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A Switching Law to Stabilize an Unstable Switched Linear System*

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Abstract Stabilization of switched systems fully composed of unstable modes is of theoretical and practical significance. In this paper, we obtain some sufficient algebraic conditions for stabilizing switched linear systems with all unstable subsystems based on the theory of spherical covering and crystal point groups. Under the proposed algebraic conditions switching laws are easy to be designed to stabilize the switched systems. Some simple examples are provided to illustrate our results.

Keywords Switched system, Unstable mode, Stabilization, Switching law, Spherical covering.

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1. Introduction

A lot of natural and artificial processes encompass several modes of operation with a different dynamical behavior in each mode. To deal with such processes, theory of hybrid dynamical systems has been developed in recent years. A typical class of hybrid systems are the so-called switched systems. A switched system is composed of a family of subsystems and rules that regulate the switching among them. Stability issue for switched systems is of great significance, which has been extensively studied [2, 4, 11–13, 15, 16, 29]. One of the early results of hybrid system stability for linear switched systems was developed by Peleties [15]. In [17], necessary and sufficient conditions are proposed for a given set of controllers to quadratically stabilize a plant and for robust stabilizability with a quadratic storage function. Also, the stability condition can be expressed from LyapunovMetzler inequalities [7]. For more details about switching in system and control, see [10, 18, 20].

Stabilization of switched systems fully consisting of unstable modes is one of the most challenging problems in the field of switched systems [5, 25, 27]. Multiple Lyapunov or Lyapunov-likes functions [11, 12] may be concatenated together to produce a nontraditional Lyapunov function and stabilize unstable switched system.

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Decarlo et al. [5] proposed conditions for the existence of switched controllers in stabilizing unstable switched systems by means of Lyapunov-like functions. The corresponding Lyapunov-like function values at every switching time form a monotonically decreasing sequence. For a switched system with all unstable subsystems, how to design appropriate switching laws to stabilize the switched system is of great interest. Except with multiple Lyapunov functions (MLF) theory, the switching signal can be designed to stabilize the unstable system with mode-dependent average dwell time (MDADT) property [30].

However, for a system, its corresponding Lyapunov function may be hard to construct even though it does exist. Therefore, studying simple sufficient conditions to ensure that a switched system with all unstable modes is stabilizable is of very importance. Motivated by [1, 10], we investigate a new easy-to-use sufficient condition for stabilization of n-dimensional switched systems with all unstable modes and obtain some novel results in two-dimensional and three-dimensional switched systems by virtue of the theory of spherical covering and crystal point groups [3,6].

This paper is organized as follows: In Section 2, we set forth a sufficient condition for stabilizing switched linear systems with all unstable subsystems. Section 3 presents a sufficient condition for stabilization of two-dimensional switched systems with all unstable modes. Two examples are provided to illustrate the ideas in this section. Section 4 presents some results about stabilization of three-dimensional switched systems with three, four and five unstable modes.

2. A new sufficient condition for stabilization

Before starting our discussions, we give some notations. Let $\mathbf{M}_{n\times n}$ denote the set of all $n \times n$ real matrices and $\mathbf{M}^u_{n\times n}$ denote the matrices in $\mathbf{M}_{n\times n}$ with at least one of its eigenvalues having positive real parts. We denote the inner product of two vectors by $\langle\cdot,\cdot\rangle$, the transpose of $\mathbf{A}$ by $\mathbf{A}^T$ and the Euclidean norm on $\mathbb{R}^n$ by $|\cdot|$. Denote the unit circle in $\mathbb{R}^2$ by S^1 and the unit sphere in $\mathbb{R}^3$ by S^2.

In this paper, we focus on the following linear switched system

$$\dot{\mathbf{x}} = \mathbf{A}_{\sigma(\mathbf{x})}\mathbf{x} \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\sigma(\mathbf{x}) : \mathbb{R}^n \to \mathcal{M} = \{1, 2, \ldots, m\}$ denotes a switching function where m is the number of modes and $\mathbf{A}_i \in \mathbf{M}^u_{n\times n}, i \in \mathcal{M}$. For system (2.1) with state-dependent switching, the state space $\mathbb{R}^n$ is partitioned into a finite number of operating regions by means of a family of switching surface.

Based on Lyapunov stability theory [17, 26], we have the definition of stabilizability as follows.

Definition 2.1. The origin of a switched system (2.1) is said to be switching stabilizable if there exists a switching law $\sigma : \mathbb{R}^n \longmapsto \{1, 2, \ldots, m\}$ under which the origin is asymptotically stable.

According to the results in [17], we have the following definition.

Definition 2.2. The collection of real symmetric matrices $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_k$ is said to be complete if for any $\mathbf{x}_0 \in \mathbb{R}^n$ there exists $i \in \{1, 2, \ldots, k\}$ such that $\mathbf{x}_0^T\mathbf{Z}_i\mathbf{x}_0 \le 0$. Furthermore, the collection $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_k$ is said to be strictly complete if for any $\mathbf{x}_0 \in \mathbb{R}^n/\{\mathbf{0}\}$ there exists $i \in \{1, 2, \ldots, k\}$ such that $\mathbf{x}_0^T\mathbf{Z}_i\mathbf{x}_0 < 0$.

Then, we have

Proposition 2.1. *The origin of system (2.1) is stabilizable if there exists a real symmetric matrix* $\boldsymbol{P}$ *such that the set of matrices*

$$\boldsymbol{Z}_i = \boldsymbol{A}_i^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i,\ i \in \mathcal{M}$$

is strictly complete.

For the proof of this proposition and more references, the reader can see [17].

It is evident to notice that system (2.1) can not be stable under arbitrary switching (Readers can see [10, 28] etc. for the stability under arbitrary switching.), but may be stable under constraint switching signals. According to Proposition 2.1, if there exists a common matrix $\mathbf{P}$ such that $\{\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i,\ i \in \mathcal{M}\}$ is complete, the system is stabilizable. Let $\Omega^s_{\mathbf{P}}(\mathbf{A}_i) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i)\mathbf{x} < 0\}, i \in \mathcal{M}$. Then, according to Proposition 2.1, we have

Proposition 2.2. *The origin of system (2.1) is stabilizable, if* $\bigcup_{i=1}^m \Omega^s_{\boldsymbol{P}}(\boldsymbol{A}_i) = \mathbb{R}^n$.

