Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 3, Number 3, September 2021, 447–463 DOI:10.12150/jnma.2021.447

Hopf Bifurcation Analysis of a Host-generalist Parasitoid Model with Diffusion Term and Time Delay*

Zijun Liu[1] and Ruizhi Yang[1,†]

Abstract In this paper, we studied a delayed host-generalist parasitoid model with Holling II functional response and diffusion term. The Turing instability and local stability are studied. The existence of Hopf bifurcation is investigated, and some explicit formulas for determining the bifurcation direction and the stability of the bifurcating periodic solution are derived by the theory of center manifold and normal form method. Some numerical simulations are carried out.

Keywords Delay, Diffusion, Turing instability, Hopf bifurcation.

MSC(2010) 34K18, 35B32.

1. Introduction

In many aspects, dynamics of population model has been studied [1,4,5,16]. Host-generalist parasitoids systems have gotten great attention in recent years. Because of the invasion of leafmicrolepidopteron attacking horse chestnut trees in Europe (in particular in France) [8], Magal et al. [3] investigated the following host-parasitoid model with Holling Type II functional response, that is

$$\begin{cases} \frac{du(t)}{dt} = r_1 u - \frac{r_1 u^2}{K_1} - \frac{\xi uv}{1+\xi hu} \\ \frac{dv(t)}{dt} = r_2 v - \frac{r_2 v^2}{K_1} + \frac{\gamma \xi uv}{1+\xi hu}, \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ denote densities of the hosts(leafminers Cameraria orhidella) and generalist parasitoids (Minotetrastichus frontalis) at time t respectively. r_1 is the intrinsic growth rate of the hosts in absence of parasitoids. r_2 represents the intrinsic growth rate of the parasitoids in absence of hosts. K_1 denotes the carrying capacity of the host population. K_2 denotes the carrying capacity of the parasitoid population. ξ is the encounter rate of hosts and parasitoids. γ is the conversion rate of parasitoids. h describes the harvesting time. r_i, $K_i (i = 1, 2)$, γ, ξ, h are all positive constants. Magal et al. analyzed the number and stability of equilibria in system (1.1) and found out that the model always predicts persistence of the

†the corresponding author.
Email address: yangruizhi529@163.com (R. Yang)
[1]Department of Mathematics, Northeast Forestry University, Harbin, Heilongjiang 150040, China
*The authors were supported by Fundamental Research Funds for the Central Universities (No. 2572019BC01) and Natural Science Foundation of Heilongjiang (No. A2018001).

parasitoid. Then, in [13], the author also considered bifurcation analysis of the following system with Holling Type II functional response. For simplicity, taking $\overline{u} = \frac{u}{K_1}$, $\overline{v} = \frac{r_2 v}{r_1 K_2}$, and $\overline{t} = r_1 t$, then (1.1) can be rewritten in the following form (still denote $\overline{u}$, $\overline{v}$, $\overline{t}$ as u, v, t respectively)

$$\begin{cases} \frac{du}{dt} = u(1 - u - \frac{bv}{a+u}), \\ \frac{dv}{dt} = v(\delta - v + \frac{cu}{a+u}), \end{cases} \tag{1.2}$$

where

$$a = \frac{1}{K_1 \xi h},\ b = \frac{K_2}{K_1 r_2 h},\ c = \frac{\gamma}{r_1 h},\ \delta = \frac{r_2}{r_1}.$$

The sufficient conditions were obtained to ensure that the equilibria are locally and globally asymptotically stable.

Time delay in population model with Holling II functional response may have significant impact on the underlying dynamics and many researchers have studied this effect [2,7,9,11,14,15,17,18]. Because of maturation time, capturing time, gestation time or other reasons, many different types of delays have been incorporated in population models. Considering the delay effect on the generalist parasitoid ,and the host and generalist parasitoid are non-homogeneous in the space. We study the following model

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + u - u^2 - \frac{buv}{a+u}, & x \in (0, l\pi),\ t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v + v(\delta - v + \frac{cu(t-\tau)}{a+u(t-\tau)}), & x \in (0, l\pi),\ t > 0, \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi, t) = v_x(l\pi, t) = 0,\ t > 0, & \\ u(x,\theta) = u_0(x,\theta) \geq 0, v(x,\theta) = v_0(x,\theta) \geq 0, & x \in [0, l\pi], \theta \in [-\tau, 0]. \end{cases} \tag{1.3}$$

where d_1 and d_2 are the diffusion coefficients of prey and predator respectively. The aim of this article is to study the local stability and Hopf bifurcation of the unique positive equilibrium for the system (1.3) by using τ as a parameter.

The rest of this paper is organized as follows: In Section 2, we study the local stability, Turing instability and the occurrence of Hopf bifurcation. In Section 3, we study the direction and stability of spatial Hopf bifurcation. In Section 4, we present some numerical simulations to illustrate the established results. Finally, a summarization is given in Section 5.

2. Analysis of the characteristic equations

By analyzing the associated characteristic equation at $P = (u_0, v_0)$, we investigate the stability and instability of $P = (u_0, v_0)$ for system (1.3). Denote

$$u_1(t) = u(\cdot, t),\quad u_2(t) = v(\cdot, t),\quad U = (u_1, u_2)^T,$$

$$X = C([0, l\pi], \mathbb{R}^2),\quad \text{and}\quad \mathscr{C}_\tau := C([-\tau, 0], X).$$

Linearizing system (1.3) at $P = (u_0, v_0)$, we have

$$\dot{U} = \mathbb{D}\Delta U(t) + L(U_t), \tag{2.1}$$

where

$$\mathbb{D}=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},\quad \mathrm{dom}(\mathbb{D}\Delta)=\{(u,v)^T: u,v\in C^2([0,l\pi],\mathbb{R}^2), u_x, v_x=0, x=0,l\pi\},$$

and $L:\mathscr{C}_\tau\mapsto X$ is defined by

$$L(\phi_t)=L_1\phi(0)+L_2\phi(-\tau),$$

for $\phi=(\phi_1,\phi_2)^T\in\mathscr{C}_\tau$ with

$$L_1=\begin{pmatrix} A & B\\ 0 & D\end{pmatrix},\qquad L_2=\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix},$$

$$\phi(t)=(\phi_1(t),\phi_2(t))^T,\qquad \phi(t)(\cdot)=(\phi_1(t+\cdot),\phi_2(t+\cdot))^T.$$

$$A:=\frac{u_0(1-a-2u_0)}{a+u_0},\quad B:=\frac{-bu_0}{(a+u_0)}<0,\quad C:=\frac{acv_0}{(a+u_0)^2}>0,\quad D:=-v_0<0 \tag{2.2}$$

