology, 1997, 42(1), 587–609.

[31] M. J. Wonham, T. De-Camino-Beck and M. A. Lewis, *An epidemiological model for West Nile virus: invasion analysis and control applications*, Proceedings of the Royal Society B: Biological ences, 2004, 271, 501–507.

[32] L. L. Yu, H. Wang and W.X. Shen, *The protection of terrestrial wildlife: habitat destruction and mechanism*, Chinese Journal of Biology, 2016, 33(4), 92–96.

[33] X. H. Zhang, S. Y. Tang and R. A. Cheke, *Models to assess how best to replace dengue virus vectors with Wolbachia-infected mosquito populations*, Mathematical Biosciences, 2015, 269, 164–177.

[34] X. Q. Zhao, *Dynamical Systems in Population Biology*, Second Edition, Springer, New Work, 2017.

[35] P. Zhou, X. L. Yang, X. G. Wang et al., *A pneumonia outbreak associated with a new coronavirus of probable bat origin*, Nature, 2020, 579(7798), 270–273.

[36] M. Zhu and Z. G. Lin, *Modeling the transmission of dengue fever with limited medical resourses and self-protection*, Discrete and Continuous Dynamical Systems Series B, 2018, 23(2), 957–974.