Note that if all eigenvalues of $\mathbf{A}_i + \mathbf{A}_i^T, i \in \mathcal{M}$ are nonnegative, then there cannot exist a real symmetric matrix that satisfies Proposition 2.1. If $\mathbf{A}_i + \mathbf{A}_i^T, \mathbf{A}_i \in \mathbf{M}^u_{n\times n}, i \in \mathcal{M}$ has at least one negative eigenvalue and $\{\mathbf{A}_i + \mathbf{A}_i^T, i \in \mathcal{M}\}$ is strictly complete, the system is stabilizable and the switching law is easy to be designed. It is strongly practical. Hence, in this paper, we focus on the case of $\mathbf{A}_i + \mathbf{A}_i^T, \mathbf{A}_i \in \mathbf{M}^u_{n\times n}, i \in \mathcal{M}$ having at least one negative eigenvalue, find new sufficient conditions for stabilizing switched system and obtain some new results that would not exist in other cases. Since $\mathbf{A}_i + \mathbf{A}_i^T, i \in \mathcal{M}$ has at least one negative eigenvalue, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T(\mathbf{A}_i + \mathbf{A}_i^T)\mathbf{x} < 0$. Let $\Omega^s(\mathbf{A}_i) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T(\mathbf{A}_i + \mathbf{A}_i^T)\mathbf{x} < 0\}, i \in \mathcal{M}$ which can be called the stable cone of the subsystem $\dot{\mathbf{x}} = \mathbf{A}_i\mathbf{x}$. Since the Euclidean norm of the solution is decreasing in this subsystem [8], then the following theorem is obvious from Proposition 2.2.

Proposition 2.3. *The origin of system (2.1) is stabilizable, if* $\bigcup_{i=1}^m \Omega^s(\boldsymbol{A}_i) = \mathbb{R}^n$.

We remark that if $\bigcup_{i=1}^m \Omega^s(\mathbf{A}_i) = \mathbb{R}^n$ with no sliding motions (If there exists sliding-like motions, then take the direction of the vector fields along the switching surfaces into consideration [21]). Then,

$$\Omega^s(\mathbf{A}_1), \Omega^s(\mathbf{A}_i) - \bigcup_{j=1}^{i-1} \Omega^s(\mathbf{A}_j), i \neq 1, i \in \mathcal{M}$$

can be regarded as the operating region of system (2.1). In each of these regions, a subsystem is given. Whenever the system trajectory hits a switching surface, a switching event can occur and a new subsystem is active. Considering the quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n$ might be conservative, but it has strong practicality and the switching laws are easy to be designed.

Let $\mathbf{C}(\mathbf{A}_i)$ denote the intersection of $\Omega^s(\mathbf{A}_i)$ and the unit sphere in $\mathbb{R}^n$. Then, from Proposition 2.3, we have the following corollary.

Corollary 2.1. *If the union of* $\boldsymbol{C}(\boldsymbol{A}_i), i \in \mathcal{M}$ *covers the unit sphere, then the origin of system (2.1) is stabilizable.*

It is straightforward to show that $\Omega^s(\mathbf{A}_i) = \{r\mathbf{x} \mid \mathbf{x} \in \mathbf{C}(\mathbf{A}_i), r \in \mathbb{R}\}$. Therefore, this corollary is proved according to Proposition 2.3.

From Corollary 2.1, we notice that the stabilization problem of system (2.1) can be sometimes transformed into the covering problem of the unit sphere of $\mathbb{R}^n$, which can be further illustrated in the next section. In the rest of this paper, we investigate a particular class of switched linear systems composed of unstable subsystems with all eigenvalues having positive real parts and study the sufficient conditions of stabilization.

3. Two-dimensional switched systems

Consider a family of switched systems in $\mathbb{R}^2$

$$\dot{\mathbf{x}} = \mathbf{A}_{\sigma(\mathbf{x})}\mathbf{x} \tag{3.1}$$

where $\sigma(\mathbf{x}) : \mathbb{R}^2 \to \mathcal{M}$ denotes the switching function and $\mathbf{A}_i \in \mathbf{M}^u_{2\times 2}, i \in \mathcal{M}$ with all eigenvalues of $\mathbf{A}_i$ having positive real parts and $\mathbf{A}_i + \mathbf{A}_i^T$ having one negative eigenvalue. Let $\mathbf{v}_1^i, \mathbf{v}_2^i$ be the unit eigenvectors corresponding to the eigenvalues λ_1^i, λ_2^i of $\mathbf{A}_i + \mathbf{A}_i^T$ where $\lambda_1^i > -\lambda_2^i > 0$. Let $\mathbf{T}_i = (\mathbf{v}_1^i \ \mathbf{v}_2^i)$. It follows from the orthogonality [9] of $\mathbf{T}_i$ that

$$\begin{aligned}\Omega^s(\mathbf{A}_i) =& \{\mathbf{T}_i\mathbf{y} \mid \mathbf{y}^T\mathbf{T}_i^T(\mathbf{A}_i + \mathbf{A}_i^T)\mathbf{T}_i\mathbf{y} < 0, \mathbf{y} \in \mathbb{R}^n\} \\ =& \{\mathbf{T}_i\mathbf{y} \mid \lambda_1^i y_1^2 + \lambda_2^i y_2^2 < 0, \mathbf{y} \in \mathbb{R}^2\}.\end{aligned} \tag{3.2}$$

In view of (3.2), we set

$$\mathbf{w}_1^i = \frac{1}{\sqrt{\lambda_1^i - \lambda_2^i}}\mathbf{T}_i\begin{pmatrix}\sqrt{-\lambda_2^i}\\ \sqrt{\lambda_1^i}\end{pmatrix},\ \mathbf{w}_2^i = \frac{1}{\sqrt{\lambda_1^i - \lambda_2^i}}\mathbf{T}_i\begin{pmatrix}-\sqrt{-\lambda_2^i}\\ \sqrt{\lambda_1^i}\end{pmatrix}.$$

Then, we have the following theorem about the stabilizability of planar switched systems.

Theorem 3.1. *Suppose that m is the number of the modes of system (3.1). The origin of system (3.1) is stabilizable providing that one of the following two conditions is satisfied.*

i) $\langle \boldsymbol{w}_1^{i+1} - \boldsymbol{w}_1^i, \boldsymbol{w}_1^{i+1} - \boldsymbol{w}_2^i\rangle < 0$ *or* $\langle \boldsymbol{w}_1^{i+1} + \boldsymbol{w}_1^i, \boldsymbol{w}_1^{i+1} + \boldsymbol{w}_2^i\rangle < 0$ *for* $i = 1, 2, \ldots, m-1$ *and either* $\langle \boldsymbol{w}_2^m - \boldsymbol{w}_1^1, \boldsymbol{w}_2^m - \boldsymbol{w}_2^1\rangle < 0$ *or* $\langle \boldsymbol{w}_2^m + \boldsymbol{w}_1^1, \boldsymbol{w}_2^m + \boldsymbol{w}_2^1\rangle < 0$ *holds;*

ii) $\langle \boldsymbol{w}_2^{i+1} - \boldsymbol{w}_1^i, \boldsymbol{w}_2^{i+1} - \boldsymbol{w}_2^i\rangle < 0$ *or* $\langle \boldsymbol{w}_2^{i+1} + \boldsymbol{w}_1^i, \boldsymbol{w}_2^{i+1} + \boldsymbol{w}_2^i\rangle < 0$ *for* $i = 1, 2, \ldots, m-1$ *and either* $\langle \boldsymbol{w}_1^m - \boldsymbol{w}_1^1, \boldsymbol{w}_1^m - \boldsymbol{w}_2^1\rangle < 0$ *or* $\langle \boldsymbol{w}_1^m + \boldsymbol{w}_1^1, \boldsymbol{w}_1^m + \boldsymbol{w}_2^1\rangle < 0$ *holds.*

Since if $\langle \mathbf{w}_1^{i+1} - \mathbf{w}_1^i, \mathbf{w}_1^{i+1} - \mathbf{w}_2^i\rangle < 0$ or $\langle \mathbf{w}_1^{i+1} + \mathbf{w}_1^i, \mathbf{w}_1^{i+1} + \mathbf{w}_2^i\rangle < 0$, then $\mathbf{w}_1^{i+1} \in \mathbf{C}(\mathbf{A}_i) = \Omega^s(\mathbf{A}_i) \cap S^1$, the proof is trivial and will be omitted.