From Wu [12], we obtain that the characteristic equation for liner system (2.1) is

$$\lambda y-d\Delta y-L(e^{\lambda}y)=0,\quad y\in\mathrm{dom}(d\Delta),\quad y\neq 0. \tag{2.3}$$

It is well-known that the eigenvalue problem

$$-\varphi''=\mu\varphi,\quad x\in(0,l\pi);\quad \varphi'(0)=\varphi'(l\pi)=0$$

has eigenvalues $\mu_n=n^2/l^2$ $(n=0,1,\cdots)$ with corresponding eigenfunctions

$$\varphi_n(x)=\cos\frac{n\pi}{l},\quad n=0,1,\cdots.$$

Substituting

$$y=\sum_{n=0}^{\infty}\begin{pmatrix} y_{1n}\\ y_{2n}\end{pmatrix}\cos\frac{n\pi}{l}$$

into the characteristic equation (2.3), it follows that

$$\begin{pmatrix} A-\frac{d_1n^2}{l^2} & B\\ Ce^{-\lambda\tau} & D-\frac{d_2n^2}{l^2}\end{pmatrix}\begin{pmatrix} y_{1n}\\ y_{2n}\end{pmatrix}=\lambda\begin{pmatrix} y_{1n}\\ y_{2n}\end{pmatrix},\ n=0,1,\cdots.$$

Therefore, the characteristic equation (2.3) is equivalent to

$$\Delta_n(\lambda,\tau)=\lambda^2+\lambda A_n+B_n-BCe^{-\lambda\tau}=0, \tag{2.4}$$

where

$$A_n=(d_1+d_2)\frac{n^2}{l^2}-A-D,\qquad B_n=d_2d_2\frac{n^4}{l^4}-(d_1D+d_2A)\frac{n^2}{l^2}+AD.$$

We make the following hypothesis,

$$(\mathbf{H})\quad AD-BC>0,\qquad A+D<0. \tag{2.5}$$

2.1. Non-delay model

When $\tau = 0$, the characteristic reduces to the following equation.

$$\lambda^2 - T_n \lambda + D_n = 0, \quad n \in \mathcal{N}_0, \tag{2.6}$$

where

$$\begin{cases} T_n = A + D - (d_1 + d_2)\frac{n^2}{l^2} \\ D_n = d_1 d_2 \frac{n^4}{l^4} - (d_2 A + d_1 D)\frac{n^2}{l^2} + AD - BC \end{cases}$$

and the eigenvalues are given by

$$\lambda_i^n(r) = \frac{T_n \pm \sqrt{T_n^2 - 4D_n}}{2}, \quad n \in \mathcal{N}_0 \tag{2.7}$$

Define some parameters $q = \frac{d_2 A + d_1 D}{2 d_1 d_2}$, $\Delta = (d_2 A - d_1 D)^2 + 4 d_1 d_2 BC$, and

$$p_{\pm} = \frac{(d_2 A + d_1 D) \pm \sqrt{(d_2 A + d_1 D)^2 - 4 d_1 d_2 (AD - BC)}}{2 d_1 d_2} \tag{2.8}$$

Theorem 2.1. *Suppose $d_1 = d_2 = 0$, $\tau = 0$, and* (**H**) *hold, then the equilibrium (u_0, v_0) is locally asymptotically stable.*

Theorem 2.2. *Suppose $d_1 > 0$, $d_2 > 0$, $\tau = 0$, and* (**H**) *hold. For the model* (1.3), *the following statements are true.*

(i) If $q \leq 0$, then the equilibrium (u_0, v_0) is locally asymptotically stable.

(ii) If $q > 0$, $\Delta < 0$, then the equilibrium (u_0, v_0) is locally asymptotically stable.

(iii) If $q > 0$, $\Delta > 0$ and there is no $k \in \mathcal{N}$ such that $\frac{n^2}{l^2} \in (p_-, p_+)$, then the equilibrium (u_0, v_0) is locally asymptotically stable.

(iv) If $q > 0$, $\Delta > 0$ and there is a $k \in \mathcal{N}$ such that $\frac{n^2}{l^2} \in (p_-, p_+)$, then the equilibrium (u_0, v_0) is Turing unstable.

Proof. By direct calculation, we can obtain $T_n < 0$ and $D_n > 0$ for $q \leq 0$. This means that all eigenvalues have negative real parts. Then, the equilibrium (u_0, v_0) is locally asymptotically stable (statement (i) is true). Similarly, statements (i)-(iii) are also true. If conditions in statement (iv) hold, then there is at least one eigenvalue root with positive real part. Then, the equilibrium (u_0, v_0) is Turing unstable. □

Fix the following parameters

$$d_1 = 0.01, d_2 = 1, a = 0.12, \ b = 1.6, \ c = 0.3, \ \delta = 0.1. \tag{2.9}$$

We choose $P(u_0, v_0) \approx (0.014, 0.13)$, and (**H**) is satisfied. If we choose $a = 0.12$, then $P(u_0, v_0)$ is Turing unstable (shown in Figure 1).

2.2. Delay model

If one of conditions (i-iii) in Theorem 2.2 and (**H**) hold, we can easily verify that $\Delta_n(0, \tau) = B_n - BC = D_n > 0$. Then, the following lemma holds.

