

Application of Almost Increasing Sequence for Absolute Riesz $|\overline{N}, p_n^{\alpha, \beta}; \delta|_k$ Summable Factor*

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Abstract In this paper, we generate an extended result by Bor and Seyhan concerning absolute Riesz summability factors. Further, we develop some well-known results from our main result.

Keywords Absolute summability, Quasi- f -power increasing sequence, Infinite series, Riesz summability.

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1. Introduction

Let $\sum a_n$ be an infinite series, $\{s_n\} = \sum_{k=0}^n a_k$ be the sequence of its partial sums and n^{th} mean of the sequence $\{s_n\}$ is given by u_n , s.t.,

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k. \quad (1.1)$$

$$(1.2)$$

Definition 1: An infinite series $\sum a_n$ is absolute summable, if

$$\lim_{n \rightarrow \infty} u_n = s$$

and

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}| < \infty, \quad (1.3)$$

Definition 2: Let $\{p_n\}$ be a sequence with $p_0 > 0$ and $p_n \geq 0$ for $n > 0$

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty. \quad (1.4)$$

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For $\alpha > -1$, $0 < \beta \leq 1$, $\alpha + \beta > 0$, define:

$$\epsilon_0^{\alpha+\beta} = 1, \quad \epsilon_n^{\alpha+\beta} = \frac{(\alpha + \beta + 1)(\alpha + \beta + 2)\dots(\alpha + \beta + n)}{n!}, \quad (n = 1, 2, 3, \dots) \quad (1.5)$$

$$p_n^{\alpha,\beta} = \sum_{v=0}^n \epsilon_{n-v}^{\alpha+\beta-1} p_v, \quad (1.6)$$

$$P_n^{\alpha,\beta} = \sum_{v=0}^n p_n^{\alpha,\beta} \rightarrow \infty, \quad n \rightarrow \infty \quad (1.7)$$

and

$$P_{-n}^{\alpha,\beta} = p_{-n}^{\alpha,\beta} = 0, \quad n \geq 1.$$

Then, the sequence-to-sequence transformation t_n defines the $(\overline{N}, p_n^{\alpha,\beta})$ mean of series $\sum a_n$ and is given by:

$$t_n = \frac{1}{P_n^{\alpha,\beta}} \sum_{k=0}^n p_k^{\alpha,\beta} s_k, \quad P_n^{\alpha,\beta} \neq 0, \quad n \in N \quad (1.8)$$

and $\lim_{n \rightarrow \infty} t_n = s$, and the series is called $(\overline{N}, p_n^{\alpha,\beta})$, formed by sequence of coefficients $\{p_n^{\alpha,\beta}\}$.

Further, if sequences $\{t_n\}$ is of bounded variation with index $k \geq 1$ i.e.

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{k-1} |\Delta t_{n-1}|^k < \infty, \quad (1.9)$$

then the series $\sum a_n$ is said to be absolutely $(R, p_n^{\alpha,\beta})_k$ summable with index k or $|\overline{N}, p_n^{\alpha,\beta}|_k$ summable to s .

Definition 3: The series is said to be $|\overline{N}, p_n^{\alpha,\beta}; \delta|_k$ summable, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k + k - 1} |\Delta t_{n-1}|^k < \infty, \quad (1.10)$$

with $k \geq 1$, $\delta \geq 0$ and

$$\Delta t_n = -\frac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}} \sum_{v=1}^n P_{v-1}^{\alpha,\beta} a_v, \quad n \geq 1. \quad (1.11)$$

Bor [1–3] generalised the result associated with Riesz summability factors. Bor and Özarşlan [4,5] established theorems using $|\overline{N}, p_n; \delta|$ summability factors. Özarşlan [11,12] used the definition of almost increasing sequence for absolute summability. Mishra et. al. [9,10] gave useful result on approximation. Also, Mishra et. al. [7,8] provided new results related to matrix summability and improper integrals. In [13], Sonker and Munjal established new theorem on absolute summability for Triangle matrices. Yildiz [14,15] determined theorems on generalized absolute matrix summability factors.

2. Known result

By using $|\overline{N}, p_n^\alpha; \delta|_k$ summability, Bor and Seyan [6] proved the following theorem.