We remark that if any arrangement of $\mathbf{A}_i, i \in \mathcal{M}$ satisfies the conditions of Theorem 3.1, then the origin of system (3.1) is stabilizable. Also, according to the process of the proof of Theorem 3.1, we notice that if system (3.1) satisfies that $\mathbf{A}_i \in \mathbf{M}^u_{2\times 2}, i \in \mathcal{M}$ and there exists a real symmetric matrix $\mathbf{P}$ such that two eigenvalues of $\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i$ are different signs and its sum is positive for every $i \in \mathcal{M}$, we can obtain a similar result where $\mathbf{w}_j^i, j = 1, 2$ related to the eigenvalues and eigenvectors of $\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i$ need to be redefined.

Based on Theorem 3.1, we have the following corollary.

Corollary 3.1. *If* $\bigcup_{i=1}^{m} \Omega^s(\boldsymbol{A}_i) = \mathbb{R}^2$, *then* $m \geq 3$.

Proof. Due to $\lambda_1^i > -\lambda_2^i > 0$, the angle of $\mathbf{C}(\mathbf{A}_i)$ is less than $\pi/2$. Therefore, any $\mathbf{C}(\mathbf{A}_i) \cup \mathbf{C}(\mathbf{A}_j), i, j \in \mathcal{M}$ can not cover S^1 which implies that at least three modes are needed. □

Next, let us consider three examples.

Example 3.1. Consider $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$. It is clear to see that the angle of $\mathbf{C}(\mathbf{A})$ is $\arctan(1/3)$, as shown in Figure 1.

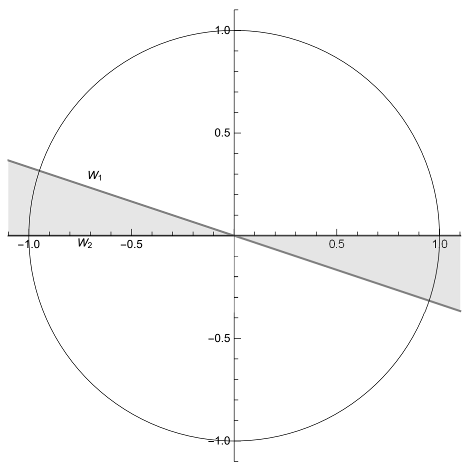

Figure 1. The intersection of the shadow region and the unit sphere refers to the points of $\mathbf{C}(\mathbf{A})$ for $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$.

Example 3.2. Consider a two-dimensional switched linear system with three modes

$$\dot{\mathbf{x}} = \mathbf{A}_{\sigma(\mathbf{x})}\mathbf{x}, \mathbf{x} \in \mathbb{R}^2 \tag{3.3}$$

where $\sigma(\mathbf{x}) \in \{1, 2, 3\}$ and

$$\mathbf{A}_1 = \begin{bmatrix} -\frac{1}{8} & \frac{3}{2} \\ -\frac{3}{2} + \frac{3\sqrt{3}}{4} & \frac{5}{8} \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -\frac{3}{8} & 1 \\ -1 - \frac{5\sqrt{3}}{4} & \frac{7}{8} \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 2 & 3 \\ -3 & -\frac{3}{2} \end{bmatrix}.$$

A simple computation gives that

$$\begin{aligned}
\mathbf{w}_1^1 &= (\frac{-3\sqrt{2} + \sqrt{3}}{6}, \frac{3 + \sqrt{6}}{6}), \\
\mathbf{w}_2^1 &= (-\frac{3\sqrt{2} + \sqrt{3}}{6}, \frac{-3 + \sqrt{6}}{6}), \\
\mathbf{w}_1^2 &= (-\frac{3\sqrt{5} + \sqrt{10}}{10}, \frac{-\sqrt{15} + \sqrt{30}}{10}), \\
\mathbf{w}_2^2 &= (\frac{-3\sqrt{5} + \sqrt{10}}{10}, -\frac{\sqrt{15} + \sqrt{30}}{10}), \\
\mathbf{w}_1^3 &= (\frac{\sqrt{21}}{7}, \frac{\sqrt{28}}{7}), \mathbf{w}_2^3 = (-\frac{\sqrt{21}}{7}, \frac{\sqrt{28}}{7}).
\end{aligned}$$

It is easy to check that $\langle \mathbf{w}_1^2 - \mathbf{w}_1^1, \mathbf{w}_1^2 - \mathbf{w}_2^1 \rangle < 0$, $\langle \mathbf{w}_1^3 + \mathbf{w}_1^2, \mathbf{w}_1^3 + \mathbf{w}_2^2 \rangle < 0$, $\langle \mathbf{w}_2^3 - \mathbf{w}_1^1, \mathbf{w}_2^3 - \mathbf{w}_2^1 \rangle < 0$. Thus, from Theorem 3.1 the origin of system (3.3) is stabilizable. Specifically, we can suppose that

$$\sigma(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega^s(\mathbf{A}_1) \\ 2, & \mathbf{x} \in \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \\ 3, & \mathbf{x} \in \Omega^s(\mathbf{A}_3) - \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \end{cases}. \tag{3.4}$$

Under this switching law, the origin is stable. Let $r(t) = \sqrt{x_1(t)^2 + x_2(t)^2}$ be the Euclidean norm of $\mathbf{x}(t) = (x_1(t), x_2(t))$. For $t_0 = 0, \mathbf{x}_0 = (1,1)$, the trajectory and the norm of $\mathbf{x}(t)$ are given in Figure 2 and Figure 3.