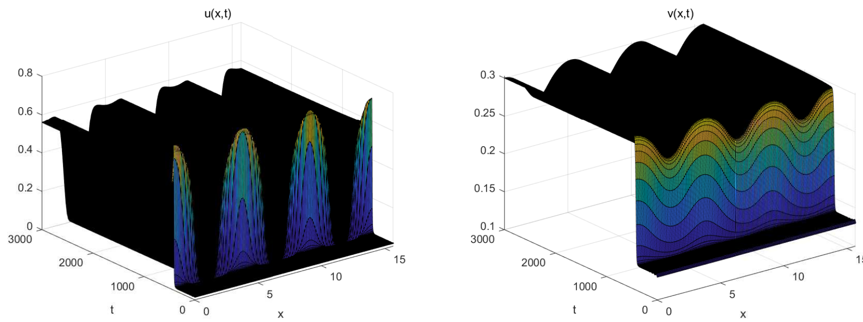

Figure 1. The numerical simulations of system (1.3) with $\tau = 0$ and the initial condition at $(0.014, 0.13)$. Left: component u (Stable). Right: component v (Stable).

Lemma 2.1. *Suppose one of conditions (i-iii) in Theorem 2.2 and* (**H**) *hold, then* $\lambda = 0$ *is not a root of equation* (2.4) *for any* $n \in \mathbb{N}_0$.

Lemma 2.2. *Suppose one of conditions (i-iii) in Theorem 2.2 and* (**H**) *hold, if* $\mathbb{S} \neq \emptyset$, *then* (2.4) *has a pair of purely imaginary roots* $\pm i\omega_n$ $(n \in \mathbb{S})$ *at*

$$\tau_n^j = \tau_n^0 + \frac{2j\pi}{\omega_n}, \ j = 0, 1, 2, \cdots, \tag{2.10}$$

where

$$\tau_n^0 = \frac{1}{\omega_n} \arccos \frac{-\omega_n^2 + B_n}{BC}, \tag{2.11}$$

$$\omega_n = \sqrt{\frac{1}{2}[-(A_n^2 - 2B_n) + \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - B^2C^2)}]}. \tag{2.12}$$

and

$$\mathbb{S} = \{n | d_1 d_2 \frac{n^4}{l^4} - (d_1 D + d_2 A)\frac{n^2}{l^2} + AD + BC < 0, \ n \in \mathbb{N}_0\}. \tag{2.13}$$

Proof. $i\omega$ is a root of (2.4), if and only if ω satisfies

$$-\omega^2 + i\omega A_n + B_n - BC(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Then, we have

$$\begin{cases} -\omega^2 + B_n - BC\cos\omega\tau = 0, \\ \omega A_n + BC\sin\omega\tau = 0, \end{cases}$$

which lead to

$$\omega^4 + \omega^2(A_n^2 - 2B_n) + B_n^2 - B^2C^2 = 0. \tag{2.14}$$

Let $z = \omega^2$, then (2.14) can be rewritten into the following form

$$z^2 + z(A_n^2 - 2B_n) + B_n^2 - B^2C^2 = 0, \tag{2.15}$$

and its roots are given by

$$z_\pm = \frac{1}{2}[-(A_n^2 - 2B_n) \pm \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - B^2C^2)}].$$

If one of conditions (i-iii) in Theorem 2.2 and (**H**) hold, we have

$$A_n^2 - 2B_n = (d_1\frac{n^2}{l^2} - A)^2 + (d_2\frac{n^2}{l^2} - D)^2 > 0,$$

and

$$B_n - BC = D_n > 0.$$

By direct computation, we have

$$B_n + BC = d_1 d_2 \frac{n^4}{l^4} - (d_1 D + d_2 A)\frac{n^2}{l^2} + AD + BC.$$

For $n \in \mathbb{S}$, $B_n + BC < 0$, then Eq (2.15) has a positive root z_+. Based on the discussion above, the statement hold, and $\omega_n = \sqrt{z_+}$.

□

Let $\lambda_n(\tau) = \alpha_n(\tau) + i\omega_n(\tau)$ be the root of (2.4) satisfying $\alpha_n(\tau_n^j) = 0$ and $\omega_n(\tau_n^j) = \omega_n$ when τ is close to τ_n^j. Then, we have the following transversality condition.

Lemma 2.3. *Suppose one of conditions (i-iii) in Theorem 2.2 and* (**H**) *hold. Then,*

$$\alpha_n'(\tau_n^j) = \frac{d\lambda}{d\tau}|_{\tau=\tau_n^j} > 0 \quad \textit{for} \quad n \in \mathbb{S} \quad \textit{and} \quad j \in \mathbb{N}_0.$$

Proof. Differentiating two sides of (2.4) with respect τ, we have

$$(\frac{d\lambda}{d\tau})^{-1} = -\frac{2\lambda + A_n + \tau BCe^{-\lambda\tau}}{\lambda BCe^{-\lambda\tau}}.$$

Then,

$$Re(\frac{d\lambda}{d\tau})^{-1}_{\tau=\tau_n^j} = \frac{A_n^2 - 2B_n + 2\omega_n^2}{B^2C^2} = \frac{\sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - B^2C^2)}}{B^2C^2} > 0.$$

Therefore, $\alpha_n'(\tau_n^j) > 0$. □

Notice that $\tau_m^j = \tau_n^k$, for some $m \neq n$ may occur. In this paper, we do not consider this case. In other words, we consider

$$\tau \in \mathcal{D} := \{\tau_n^j : \ \tau_m^j \neq \tau_n^k, \ m \neq n, \ m, n \in \mathbb{S}, \ j, k \in \mathbb{N}_0\}.$$

Define $\tau_* = \min\{\tau \in \mathcal{D}\}$. According to the above analysis, we have the following theorem.

Theorem 2.3. *For system* (1.3), *suppose one of conditions (i-iii) in Theorem 2.2 and* (**H**) *hold, then the following statements are true.*

(i) If $\mathbb{S} = \emptyset$, then the equilibrium $P(u_0, v_0)$ is locally asymptotically stable for $\tau \geq 0$.

(ii) If $\mathbb{S} \neq \emptyset$, $\tau \in [0, \tau_*)$, then the equilibrium $P(u_0, v_0)$ is locally asymptotically stable, and unstable for $\tau > \tau_*$.