Theorem 2.1. [6] *Let p_n be a sequence of +ve numbers s.t.:*

$$P_n = O(np_n) \text{ as } n \rightarrow \infty. \tag{2.1}$$

By using $|\overline{N}, p_n^\alpha; \delta|_k$ summability, Bor and Seyan [6] proved the following theorem.
 Let (X_n) be an almost increasing sequence and assuming (ξ_n) and (λ_n) are s.t.:

$$|\Delta\lambda_n| \leq \xi_n, \tag{2.2}$$

$$\xi_n \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n|\Delta\xi_n|X_n \leq \infty, \tag{2.4}$$

$$|\lambda_n|X_n = O(1) \text{ as } n \rightarrow \infty, \tag{2.5}$$

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right), \tag{2.6}$$

$$\sum_{n=1}^m \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \text{ as } m \rightarrow \infty. \tag{2.7}$$

Then, $\sum a_n \lambda_n$ is $|\overline{N}, p_n; \delta|_k$ summable where, $k \geq 1$ and $0 \leq \delta \leq \frac{1}{k}$.

3. Main result

A sequence is of bounded variation i.e. $(\lambda_n) \in BV$, if :

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| = |\lambda_n - \lambda_{n-1}| < \infty.$$

Theorem 3.1. *Let (X_n) , (ξ_n) and (λ_n) be as defined in theorem 2.1 and verify 2.2-2.5. If the following conditions also satisfy:*

$$\sum_{n=v+1}^{\infty} \frac{1}{P_{n-1}^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{\delta k-1} = O\left\{\frac{1}{P_v^{\alpha,\beta}} \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}}\right)^{\delta k}\right\}, \tag{3.1}$$

$$\sum_{n=1}^m \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{\delta k-1} |t_n|^k = O(X_m), \tag{3.2}$$

$$\sum_{n=1}^m \frac{|\lambda_n|}{n} = O(1) \tag{3.3}$$

and

$$\sum_{n=1}^m \frac{1}{n} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{\delta k} |t_n|^k = O(X_m) \text{ as } m \rightarrow \infty. \tag{3.4}$$

then, $\sum a_n \lambda_n$ is $|\overline{N}, p_n^{\alpha,\beta}; \delta|_k$ summable where $k \geq 1$ and $0 \leq \delta \leq \frac{1}{k}$.

Proof. Let Y_n denote the $(\bar{N}, p_n^{\alpha, \beta})$ mean of $\sum a_n \lambda_n$. We have:

$$Y_n = \frac{1}{P_n^{\alpha, \beta}} \sum_{v=0}^n p_v^{\alpha, \beta} \sum_{i=0}^v a_i \lambda_i. \quad (3.5)$$

For $n \geq 1$,

$$\begin{aligned} \Delta Y_n = Y_n - Y_{n-1} &= \frac{P_n^{\alpha, \beta}}{P_n^{\alpha, \beta} P_{n-1}^{\alpha, \beta}} \sum_{v=1}^n P_{v-1}^{\alpha, \beta} a_v \lambda_v = \frac{P_n^{\alpha, \beta}}{P_n^{\alpha, \beta} P_{n-1}^{\alpha, \beta}} \sum_{v=1}^n \frac{P_{v-1} \lambda_v}{v} v a_v. \\ \Delta Y_n &= \frac{n+1}{n P_n^{\alpha, \beta}} p_n^{\alpha, \beta} t_n \lambda_n \\ &\quad - \frac{p_n^{\alpha, \beta}}{P_n^{\alpha, \beta} P_{n-1}^{\alpha, \beta}} \sum_{v=1}^{n-1} p_v^{\alpha, \beta} t_v \lambda_v \frac{v+1}{v} \\ &\quad + \frac{p_n^{\alpha, \beta}}{P_n^{\alpha, \beta} P_{n-1}^{\alpha, \beta}} \sum_{v=1}^{n-1} P_v^{\alpha, \beta} t_v \Delta \lambda_v \frac{v+1}{v} \\ &\quad + \frac{p_n^{\alpha, \beta}}{P_n^{\alpha, \beta} P_{n-1}^{\alpha, \beta}} \sum_{v=1}^{n-1} P_v^{\alpha, \beta} t_v \lambda_{v+1} \frac{1}{v} \\ &= Y_1 + Y_2 + Y_3 + Y_4. \end{aligned} \quad (3.6)$$