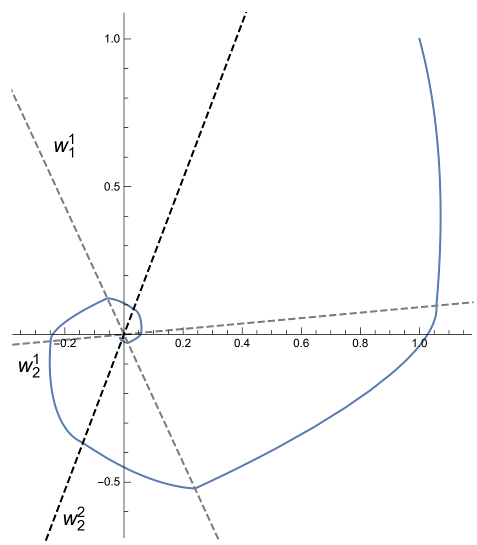

Figure 2. The trajectory of the solution of system (3.3) with the initial value $\mathbf{x}(0) = (1,1)$. We integrated system (3.3) from $t = 0$ to $t = 10$. For switching function $\sigma(\mathbf{x})$ given by (3.4), $k_1\mathbf{w}_1^1, k_2\mathbf{w}_2^1, k_3\mathbf{w}_2^2$ forms a family of switching surface where $k_1, k_2, k_3 \in \mathbb{R}$.

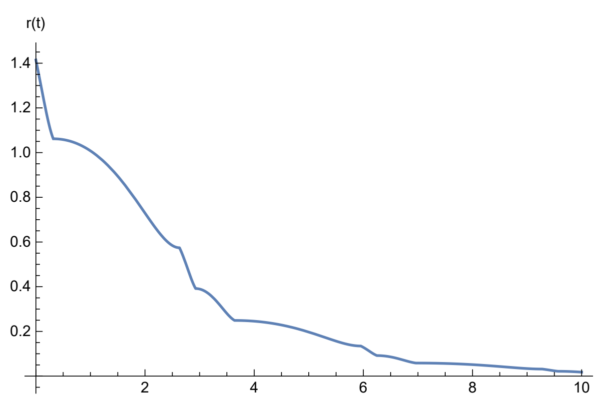

Figure 3. The Euclidean norm $r(t)$ of the solution of system (3.3) with the initial value $\mathbf{x}(0) = (1,1)$. We integrated system (3.3) from $t = 0$ to $t = 10$.

Example 3.3. Consider a two-dimensional switched linear system with four modes

$$\dot{\mathbf{x}} = \mathbf{B}_{\sigma(\mathbf{x})}\mathbf{x},\ \ \mathbf{x} \in \mathbb{R}^2, \tag{3.5}$$

where $\sigma(\mathbf{x}) \in \{1, 2, 3, 4\}$ and

$$\mathbf{B}_1 = \begin{bmatrix} 3/4 & 5 \\ -3/2 & 3/4 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} -2 & 4 \\ -4 & 5 \end{bmatrix},$$
$$\mathbf{B}_3 = \begin{bmatrix} 3 & 1 \\ -13 & 3 \end{bmatrix}, \mathbf{B}_4 = \begin{bmatrix} 25/2 & 8 \\ -8 & -4 \end{bmatrix}.$$

It is not difficult to check that $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4$ satisfies the conditions of Theorem 3.1. Therefore, the origin of system (3.5) is stabilizable. Without loss of generality, we suppose that

$$\sigma(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega^s(\mathbf{A}_1) \\ 2, & \mathbf{x} \in \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \\ 3, & \mathbf{x} \in \Omega^s(\mathbf{A}_3) - \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \\ 4, & \mathbf{x} \in \Omega^s(\mathbf{A}_4) - \Omega^s(\mathbf{A}_3) - \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \end{cases}. \tag{3.6}$$

For switching function $\sigma(\mathbf{x})$ given by (3.6), $k_1\mathbf{w}_1^1, k_2\mathbf{w}_2^1$, $k_3\mathbf{w}_2^2, k_4\mathbf{w}_2^3$ forms a family of switching surface, where $k_1, k_2, k_3, k_4 \in \mathbb{R}$ and

$$\begin{aligned} \mathbf{w}_1^1 =& \left(\frac{-\sqrt{70}+2\sqrt{7}}{14}, \frac{\sqrt{70}+2\sqrt{7}}{14}\right), \\ \mathbf{w}_2^1 =& \left(\frac{-\sqrt{70}-2\sqrt{7}}{14}, \frac{\sqrt{70}-2\sqrt{7}}{14}\right), \\ \mathbf{w}_2^2 =& (-\sqrt{35}/7, -\sqrt{14}/7),\ \ \mathbf{w}_2^3 = (-1/2, \sqrt{3}/2). \end{aligned}$$

It is easy to obtain that any solution of system (3.5) shall asymptotically approach the origin under this switching function. For $t_0 = 0, \mathbf{x}_0 = (-1, 1)$, the trajectory of the solution $\mathbf{x}(t)$ is shown in Figure 4.

From Example 3.2 and Example 3.3, we remark that the switching law $\sigma(\mathbf{x})$ is not unique to stabilize the corresponding switched system if the state space can be covered by some stable cones.

The results in two-dimensional switched systems give rise to some new questions, and under what conditions a three-dimensional switched system with all unstable modes is stabilizable? In the next section, we give some discussions.

4. Three-dimensional switched systems

It is very hard to research the covering problem of stable cones of switched systems. Inspired by some results in [6, 19], we consider some special cases and try to get some new results. Now, we focus on a family of switched systems in $\mathbb{R}^3$

$$\dot{\mathbf{x}} = \mathbf{A}_{\sigma(\mathbf{x})}\mathbf{x} \tag{4.1}$$

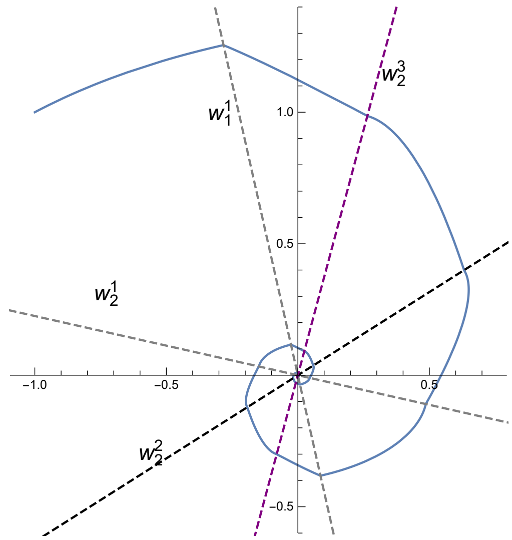

Figure 4. The trajectory of the solution of system (3.5) with the initial value $\mathbf{x}(0) = (-1, 1)$. We integrated system (3.5) from $t = 0$ to $t = 10$.