(iii) $\tau = \tau_0^j$ ($j \in \mathbb{N}_0$) are Hopf bifurcation values of system (1.3),

and the bifurcating periodic solutions are spatially homogeneous, which coincide with the periodic solutions of the corresponding FDE system; when $\tau \in \mathcal{D}/\{\tau_0^k : k \in \mathbb{N}_0\}$, system (1.3) *also undergoes a Hopf bifurcation and the bifurcating periodic solutions are spatially non-homogeneous.*

3. Direction and stability of spatial Hopf bifurcation

In this section, we shall study the direction of Hopf bifurcation and stability of the bifurcating periodic solution by applying center manifold theorem and normal form theorem of partial functional differential equations [6, 10, 12]. For fixed $j \in \mathbb{N}_0$ and $n \in \mathbb{S}$, we denote $\tilde{\tau} = \tau_n^j$. Let $\tilde{u}(x,t) = u(x, \tau t) - u_0$ and $\tilde{v}(x,t) = v(x, \tau t) - v_0$. For convenience, we drop the tilde. Then, the system (1.3) can be transformed into

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau[d_1 \Delta u + u + u_0 - (u+u_0)^2 - \dfrac{b(u+u_0)(v+v_0)}{a+(u+u_0)}], \\ \dfrac{\partial v}{\partial t} = \tau[d_2 \Delta v + \delta(v+v_0) - (v+v_0)^2 + \dfrac{c(u(t-1)+u_0)(v+v_0)}{a+(u(t-1)+u_0)}], \end{cases} \tag{3.1}$$

for $x \in (0, l\pi)$, and $t > 0$. Let

$$\tau = \tilde{\tau} + \mu, \quad u_1(t) = u(\cdot, t), \quad u_2(t) = v(\cdot, t) \quad \text{and} \quad U = (u_1, u_2)^T.$$

Then, (3.1) can be rewritten in an abstract form in the phase space $\mathscr{C}_1 := C([-1,0], X)$

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t, \mu), \tag{3.2}$$

where $L_\mu(\phi)$ and $F(\phi, \mu)$ are defined by

$$L_\mu(\phi) = \mu \begin{pmatrix} A\phi_1(0) + B\phi_2(0) \\ C\phi_1(-1) + D\phi_2(0) \end{pmatrix} \tag{3.3}$$

$$F(\phi, \mu) = \mu D \Delta \phi + L_\mu(\phi) + f(\phi, \mu), \tag{3.4}$$

with

$$f(\phi,\mu) = (\tilde{\tau} + \mu)(F_1(\phi,\mu), F_2(\phi,\mu))^T,$$

$$F_1(\phi,\mu) = \phi_1(0) + u_0 - (\phi_1(0)+u_0)^2 - \frac{b(\phi_1(0)+u_0)(\phi_2(0)+v_0)}{a+(\phi_1(0)+u_0)} - A\phi_1(0) - B\phi_2(0),$$

$$F_2(\phi,\mu) = \delta(\phi_2(0)+v_0) - (\phi_2(0)+v_0)^2 + \frac{c(\phi_1(-1)+u_0)(\phi_2(0)+v_0)}{a+(\phi_1(-1)+u_0)^2} - C\phi_1(-1) - D\phi_2(0).$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathscr{C}_1$.

Consider the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t). \tag{3.5}$$

According to the results in Section 2, we know that $\Lambda_n := \{i\omega_n\tilde{\tau}, -i\omega_n\tilde{\tau}\}$ are characteristic values of system (3.5) and the liner functional differential equation

$$\frac{dz(t)}{dt} = -\tilde{\tau} D \frac{n^2}{l^2} z(t) + L_{\tilde{\tau}}(z_t). \tag{3.6}$$

By Riesz representation theorem, there exists a 2×2 matrix function $\eta^n(\sigma, \tilde{\tau})$ $-1 \leq \sigma \leq 0$, whose elements are of bounded variation functions such that

$$-\tilde{\tau} D \frac{n^2}{l^2} \phi(0) + L_{\tilde{\tau}}(\phi) = \int_{-1}^{0} d\eta^n(\sigma, \tau)\phi(\sigma)$$

for $\phi \in C([-1,0],\mathbb{R}^2)$.

In fact, we can choose

$$\eta^n(\sigma,\tau) = \begin{cases} \tau E & \sigma = 0, \\ 0 & \sigma \in (-1,0), \\ -\tau F & \sigma = -1, \end{cases} \tag{3.7}$$

where

$$E = \begin{pmatrix} A - d_1\frac{n^2}{l^2} & B \\ 0 & D - d_2\frac{n^2}{l^2} \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}. \tag{3.8}$$

Let $A(\tilde{\tau})$ denote the infinitesimal generators of semigroup included by the solutions of equation (3.6) and A^* be the formal adjoint of $A(\tilde{\tau})$ under the bilinear paring

$$\begin{aligned} (\psi,\phi) &= \psi(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\sigma} \psi(\xi-\sigma)d\eta^n(\sigma,\tilde{\tau})\phi(\xi)d\xi \\ &= \psi(0)\phi(0) + \tilde{\tau}\int_{-1}^{0} \psi(\xi+1)F\phi(\xi)d\xi. \end{aligned} \tag{3.9}$$

for $\phi \in C([-1,0],\mathbb{R}^2)$, $\psi \in C([-1,0],\mathbb{R}^2)$. $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_n\tilde{\tau}$, and they are also eigenvalues of A^*. Let P and P^* be the center subspace, that is, the generalized eigenspace of $A(\tilde{\tau})$ and A^* associated with Λ_n respectively. Then, P^* is the adjoint space of P and $\dim P = \dim P^* = 2$.

It can be verified that $p_1(\theta) = (1,\xi)^T e^{i\omega_n\tilde{\tau}\sigma}$ $(\sigma \in [-1,0])$, $p_2(\sigma) = \overline{p_1(\sigma)}$ is a basis of $A(\tilde{\tau})$ with Λ_n and $q_1(r) = (1,\eta)e^{-i\omega_n\tilde{\tau}r}$ $(r \in [0,1])$, $q_2(r) = \overline{q_1(r)}$ is a basis of A^* with Λ_n, where

$$\xi = \frac{1}{B}\left(A - \frac{d_1 n^2}{l^2} - i\omega_n\right) = \frac{ce^{-i\tilde{\tau}\omega_n}}{i\omega_n + \frac{d_2 n^2}{l^2} - D}, \eta = \frac{B}{i\omega_n - d_2\frac{n^2}{l^2} + D} = \frac{-i\omega_n - A + \frac{d_1 n^2}{l^2}}{ce^{i\tilde{\tau}\omega_n}}.$$