To prove the Theorem 3.1, it is enough to prove

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k + k - 1} |\Delta Y_n|^k < \infty. \quad (3.7)$$

Using Minkowski's inequality,

$$|Y_1 + Y_2 + Y_3 + Y_4|^k \leq 4^k (|Y_1|^k + |Y_2|^k + |Y_3|^k + |Y_4|^k).$$

Then, equation 3.7 reduces to:

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k + k - 1} |Y_r|^k < \infty \text{ for } r = 1, 2, 3, 4. \quad (3.8)$$

Now, the L. H. S. of equation 3.8 is given as:

$$\begin{aligned} \sum_{n=1}^m \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k + k - 1} |Y_1|^k &= \sum_{n=1}^m \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k + k - 1} \left| \frac{n+1}{n P_n^{\alpha, \beta}} p_n^{\alpha, \beta} t_n \lambda_n \right|^k \\ &= \sum_{n=1}^m \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k - 1} |t_n|^k |\lambda_n| \\ &= O(1) |\lambda_m| \sum_{n=1}^m \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k - 1} |t_n|^k \\ &\quad + O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n \left(\frac{P_v^{\alpha, \beta}}{p_v^{\alpha, \beta}} \right)^{\delta k - 1} |t_v|^k \\ &= O(1) |\lambda_m| X_m + O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n \\ &= O(1) \text{ as } m \rightarrow \infty, \end{aligned} \quad (3.9)$$

$$\sum_{n=2}^{m+1} \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k + k - 1} |Y_2|^k = O(1) \sum_{n=2}^{m+1} \frac{1}{P_{n-1}^{\alpha, \beta}} \left(\frac{P_n^{\alpha, \beta}}{p_n^{\alpha, \beta}} \right)^{\delta k - 1} \times$$

$$\begin{aligned}
 & \times \sum_{v=1}^{n-1} p_v^{\alpha,\beta} |t_v|^k |\lambda_v| \left(\frac{1}{P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n-1} p_v^{\alpha,\beta} \right)^{k-1} \\
 & = O(1) \sum_{v=1}^m p_v^{\alpha,\beta} |t_v|^k |\lambda_v| \times \\
 & \times \sum_{n=v+1}^{m+1} \frac{1}{P_{n-1}^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k-1} \\
 & = O(1) \sum_{v=1}^m p_v^{\alpha,\beta} |t_v|^k |\lambda_v| \frac{1}{P_v^{\alpha,\beta}} \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k} \\
 & = O(1) |\lambda_m| \sum_{n=1}^m \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k-1} |t_n|^k \\
 & + O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k-1} |t_v|^k \\
 & = O(1) |\lambda_m| X_m + O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n \\
 & = O(1) \text{ as } m \rightarrow \infty, \tag{3.10}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k+k-1} |Y_3|^k & = O(1) \sum_{n=2}^{m+1} \frac{1}{P_{n-1}^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k-1} \times \\
 & \times \sum_{v=1}^{n-1} P_v^{\alpha,\beta} |t_v|^k \xi_v \left(\frac{1}{P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n-1} P_v^{\alpha,\beta} \xi_v \right)^{k-1} \\
 & = O(1) \sum_{v=1}^m P_v^{\alpha,\beta} \xi_v |t_v|^k \times \\
 & \times \sum_{n=v+1}^{m+1} \frac{1}{P_{n-1}^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k-1} \\
 & = O(1) \sum_{v=1}^m P_v^{\alpha,\beta} |t_v|^k \xi_v \frac{1}{P_v^{\alpha,\beta}} \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k} \\
 & = m \xi_m \sum_{v=1}^m \frac{1}{v} \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k} |t_v|^k \\
 & + O(1) \sum_{v=1}^{m-1} \Delta(v \xi_v) \sum_{i=1}^v \frac{1}{i} \left(\frac{P_i^{\alpha,\beta}}{p_i^{\alpha,\beta}} \right)^{\delta k} |t_i|^k \\
 & = O(1) m \xi_m X_m + O(1) \sum_{v=1}^{m-1} |\Delta(v \xi_v)| X_v \\
 & = O(1) m \xi_m X_m + O(1) \sum_{v=1}^{m-1} |\Delta \xi_v| X_v \\
 & + O(1) \sum_{v=1}^{m-1} \xi_{v+1} X_{v+1} \\
 & = O(1) \text{ as } m \rightarrow \infty, \tag{3.11}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=1}^m \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k+k-1} |Y_4|^k & = O(1) \sum_{n=2}^{m+1} \frac{1}{P_{n-1}^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k-1} \times \\
 & \times \sum_{v=1}^{n-1} P_v^{\alpha,\beta} \frac{|\lambda_{v+1}|}{v} |t_v|^k \left(\frac{1}{P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n-1} P_v^{\alpha,\beta} \frac{|\lambda_{v+1}|}{v} \right)^{k-1}
 \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{v=1}^m P_v^{\alpha,\beta} \frac{|\lambda_{v+1}|}{v} |t_v|^k \times \\
&\times \sum_{n=v+1}^{m+1} \frac{1}{P_n^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k-1} \\
&= O(1) \sum_{v=1}^m P_v^{\alpha,\beta} \frac{|\lambda_{v+1}|}{v} |t_v|^k \frac{1}{P_v^{\alpha,\beta}} \times \\
&\times \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k} \\
&= O(1) \sum_{v=1}^m |\lambda_{v+1}| \frac{1}{v} \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k} |t_v|^k \\
&= O(1) |\lambda_{m+1}| \sum_{v=1}^m \frac{1}{v} \left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}} \right)^{\delta k} |t_v|^k \\
&+ O(1) \sum_{v=1}^{m-1} \Delta |\lambda_{v+1}| \times \\
&\times \sum_{i=1}^v \frac{1}{i} \left(\frac{P_i^{\alpha,\beta}}{p_i^{\alpha,\beta}} \right)^{\delta k} |t_i|^k \\
&= O(1) |\lambda_{m+1}| X_m + O(1) \sum_{v=1}^{m-1} \Delta |\lambda_{v+1}| X_v \\
&= O(1) \text{ as } m \rightarrow \infty. \tag{3.12}
\end{aligned}$$