where $\sigma(\mathbf{x}) : \mathbb{R}^3 \to \mathcal{M}$ denotes the switching function and $\mathbf{A}_i \in \mathbf{M}^u_{3\times 3}, i \in \mathcal{M}$ with all eigenvalues of $\mathbf{A}_i$ having positive real parts and $\mathbf{A}_i + \mathbf{A}_i^T$ having at least one negative eigenvalue and at most two negative eigenvalues. In this paper, we consider the case of one negative eigenvalue. Hence, let us set that $\mathbf{v}_1^i, \mathbf{v}_2^i, \mathbf{v}_3^i$ are the unit eigenvectors corresponding to the eigenvalues $\lambda_1^i, \lambda_2^i, \lambda_3^i$ of $\mathbf{A}_i + \mathbf{A}_i^T$ where $\lambda_1^i, \lambda_2^i > 0$ and $\lambda_1^i + \lambda_2^i > -\lambda_3^i > 0$. Let $\mathbf{T}_i = (\mathbf{v}_1^i\ \mathbf{v}_2^i\ \mathbf{v}_3^i)$, then similar to (3.2), we have

$$\Omega^s(\mathbf{A}_i) = \{\mathbf{T}_i\mathbf{y} \mid \lambda_1^i y_1^2 + \lambda_2^i y_2^2 + \lambda_3^i y_3^2 < 0, \mathbf{y} \in \mathbb{R}^3\} \tag{4.2}$$

and

$$\mathbf{C}(\mathbf{A}_i) = \Omega^s(\mathbf{A}_i) \cap S^2 = \{\mathbf{T}_i\mathbf{y} \mid \lambda_1^i y_1^2 + \lambda_2^i y_2^2 + \lambda_3^i y_3^2 < 0, y_1^2 + y_2^2 + y_3^2 = 1\}. \tag{4.3}$$

For simplicity of presentation, let $a_i = -\lambda_1^i/\lambda_3^i, b_i = -\lambda_2^i/\lambda_3^i$. From $\lambda_1^i, \lambda_2^i > 0$ and $\lambda_1^i + \lambda_2^i > -\lambda_3^i > 0$, we obtain $a_i, b_i > 0$ and $a_i + b_i > 1$. It follows that

$$\mathbf{C}(\mathbf{A}_i) = \{\mathbf{T}_i\mathbf{y} \mid a_i y_1^2 + b_i y_2^2 < y_3^2, y_1^2 + y_2^2 + y_3^2 = 1, \mathbf{y} \in \mathbb{R}^3\}.$$

Related to the area of $\mathbf{C}(\mathbf{A}_i)$, we have the following lemma.

Lemma 4.1. *Suppose that $a, b > 0$ and $a + b \geq 1$. Then, the area of $\mathcal{R} = \{\boldsymbol{y} \in \mathbb{R}^3 \mid ay_1^2 + by_2^2 \leq y_3^2, y_1^2 + y_2^2 + y_3^2 = 1\}$ reaches maximal value if $a + b = 1$ for any given a or b. Moreover, the area of $\mathcal{R}$ achieves the minimum if $a = 1/2$ for $a + b = 1$.*

Proof. Based on the theory of surface integral, we obtain the area of $\mathcal{R}$ as follows:

$$\iint_{\mathcal{R}} ds = 4\pi - 2\int_0^{2\pi} \sqrt{\frac{a\cos^2\theta + b\sin^2\theta}{1 + a\cos^2\theta + b\sin^2\theta}}\, d\theta. \tag{4.4}$$

For simplicity of presentation, we denote (4.4) by the function $g(a, b)$. Consider the partial derivative of $g(a, b)$ with respect to a and b as follows:

$$\frac{\partial}{\partial a} g(a, b) = \int_0^{2\pi} \frac{-\cos^2\theta}{\sqrt{a\cos^2\theta + b\sin^2\theta}(1 + a\cos^2\theta + b\sin^2\theta)^{3/2}} d\theta,$$

$$\frac{\partial}{\partial b}g(a,b)=\int_0^{2\pi}\frac{-\sin^2\theta}{\sqrt{a\cos^2\theta+b\sin^2\theta}(1+a\cos^2\theta+b\sin^2\theta)^{3/2}}d\theta.$$

Thus, we obtain that for any $a>0$, $g(a,b)$ decreases monotonically with respect to b when $b\geq 1-a$. Similarly, for any $b>0$, $g(a,b)$ decreases monotonically with respect to a when $a\geq 1-b$.

If $a+b=1$, then we have

$$\frac{\partial}{\partial a}g(a,1-a)=\int_0^{2\pi}-\frac{4\cos(\theta)}{\sqrt{1+(2a-1)\cos(\theta)}(3+(2a-1)\cos(\theta))^{3/2}}d\theta.$$

Since

$$\frac{\partial}{\partial a}g(a,1-a)|_{a=\frac{1}{2}}=0,\ \frac{\partial^2}{\partial a^2}g(a,1-a)|_{a=\frac{1}{2}}>0,$$

we obtain that $a=1/2$ is a minimum point of $g(a,1-a)$. Therefore, we complete the proof of this lemma. □

Let $\mathbf{p}_i=\mathbf{T}_i(0,0,1)^T, -\mathbf{p}_i=\mathbf{T}_i(0,0,-1)^T$ denote the centres of $\mathbf{C}(\mathbf{A}_i)$. Taking account of Lemma 4.1, we have the following proposition.

Proposition 4.1. *Suppose the number of modes of system (4.1) is three and $a_1=a_2=a_3=\bar{a}>0, b_1=b_2=b_3=\bar{b}>0$. Then, the area of uncovered surface of the sphere $S^2-(\boldsymbol{C}(\boldsymbol{A}_1)\cup\boldsymbol{C}(\boldsymbol{A}_2)\cup\boldsymbol{C}(\boldsymbol{A}_3))$ approaches zero as $\bar{a}+\bar{b}$ approaches one from right provided that $\{\pm\boldsymbol{p}_i, i=1,2,3\}$ are distributed with octahedron symmetry group O_h (see [3]) where $\pm\boldsymbol{p}_i$ are the centres of $\boldsymbol{C}(\boldsymbol{A}_i)$.*

Proof. Since $\{\pm\mathbf{p}_i, i=1,2,3\}$ are distributed with octahedron O_h, without loss of generality, we consider the case $\mathbf{p}_1=(0,0,1)^T, \mathbf{p}_2=(0,1,0)^T, \mathbf{p}_3=(1,0,0)^T$ which gives that

$$\begin{aligned}\mathbf{C}(\mathbf{A}_1)=&\{\mathbf{y}\in\mathbb{R}^3\mid \bar{a}y_1^2+\bar{b}y_2^2<y_3^2, y_1^2+y_2^2+y_3^2=1\},\\ \mathbf{C}(\mathbf{A}_2)=&\{\mathbf{y}\in\mathbb{R}^3\mid \bar{a}y_1^2+\bar{b}y_3^2<y_2^2, y_1^2+y_2^2+y_3^2=1\},\\ \mathbf{C}(\mathbf{A}_3)=&\{\mathbf{y}\in\mathbb{R}^3\mid \bar{a}y_3^2+\bar{b}y_2^2<y_1^2, y_1^2+y_2^2+y_3^2=1\}.\end{aligned}$$

According to Lemma 4.1, the area of $\mathbf{C}(\mathbf{A}_i), i=1,2,3$ decreases as $\bar{a}+\bar{b}$ increases for $\bar{a}+\bar{b}>1$. Therefore, we shall consider the limiting case $\bar{a}+\bar{b}=1$.