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with

$$\Phi_1(\sigma) = \frac{p_1(\sigma)+p_2(\sigma)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{i\omega_n\tilde{\tau}\sigma}\right) \\ \mathrm{Re}\left(\xi e^{i\omega_n\tilde{\tau}\sigma}\right) \end{pmatrix} = \begin{pmatrix} \cos(\omega_n\tilde{\tau}\sigma) \\ \frac{1}{B}(A - d_1\frac{n^2}{l^2})\cos\sigma\tilde{\tau}\omega_n + \frac{\omega_n}{B}\sin\sigma\tilde{\tau}\omega_n \end{pmatrix},$$

$$\Phi_2(\sigma) = \frac{p_1(\sigma)-p_2(\sigma)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{i\omega_n\tilde{\tau}\sigma}\right) \\ \mathrm{Im}\left(\xi e^{i\omega_n\tilde{\tau}\sigma}\right) \end{pmatrix} = \begin{pmatrix} \sin(\omega_n\tilde{\tau}\sigma) \\ -\frac{\omega_n}{B}\cos\sigma\tilde{\tau}\omega_n - \frac{1}{B}\left(d_1\frac{n^2}{l^2} - A\right)\sin\sigma\tilde{\tau}\omega_n \end{pmatrix},$$

for $\theta \in [-1,0]$, and

$$\Psi_1^*(r) = \frac{q_1(r)+q_2(r)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{-i\omega_n\tilde{\tau}r}\right) \\ \mathrm{Re}\left(\eta e^{-i\omega_n\tilde{\tau}r}\right) \end{pmatrix} = \begin{pmatrix} \cos(\omega_n\tilde{\tau}r) \\ \frac{1}{B}\cos(\omega_n\tilde{\tau}r)(A - d_1\frac{n^2}{l^2}) - \frac{1}{B}\omega_n\sin r\tilde{\tau}\omega_n \end{pmatrix},$$

$$\Psi_2^*(r) = \frac{q_1(r)-q_2(r)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{-i\omega_n\tilde{\tau}r}\right) \\ \mathrm{Im}\left(\eta e^{-i\omega_n\tilde{\tau}r}\right) \end{pmatrix} = \begin{pmatrix} -\sin(\omega_n\tilde{\tau}r) \\ \frac{\omega_n}{B}\cos r\tilde{\tau}\omega_n + \frac{1}{B}(A - d_1\frac{n^2}{l^2})\sin r\tilde{\tau}\omega_n \end{pmatrix},$$

for $r \in [0,1]$. Then, we can compute by (3.9)

$$D_1^* := (\Psi_1^*, \Phi_1),\ D_2^* := (\Psi_1^*, \Phi_2),\ D_3^* := (\Psi_2^*, \Phi_1),\ D_4^* := (\Psi_2^*, \Phi_2).$$

Define $(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}$ and construct a new basis Ψ for P^* by

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1}\Psi^*.$$

Then, $(\Psi, \Phi) = I_2$. In addition, define $f_n := (\beta_n^1, \beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix} \cos\frac{n}{l}x \\ 0 \end{pmatrix}, \quad \beta_n^2 = \begin{pmatrix} 0 \\ \cos\frac{n}{l}x \end{pmatrix}.$$

We also define

$$c \cdot f_n = c_1\beta_n^1 + c_2\beta_n^2, \quad \text{for} \quad c = (c_1, c_2)^T \in \mathscr{C}_1.$$

Thus, the center subspace of linear equation (3.5) is given by $P_{CN}\mathscr{C}_1 \oplus P_S\mathscr{C}_1$ and $P_S\mathscr{C}_1$ denotes the complement subspace of $P_{CN}\mathscr{C}_1$ in $\mathscr{C}_1$,

$$< u, v >:= \frac{1}{l\pi}\int_0^{l\pi} u_1\overline{v_1}dx + \frac{1}{l\pi}\int_0^{l\pi} u_2\overline{v_2}dx$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u, v \in X$ and $< \phi, f_0 >= (< \phi, f_0^1 >, < \phi, f_0^2 >)^T$. Let $A_{\tilde{\tau}}$ denote the infinitesimal generator of an analytic semigroup induced by the linear system (3.5), and equation (3.1) can be rewritten as the following abstract form

$$\frac{dU(t)}{dt} = A_{\tilde{\tau}}U_t + R(U_t, \mu), \tag{3.10}$$

where

$$R(U_t, \mu) = \begin{cases} 0, & \theta \in [-1, 0); \\ F(U_t, \mu), & \theta = 0. \end{cases} \tag{3.11}$$

By the decomposition of $\mathscr{C}_1$, the solution above can be written as

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n + h(x_1, x_2, \mu), \tag{3.12}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, < U_t, f_n >),$$

and

$$h(x_1, x_2, \mu) \in P_S\mathscr{C}_1, \quad h(0,0,0) = 0, \quad Dh(0,0,0) = 0.$$

In particular, the solution of (3.2) on the center manifold is given by

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, 0). \tag{3.13}$$

Let $z = x_1 - ix_2$, and notice that $p_1 = \Phi_1 + i\Phi_2$. Then we have

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z+\overline{z}}{2} \\ \frac{i(z-\overline{z})}{2} \end{pmatrix} f_n = \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n,$$

and

$$h(x_1, x_2, 0) = h(\frac{z+\overline{z}}{2}, \frac{i(z-\overline{z})}{2}, 0).$$

Hence, equation (3.13) can be transformed into

$$\begin{aligned} U_t &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + h(\frac{z+\overline{z}}{2}, \frac{i(z-\overline{z})}{2}, 0) \\ &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + W(z, \overline{z}), \end{aligned} \tag{3.14}$$

where

$$W(z, \overline{z}) = h(\frac{z+\overline{z}}{2}, \frac{i(z-\overline{z})}{2}, 0).$$

From [12], z satisfies

$$\dot{z} = i\omega_n \tilde{\tau} z + g(z, \overline{z}), \tag{3.15}$$

where

$$g(z, \overline{z}) = (\Psi_1(0) - i\Psi_2(0)) < F(U_t, 0), f_n > . \tag{3.16}$$

Let

$$W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11} z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots , \tag{3.17}$$