Collecting 3.5-3.12, we have

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{\delta k+k-1} |T_{n,r}|^k < \infty \text{ for } r = 1, 2, 3, 4. \tag{3.13}$$

Hence, the theorem is proved. \square

Corollary 3.1. *Let (X_n) , (ξ_n) and (λ_n) are s.t. conditions 2.2-2.5 of theorem 2.1, condition 3.3 of theorem 3.1,*

$$\sum_{n=v+1}^{\infty} \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} = O\left(\frac{1}{P_v^\alpha}\right), \tag{3.14}$$

$$\sum_{n=1}^m \frac{p_n^\alpha}{P_n^\alpha} |t_n|^k = O(X_m) \text{ as } m \rightarrow \infty \tag{3.15}$$

and

$$\sum_{n=1}^m \frac{1}{n} |t_n|^k = O(X_m) \text{ as } m \rightarrow \infty \tag{3.16}$$

holds. Then, $\sum a_n \lambda_n$ is $|\overline{N}, p_n^\alpha|_k$ summable for $k \geq 1$.

Proof: By using $\beta = 1$ and $\delta = 0$ in main theorem, we will get 3.14, 3.15 and 3.16. The proof is same as the main theorem 3.1, but here we used equations 3.14, 3.15 and 3.16 instead of equations 3.1, 3.2 and 3.3.

Corollary 3.2. *Let (X_n) , (ξ_n) and (λ_n) are s.t. conditions 2.2-2.5 of theorem 2.1, condition 3.3 of theorem 3.1 and 3.14 - 3.16 holds. Then, $\sum a_n \lambda_n$ is $|\overline{N}, p_n^\alpha|$ summable.*

Proof: By using $\beta = 1$, $k = 1$ and $\delta = 0$ in main theorem and equations 3.14 - 3.16, we get this result.

4. Conclusion

The negligible set of conditions has been obtained for the infinite series in this paper. By the examination, we may infer that our hypothesis is a summed up variant which can be diminished for a few notable summabilities as appeared in corollaries.

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