For $\bar{a}+\bar{b}=1$, the points in $S^2-(\mathbf{C}(\mathbf{A}_1)\cup\mathbf{C}(\mathbf{A}_2)\cup\mathbf{C}(\mathbf{A}_3))$ satisfy that

$$\begin{cases} y_1^2+y_2^2+y_3^2=1,\\ \bar{a}y_1^2+(1-\bar{a})y_2^2\geq y_3^2,\\ \bar{a}y_1^2+(1-\bar{a})y_3^2\geq y_2^2,\\ \bar{a}y_3^2+(1-\bar{a})y_2^2\geq y_1^2.\end{cases}\tag{4.5}$$

Some calculation leads to the solutions of (4.5) as follows:

$$\{(y_1,y_2,y_3)\mid y_1^2=y_2^2=y_3^2=\frac{1}{3}\}.$$

Therefore, the proof of Proposition 4.1 is completed. □

We remark that O_h in Proposition 4.1 refers to regular octahedral crystal group, which is the centres of spherical caps compose the vertices of a regular octahedron. As the consequence of the above proposition, we obtain the following corollary:

Corollary 4.1. *If* $\bigcup_{i=1}^{m} \Omega^s(\boldsymbol{A}_i) = \mathbb{R}^3$, *then* $m \geq 4$.

Now, let us review the following definition.

Definition 4.1. Let $C(\mathbf{p}, r), \mathbf{p} \in S^2$ denote the spherical cap with the centre $\mathbf{p}$ of angular radius r composed of the points whose geodesic distance from $\mathbf{p}$ is less than r, as shown in Figure 5.

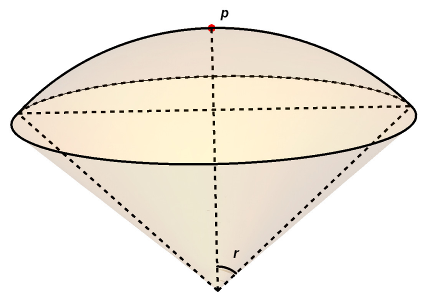

Figure 5. A spherical cap with the centre $\mathbf{p}$ of angular radius r

It is known that spherical cap refers to the intersection of the circular cone and the unit sphere. Owing to the max-min theorem, it is obvious to see that for any $\mathbf{C}(\mathbf{A}_i), i \in \mathcal{M}$, there exist two spherical caps $C(\mathbf{p}_i, r_i)$ and $C(-\mathbf{p}_i, r_i)$ such that $C(\mathbf{p}_i, r_i) \cup C(-\mathbf{p}_i, r_i) \subset \mathbf{C}(\mathbf{A}_i)$. Thus, we have the following proposition.

Proposition 4.2. *If the union of spherical caps* $C(\boldsymbol{p}_i, r_i) \cup C(-\boldsymbol{p}_i, r_i), i \in \mathcal{M}$ *covers the unit sphere, the origin of system (4.1) is stabilizable.*

According to Corollary 2.1, it is evident to see that the proposition holds.

To find the conditions easy to verify, we consider the simplest case. Hence, we shall ask how to arrange n equal spherical caps on the surface of a sphere so that the area covered by the circles will be as large as possible. The packing and covering problems related to the above question have a vast literature which is studied by Fejes Tóth, L. [23, 24], Fejes Tóth, G. [22] and Moser & Pach [14]. Contact polyhedra [6] is one of research tools. An arrangement of n equal spherical caps on sphere corresponds to a polyhedra whose vertices are the centres of these spherical caps and whose edges are composed of the lines between the centres of two overlapping spherical caps. Symmetry groups and edge counts of the polyhedra are associated with the arrangements of n equal spherical caps. For the convenience of readers, we summarize the results of $n = 8, 12$ in [6] in the following lemma.

Lemma 4.2. *Suppose* n *equal spherical caps of angular radius* r *can cover the surface of the unit sphere. Then, we have*

i) for $n = 8$, r *reaches minimum* 0.840193, *if the contact polyhedra satisfies symmetry group* D_{2d};

ii) for $n = 12$, r *reaches minimum* 0.652359, *if the contact polyhedra satisfies symmetry group icosahedron* I_h.

The proof of this lemma is omitted. The more detailed results can be seen in [6, 23] and more discussions about point group can be seen in [3]. We remark that for $n = 8$, the trigonal dodecahedron satisfying symmetry group D_{2d} as shown in Figure 6 is the best arrangement of the spherical caps such that the greatest distance between a point of the spherical surface and the nearest of the centres of these eight spherical caps. Similarly, for $n = 12$, the regular icosahedron as shown in Figure 7 is the best arrangement of the spherical caps.

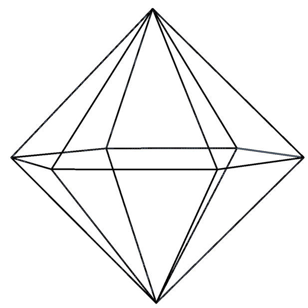

Figure 6. Symmetry D_{2d} of the contact polyhedra formed by the centres of 8 spherical caps

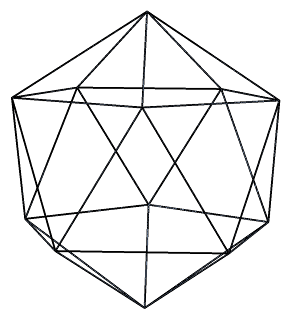

Figure 7. Symmetry I_h of the contact polyhedra formed by the centres of 12 spherical caps

Since each subsystem corresponds to two maximum spherical caps, combined with the above discussions we conclude the following propositions:

Proposition 4.3. *Suppose switched system (4.1) has four subsystems. If for each* $\boldsymbol{C}(\boldsymbol{A}_i), i = 1, 2, 3, 4$*, there exists two spherical caps* $C(\boldsymbol{p}_i, r_i)$ *and* $C(-\boldsymbol{p}_i, r_i)$ *included in* $\boldsymbol{C}(\boldsymbol{A}_i)$ *such that* $r_i \geq 0.841$ *and* $\{\pm\boldsymbol{p}_i, i = 1, 2, 3, 4\}$ *are distributed with point group symmetry* D_{2d}*. Then, the origin of this system is stabilizable.*

Proposition 4.4. *Suppose switched system (4.1) has six subsystems. Then, if for each* $\boldsymbol{C}(\boldsymbol{A}_i), i = 1, 2, 3, 4, 5, 6$ *there exists two spherical caps* $C(\boldsymbol{p}_i, r_i)$ *and* $C(-\boldsymbol{p}_i, r_i)$ *included in* $\boldsymbol{C}(\boldsymbol{A}_i)$ *such that* $r_i \geq 0.653$ *and* $\{\pm\boldsymbol{p}_i, i = 1, 2, 3, 4, 5, 6\}$ *are distributed with point group symmetry* I_h*, the origin of this system is stabilizable.*