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots , \tag{3.18}$$

from equation (3.14) and (3.17), we have

$$u_t(0) = \frac{1}{2}(z + \overline{z}) \cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots ,$$

$$v_t(0) = \frac{1}{2}(\xi + \overline{\xi}\overline{z}) \cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots ,$$

$$u_t(-1) = \frac{1}{2}(ze^{-i\omega_n\tilde{\tau}} + \overline{z}e^{i\omega_n\tilde{\tau}}) \cos(\frac{nx}{l}) + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots ,$$

and

$$\begin{aligned} \overline{F}_1(U_t, 0) &= \frac{1}{\tilde{\tau}} F_1 = \frac{1}{2} f_{uu} u_t^2(0) + f_{uv} u_t(0) v_t(0) + \frac{1}{2} f_{vv} v_t^2(0) \\ &+ \frac{1}{6} f_{uuu} u_t^3(0) + \frac{1}{2} f_{uuv} u_t^2(0) v_t(0) + \frac{1}{2} f_{uvv} u_t(0) v_t^2(0) + \frac{1}{6} f_{vvv} v_t^3(0) + O(4), \end{aligned} \tag{3.19}$$

$$\overline{F}_2(U_t,0)=\frac{1}{\tilde{\tau}}F_2=\frac{1}{2}g_{uu}u_t^2(-1)+g_{uv}u_t(-1)v_t(0)+\frac{1}{2}g_{vv}v_t^2(0)$$
$$+\frac{1}{6}g_{uuu}u_t^3(-1)+\frac{1}{6}g_{uuv}u_t^2(-1)v_t(0)+\frac{1}{6}g_{uvv}u_t(-1)v_t^2(0)+\frac{1}{6}g_{vvv}v_t^3(0) \quad (3.20)$$
$$+O(4),$$

with

$$f_{uu}=-2+\frac{2abv_0}{(a+u_0)^3},\quad f_{uv}=-\frac{ba}{(a+u_0)^3},\quad g_{uu}=\frac{2cu_0v_0}{(a+u_0)^3},\quad g_{uv}=-\frac{cu_0}{(a+u_0)^2},$$
$$f_{uuu}=-\frac{6abv_0}{(a+u_0)^4},\quad f_{uuv}=\frac{2ba}{(a+u_0)^3},\quad g_{vv}=-2$$
$$f_{vv}=f_{uvv}=f_{vvv}=g_{uvv}=g_{vvv}=0.$$

Hence,

$$\overline{F}_1(U_t,0)=\cos^2(\frac{nx}{l})(\frac{z^2}{2}\chi_{20}+z\overline{z}\chi_{11}+\frac{\overline{z}^2}{2}\overline{\chi}_{20})+\frac{z^2\overline{z}}{2}\cos\frac{nx}{l}\varsigma_{11}+\frac{z^2\overline{z}}{2}\cos^3\frac{nx}{l}\varsigma_{12} \quad (3.21)$$
$$+\cdots,$$

$$\overline{F}_2(U_t,0)=\cos^2(\frac{nx}{l})(\frac{z^2}{2}\varrho_{20}+z\overline{z}\varrho_{11}+\frac{\overline{z}^2}{2}\overline{\varrho}_{20})+\frac{z^2\overline{z}}{2}\cos\frac{nx}{l}\varsigma_{21}+\frac{z^2\overline{z}}{2}\cos^3\frac{nx}{l}\varsigma_{22} \quad (3.22)$$
$$+\cdots,$$

$$< F(U_t,0),f_n >=\tilde{\tau}(\overline{F}_1(U_t,0)f_n^1+\overline{F}_2(U_t,0)f_n^2)$$
$$=\frac{z^2}{2}\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\Gamma+z\overline{z}\tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varsigma_{11}\end{pmatrix}\Gamma+\frac{\overline{z}^2}{2}\tilde{\tau}\begin{pmatrix}\overline{\chi}_{20}\\ \overline{\varsigma}_{20}\end{pmatrix}\Gamma+\frac{z^2\overline{z}}{2}\tilde{\tau}\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix}$$
$$+\cdots. \quad (3.23)$$

with

$$\Gamma=\frac{1}{l\pi}\int_0^{l\pi}\cos^3(\frac{nx}{l})dx,$$

$$\chi_{20}=\frac{1}{4}\left(f_{uu}+2\xi f_{uv}\right),\quad \chi_{11}=\frac{1}{4}(f_{uu}+f_{uv}(\xi+\bar{\xi}))$$
$$\varsigma_{11}=\frac{1}{2}(f_{uu}(2W_{11}^1(0)+W_{20}^1(0))+f_{uv}(2\xi W_{11}^1(0)+2W_{11}^2(0)+\bar{\xi}W_{20}^1(0)+W_{20}^2(0)))$$
$$\varsigma_{12}=\frac{1}{8}(f_{uuu}+f_{uuv}(\bar{\xi}+2\xi))$$
$$\varrho_{20}=\frac{1}{4}e^{-2i\tilde{\tau}\omega_n}\left(g_{uu}+e^{i\tilde{\tau}\omega_n}\xi 2g_{uv}+g_{vv}\xi e^{i\tilde{\tau}\omega_n}\right),\quad \varrho_{11}=\frac{1}{4}g_{uu}+\frac{1}{4}e^{-i\tilde{\tau}\omega_n}\bar{\xi}g_{uv}+$$
$$\frac{1}{4}e^{i\tilde{\tau}\omega_n}\xi g_{uv}$$
$$\varsigma_{21}=W_{11}^1(-1)\left(g_{uv}\xi+g_{uu}e^{-i\tilde{\tau}\omega_n}\right)+\frac{1}{2}W_{20}^1(-1)\left(\bar{\xi}g_{uv}+g_{uu}e^{i\tau\omega_n}\right)$$
$$+W_{11}^2(0)\left(g_{vv}\xi+g_{uv}e^{-i\tilde{\tau}\omega_n}\right)+\frac{1}{2}W_{20}^2(0)\left(\bar{\xi}g_{vv}+g_{uv}e^{i\tilde{\tau}\omega_n}\right)$$
$$\varsigma_{22}=\frac{1}{8}e^{-2i\tilde{\tau}\omega_n}\left(g_{uuu}e^{i\tilde{\tau}\omega_n}+\bar{\xi}g_{uuv}+2g_{uuv}\xi e^{2i\tilde{\tau}\omega_n}\right). \quad (3.24)$$

$$\kappa_1=\varsigma_{11}\frac{1}{l\pi}\int_0^{l\pi}\cos^2(\frac{nx}{l})dx+\varsigma_{12}\frac{1}{l\pi}\int_0^{l\pi}\cos^4(\frac{nx}{l})dx,$$

$$\kappa_2 = \varsigma_{21}\frac{1}{l\pi}\int_0^{l\pi}\cos^2(\frac{nx}{l})dx + \varsigma_{22}\frac{1}{l\pi}\int_0^{l\pi}\cos^4(\frac{nx}{l})dx$$

Denote

$$\Psi_1(0) - i\Psi_2(0) := (\gamma_1 \quad \gamma_2).$$

Notice that

$$\frac{1}{l\pi}\int_0^{l\pi}\cos^3(\frac{nx}{l})dx = 0, \quad n = 1, 2, 3, \cdots,$$

and we have

$$\begin{aligned}(\Psi_1(0) - i\Psi_2(0)) &< F(U_t, 0), f_n >= \\ &\frac{z^2}{2}(\gamma_1\chi_{20} + \gamma_2\varsigma_{20})\Gamma\tilde{\tau} + z\overline{z}(\gamma_1\chi_{11} + \gamma_2\varsigma_{11})\Gamma\tilde{\tau} + \frac{\overline{z}^2}{2}(\gamma_1\overline{\chi}_{20} + \gamma_2\overline{\varsigma}_{20})\Gamma\tilde{\tau} \\ &+ \frac{z^2\overline{z}}{2}\tilde{\tau}[\gamma_1\kappa_1 + \gamma_2\kappa_2] + \cdots,\end{aligned} \tag{3.25}$$

Then, by (3.16), (3.18) and (3.25), we have $g_{20} = g_{11} = g_{02} = 0$, for $n = 1, 2, 3, \cdots$. If $n = 0$, we have the following quantities:

$$g_{20} = \gamma_1\tilde{\tau}\chi_{20} + \gamma_2\tilde{\tau}\varrho_{20}, \qquad g_{11} = \gamma_1\tilde{\tau}\chi_{11} + \gamma_2\tilde{\tau}\varrho_{11}, \qquad g_{02} = \gamma_1\tilde{\tau}\overline{\chi}_{20} + \gamma_2\tilde{\tau}\overline{\varrho}_{20},$$

and for $n \in \mathbb{N}_0$, $g_{21} = \tilde{\tau}(\gamma_1\kappa_1 + \gamma_2\kappa_2)$. Now, a complete description for g_{21} depends on the algorithm for $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0]$ which we shall compute. From [12], we have

$$\dot{W}(z, \overline{z}) = W_{20}z\dot{z} + W_{11}\dot{z}\overline{z} + W_{11}z\dot{\overline{z}} + W_{02}\overline{z}\dot{\overline{z}} + \cdots,$$

$$A_{\tilde{\tau}}W(z, \overline{z}) = A_{\tilde{\tau}}W_{20}\frac{z^2}{2} + A_{\tilde{\tau}}W_{11}z\overline{z} + A_{\tilde{\tau}}W_{02}\frac{\overline{z}^2}{2} + \cdots,$$

and $W(z, \overline{z})$ satisfies

$$\dot{W}(z, \overline{z}) = A_{\tilde{\tau}}W + H(z, \overline{z}),$$

where

$$\begin{aligned}H(z, \overline{z}) &= H_{20}\frac{z^2}{2} + W_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots \\ &= X_0F(U_t, 0) - \Phi(\Psi, < X_0F(U_t, 0), f_n > \cdot f_n).\end{aligned} \tag{3.26}$$

Hence, we have

$$(2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{20} = H_{20}, \quad -A_{\tilde{\tau}}W_{11} = H_{11}, \quad (-2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{02} = H_{02}, \tag{3.27}$$

that is

$$W_{20} = (2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{20}, \quad W_{11} = -A_{\tilde{\tau}}^{-1}H_{11}, \quad W_{02} = (-2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{02}. \tag{3.28}$$

By (3.25), we have that for $\theta \in [-1, 0)$,

$$\begin{aligned}H(z,\overline{z}) &= -\Phi(0)\Psi(0) < F(U_t, 0), f_n > \cdot f_n \\ &= -(\frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2i})\begin{pmatrix}\Phi_1(0) \\ \Phi_2(0)\end{pmatrix} < F(U_t, 0), f_n > \cdot f_n \\ &= -\frac{1}{2}[p_1(\theta)(\Phi_1(0) - i\Phi_2(0)) + p_2(\theta)(\Phi_1(0) + i\Phi_2(0))] < F(U_t, 0), f_n > \cdot f_n \\ &= -\frac{1}{2}[(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02})\frac{z^2}{2} + (p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11})z\overline{z} + (p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20})\frac{\overline{z}^2}{2}] \\ &+ \cdots.\end{aligned}$$

Therefore, by (3.26), for $\theta \in [-1, 0)$,

$$H_{20}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}) \cdot f_0 & n = 0, \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_0 & n = 0, \end{cases}$$

$$H_{02}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20}) \cdot f_0 & n = 0, \end{cases}$$

and

$$H(z, \overline{z})(0) = F(U_t, 0) - \Phi(\Psi, < F(U_t, 0), f_n >) \cdot f_n,$$

where

$$H_{20}(0) = \begin{cases} \tilde{\tau} \begin{pmatrix} \chi_{20} \\ \varrho_{20} \end{pmatrix} \cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau} \begin{pmatrix} \chi_{20} \\ \varrho_{20} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\overline{g}_{02}) \cdot f_0, & n = 0. \end{cases} \tag{3.29}$$

$$H_{11}(0) = \begin{cases} \tilde{\tau} \begin{pmatrix} \chi_{11} \\ \varrho_{11} \end{pmatrix} \cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau} \begin{pmatrix} \chi_{11} \\ \varrho_{11} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{11} + p_2(0)\overline{g}_{11}) \cdot f_0, & n = 0. \end{cases} \tag{3.30}$$