Proposition 4.3 and Proposition 4.4 follows immediately from Lemma 4.2. Since for 10 spherical caps the contact polyhedra of the best arrangement is not symmetric about the center of the sphere, we can not obtain sufficient conditions of stabilization from the minimum covering configuration. Hence, we have the following specific result:

Proposition 4.5. *Suppose switched system (4.1) has five subsystems. If for each* $\boldsymbol{C}(\boldsymbol{A}_i), i = 1, 2, 3, 4, 5$*, there exists two spherical caps* $C(\boldsymbol{p}_i, r_i)$ *and* $C(-\boldsymbol{p}_i, r_i)$ *including in* $\boldsymbol{C}(\boldsymbol{A}_i)$ *such that* $r_i \geq 0.825$ *and* $\{\pm\boldsymbol{p}_i, i = 1, 2, 3, 4, 5\}$ *satisfy that one is* *situated at the north pole, another one at the south pole and the remaining centres equidistant along the equator after an orthogonal transformation. Then, the origin of this system is stabilizable.*

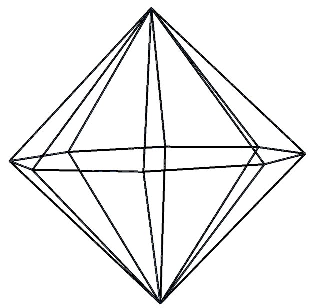

Figure 8. An arrangement of the centres of 10 spherical caps

We remark that the centres in Proposition 4.5 can be arranged as shown in Figure 8. Based on the results of the covering problem, more sufficient conditions for stabilization can be obtained, which is worth studying further.

Next, let us consider the following simple example.

Example 4.1. Consider a three-dimensional switched linear system with four modes

$$\dot{\mathbf{x}} = \mathbf{A}_{\sigma(\mathbf{x})}\mathbf{x},\ \ \mathbf{x} \in \mathbb{R}^3, \tag{4.6}$$

where $\sigma(\mathbf{x}) \in \{1, 2, 3, 4\}$ and

$$\mathbf{A}_1 = \begin{bmatrix} 0.425675 & 0.316797 & 0.433413 \\ -0.316797 & 0.425675 & 0.454755 \\ -0.433413 & -0.454755 & -0.574325 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -0.574325 & 0.94177 & 0.628207 \\ -0.94177 & 0.425675 & 0.532595 \\ -0.628207 & -0.532595 & 0.425675 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} 0.175675 & 1.35493 & 0.202179 \\ -0.488904 & -0.324325 & 0.753435 \\ -0.202179 & -0.753435 & 0.425675 \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} 0.175675 & 0.283498 & 0.580803 \\ -1.14952 & -0.324325 & 0.185781 \\ -0.580803 & -0.185781 & 0.425675 \end{bmatrix}.$$

Some simple calculations give

$$\Omega^s(\mathbf{A}_1) = \{\mathbf{y} \in \mathbb{R}^3 \mid y_1^2 + y_2^2 - 1.34921 y_3^2 < 0\},$$

$$\Omega^s(\mathbf{A}_2) = \{\mathbf{y} \in \mathbb{R}^3 \mid -y_1^2 + 0.741173y_2^2 + 0.741173y_3^2 < 0\},$$
$$\Omega^s(\mathbf{A}_3) = \{\mathbf{y} \in \mathbb{R}^3 \mid y_1^2 + 4.92971y_1y_2 - 1.84617y_2^2 + 2.42309y_3^2 < 0\},$$
$$\Omega^s(\mathbf{A}_4) = \{\mathbf{y} \in \mathbb{R}^3 \mid y_1^2 - 4.92971y_1y_2 - 1.84617y_2^2 + 2.42309y_3^2 < 0\}.$$

Using the method in [6], we obtain that $\mathbf{C}(\mathbf{A}_1) \cup \mathbf{C}(\mathbf{A}_2) \cup \mathbf{C}(\mathbf{A}_3) \cup \mathbf{C}(\mathbf{A}_4)$ can cover the unit sphere. Thus, if we assume that

$$\sigma(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega^s(\mathbf{A}_1) \\ 2, & \mathbf{x} \in \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \\ 3, & \mathbf{x} \in \Omega^s(\mathbf{A}_3) - \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \\ 4, & \mathbf{x} \in \Omega^s(\mathbf{A}_4) - \Omega^s(\mathbf{A}_3) - \Omega^s(\mathbf{A}_2) - \Omega^s(\mathbf{A}_1) \end{cases}, \quad (4.7)$$

then the origin of system (4.6) is stabilizable for switching function $\sigma(\mathbf{x})$ given by (4.7) according to Proposition 4.2. For $t_0 = 0, \mathbf{x}_0 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, the trajectory of the solution $\mathbf{x}(t)$ is shown in Figure 9 and the Euclidean norm of the solution is shown in Figure 10.

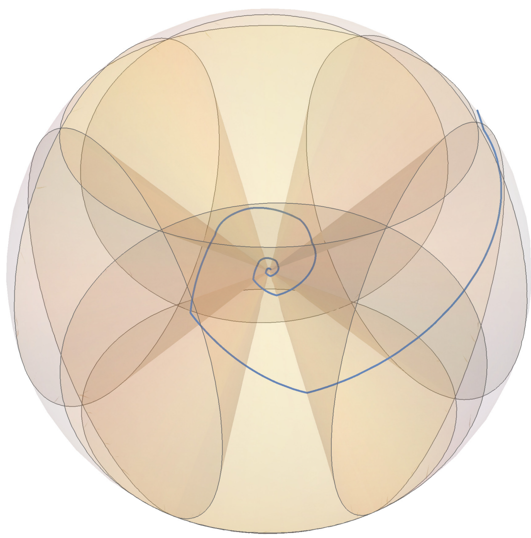

Figure 9. The trajectory of the solution of system (4.6) with the initial value $\mathbf{x}(0) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. We integrated system (4.6) from $t = 0$ to $t = 20$. The different shadow regions illustrate the intersection of $\Omega^s(\mathbf{A}_i)$ and the unit sphere, $i = 1, 2, 3, 4$.

5. Conclusions

In this paper, we have investigated some new sufficient conditions for stabilization of n-dimensional switched systems with all unstable modes. Consequently, we get a simple sufficient condition for stabilization of two-dimensional switched systems with all unstable modes. Moreover, we also obtain some interesting sufficient conditions for stabilization of three-dimensional switched systems with all unstable modes by means of the theory of spherical covering and crystal point groups. Some examples are provided to illustrate our results. We hope that our results can provide a new thread of studying stabilization of switched systems with all unstable modes.

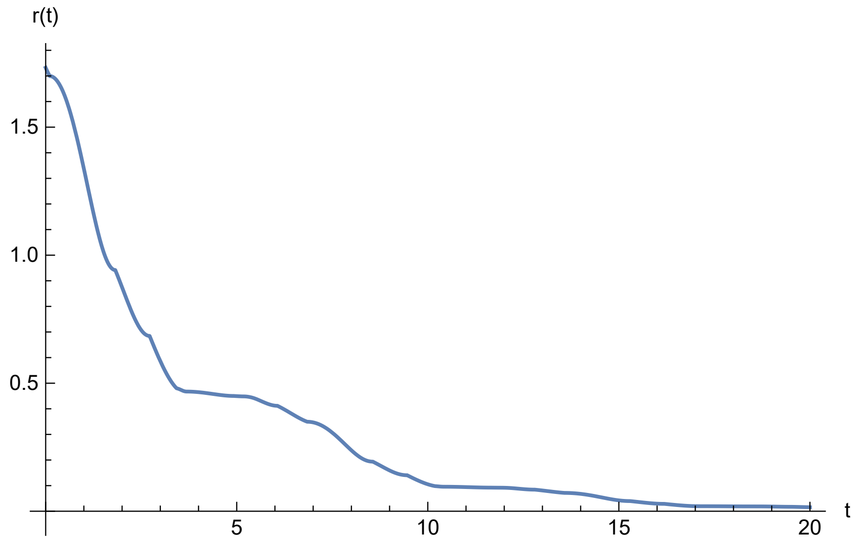

Figure 10. The Euclidean norm of the solution of system (4.6) with the initial value $\mathbf{x}(0) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. We integrated system (4.6) from $t = 0$ to $t = 20$.

References

[1] M. S. Branicky, *Stability of switched and hybrid systems*, in *Decision and Control, 1994, Proceedings of the 33rd IEEE Conference*, IEEE, 1994, 4, 3498–3503.

[2] M. S. Branicky, *Multiple lyapunov functions and other analysis tools for switched and hybrid systems*, IEEE Transactions on Automatic Control, 1998, 43(4), 475–482.

[3] F. A. Cotton, *Chemical applications of group theory*, John Wiley & Sons, New York, 2003.

[4] J. Daafouz, P. Riedinger and C. Iung, *Stability analysis and control synthesis for switched systems: A switched lyapunov function approach*, IEEE Transactions on Automatic Control, 2002, 47(11), 1883–1887.

[5] R. A. Decarlo, M. S. Branicky, S. Pettersson and B. Lennartson, *Perspectives and results on the stability and stabilizability of hybrid systems*, Proceedings of the IEEE, 2000, 88(7), 1069–1082.

[6] P. W. Fowler and T. Tarnai, *Transition from spherical circle packing to covering: Geometrical analogues of chemical isomerization*, Proceedings of the Royal Society A, 1996, 452(1952), 2043–2064.

[7] J. C. Geromel and P. Colaneri, *Stability and stabilization of continuous-time switched linear systems*, SIAM Journal on Control and Optimization, 2006, 45(5), 1915–1930.

[8] M. W. Hirsch, R. L. Devaney and S. Smale, *Differential equations, Dynamical Systems, and Linear Algebra*, 60, Academic Press, Salt Lake City, 1974.

[9] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd Edition, Cambridge University Press, Cambridge, 2013.

[10] D. Liberzon, *Switching in Systems and Control*, Springer Science & Business Media, New York, 2003.

[11] D. Liberzon and A. S. Morse, *Basic problems in stability and design of switched systems*, IEEE Control systems, 1999, 19(5), 59–70.

[12] H. Lin and P. J. Antsaklis, *Stability and stabilizability of switched linear systems: A survey of recent results*, IEEE Transactions on Automatic Control, 2009, 54(2), 308–322.

[13] M. Margaliot, *Stability analysis of switched systems using variational principles: An introduction*, Automatica, 2006, 42(12), 2059–2077.

[14] W. Moser and J. Pach, *Research problems in discrete geometry: Packing and covering*, D1MACS Technical Report, 1993, 93–32.

[15] P. Peleties and R. DeCarlo, *Asymptotic stability of m-switched systems using lyapunov-like functions*, in *1991 American Control Conference*, IEEE, 1991, 1679–1684.

[16] R. N. Shorten and K. S. Narendra, *On the stability and existence of common lyapunov functions for linear switching systems*, in *Proceedings of the 37th IEEE Conference on Decision and Control*, 4, IEEE, 1998, 3723–3724.

[17] E. Skafidas, R. J. Evans, A. V. Savkin and I. R. Petersen, *Stability results for switched controller systems*, Automatica, 1999, 35(4), 553–564.

[18] Z. Sun and S. Ge, *Stability Theory of Switched Dynamical Systems*, 1, Springer Science & Business Media, New York, 2011.

[19] T. Tarnai and Z. Gáspár, *Covering a sphere by equal circles, and the rigidity of its graph*, Mathematical Proceedings of the Cambridge Philosophical Society, 1991, 110(1), 71–89.

[20] W. J. Terrell, *Stability and stabilization: an introduction*, Princeton University Press, Princeton, 2009.

[21] L. Q. Thuan and M. K. Camlibel, *On the existence, uniqueness and nature of carathodory and filippov solutions for bimodal piecewise affine dynamical systems*, Systems & control letters, 2014, 68, 76–85.

[22] G. F. Tóth and W. Kuperberg, *Packing and covering with convex sets*, in *Handbook of Convex Geometry*, Elsevier, 1993, 799–860.

[23] L. F. Tóth, *On covering a spherical surface with equal spherical caps*, Matematikai és Fizikai Lapok, 1943, 50, 40–46.

[24] L. F. Tóth, *Perfect distribution of points on a sphere*, Periodica Mathematica Hungarica, 1971, 1(1), 25–33.

[25] M. Wicks, P. Peleties and R. Decarlo, *Switched controller design for the quadratic stabilization of a pair of unstable linear systems*, European Journal of Control, 1998, 4(98), 140–147.

[26] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, 2, Springer Science & Business Media, New York, 2003.

[27] W. Xiang and J. Xiao, *Stabilization of switched continuous-time systems with all modes unstable via dwell time switching*, Automatica, 2014, 50(3), 940–945.

[28] H. Yang, B. Jiang and V. Cocquempot, *A survey of results and perspectives on stabilization of switched nonlinear systems with unstable modes*, Nonlinear Analysis Hybrid Systems, 2014, 13, 45–60.

[29] H. Ye, A. N. Michel and L. Hou, *Stability theory for hybrid dynamical systems*, IEEE Trans on Automatic Control, 1998, 43(4), 461–474.

[30] X. Zhao, S. Yin, H. Li and B. Niu, *Switching stabilization for a class of slowly switched systems*, IEEE Transactions on Automatic Control, 2015, 60(1), 221–226.