By the definition of $A_{\tilde{\tau}}$ and (3.27), we have

$$\dot{W}_{20} = A_{\tilde{\tau}} W_{20} = 2i\omega_n \tilde{\tau} W_{20} + \frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}) \cdot f_n, \quad -1 \le \theta < 0.$$

That is,

$$W_{20}(\theta) = \frac{i}{2i\omega_n \tilde{\tau}}(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)) \cdot f_n + E_1 e^{2i\omega_n \tilde{\tau}\theta},$$

where

$$E_1 = \begin{cases} W_{20}(0) & n = 1, 2, 3, \cdots, \\ W_{20}(0) - \frac{i}{2i\omega_n \tilde{\tau}}(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)) \cdot f_0 & n = 0. \end{cases}$$

Using the definition of $A_{\tilde{\tau}}$ and (3.27), we have that for $-1 \le \theta < 0$

$$\begin{aligned} & - (g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)) \cdot f_0 + 2i\omega_n \tilde{\tau} E_1 - A_{\tilde{\tau}}(\frac{i}{2\omega_n \tilde{\tau}}(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)) \cdot f_0) \\ & - A_{\tilde{\tau}} E_1 - L_{\tilde{\tau}}(\frac{i}{2\omega_n \tilde{\tau}}(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)) \cdot f_n + E_1 e^{2i\omega_n \tilde{\tau}\theta}) \\ & = \tilde{\tau} \begin{pmatrix} \chi_{20} \\ \varrho_{20} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\overline{g}_{02}) \cdot f_0. \end{aligned}$$

As

$$A_{\tilde{\tau}}p_1(0) + L_{\tilde{\tau}}(p_1 \cdot f_0) = i\omega_0 p_1(0) \cdot f_0,$$

and

$$A_{\tilde{\tau}}p_2(0) + L_{\tilde{\tau}}(p_2 \cdot f_0) = -i\omega_0 p_2(0) \cdot f_0,$$

we have

$$2i\omega_n E_1 - A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}E_1 e^{2i\omega_n} = \tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varrho_{20}\end{pmatrix}\cos^2(\frac{nx}{l}), \quad n = 0, 1, 2, \cdots.$$

That is,

$$E_1 = \tilde{\tau}E\begin{pmatrix}\chi_{20}\\ \varrho_{20}\end{pmatrix}\cos^2(\frac{nx}{l})$$

where

$$E = \begin{pmatrix} 2i\omega_n\tilde{\tau} + d_1\frac{n^2}{l^2} - A & -B \\ -Ce^{-2i\omega_n\tilde{\tau}} & -D + 2i\omega_n\tilde{\tau} + d_2\frac{n^2}{l^2}\end{pmatrix}^{-1}.$$

Similarly, from (3.28), we have

$$-\dot{W}_{11} = \frac{i}{2\omega_n\tilde{\tau}}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_n, \quad -1 \le \theta < 0.$$

That is,

$$W_{11}(\theta) = \frac{i}{2i\omega_n\tilde{\tau}}(p_1(\theta)\overline{g}_{11} - p_1(\theta)g_{11}) + E_2.$$

Similar to the procedure of computing W_{20}, we have

$$E_2 = \tilde{\tau}E^*\begin{pmatrix}\chi_{11}\\ \varrho_{11}\end{pmatrix}\cos^2(\frac{nx}{l}),$$

where

$$E^* = \begin{pmatrix} d_1\frac{n^2}{l^2} - A & -B \\ -C & -D + d_2\frac{n^2}{l^2}\end{pmatrix}^{-1}.$$

Thus, we can compute the following quantities which determine the direction and stability of bifurcating periodic orbits:

$$\begin{cases} c_1(0) = \frac{i}{2\omega_n\tilde{\tau}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{1}{2}g_{21}, \ \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_n^j))}, \\ T_2 = -\frac{1}{\omega_n\tilde{\tau}}[Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_n^j))], \qquad \beta_2 = 2Re(c_1(0)). \end{cases} \tag{3.31}$$

Then, we have the following theorem.

Theorem 3.1. *For any critical value τ_n^j, we have*

(i) μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ (respectively <0), then the Hopf bifurcation is forward (respectively backward). That is, the bifurcating periodic solutions exists for $\tau > \tau_n^j$ (respectively $\tau < \tau_n^j$) ;

(ii) β_2 *determines the stability of the bifurcating periodic solutions on the center manifold: if* $\beta_2 < 0$ *(respectively >0), then the bifurcating periodic solutions are orbitally asymptotically stable (respectively unstable).*

(iii) T_2 *determines the period of bifurcating periodic solutions: if* $T_2 > 0$ *(respectively* $T_2 < 0$*), then the period increases (respectively decreases).*

4. Numerical simulations

In this section, to illustrate the results found in the previous sections, some examples and numerical results are presented. We use Matlab to simulate and plot numerical graphs. For the system (1.3), we choose parameters:

$$d_1 = 2,\ d_2 = 2,\ a = 0.2,\ b = 1.6,\ c = 0.3,\ \delta = 0.1. \tag{4.1}$$

By direct computation, we have $u_0 \approx 0.03$, $v_0 \approx 0.14$. Hence, (**H**) holds. From (2.11) and (2.12), we have $\tau_* = \tau_0^0 \approx 1.2769$ and $\omega_0 \approx 2.14$. By Theorem 2.1 (i), we know that if $\tau \in [0, \tau_*)$, then the equilibrium $P(u_0, v_0)$ is locally asymptotically stable. This is shown in Figure 2, where we choose $\tau = 2$ and the initial condition at $(0.03, 0.14)$. By Theorem 2.1 (iii), we conclude that the equilibrium $P(u_0, v_0)$ loses its stability and Hopf bifurcation occurs when τ crosses τ_0^0. By Theorem 3.2,

$$\mu_2 \approx 0.0219829 > 0, \quad \beta_2 \approx -137.48 < 0, \quad \text{and} \quad T_2 \approx 47.0227 > 0.$$

Hence, the direction of the bifurcation is forward, and the bifurcating period solutions are locally asymptotically stable. In addition, the period of bifurcating periodic solutions increase. This is shown in Figure 3, where we choose $\tau = 2$ and the initial condition at $(0.03, 0.14)$.

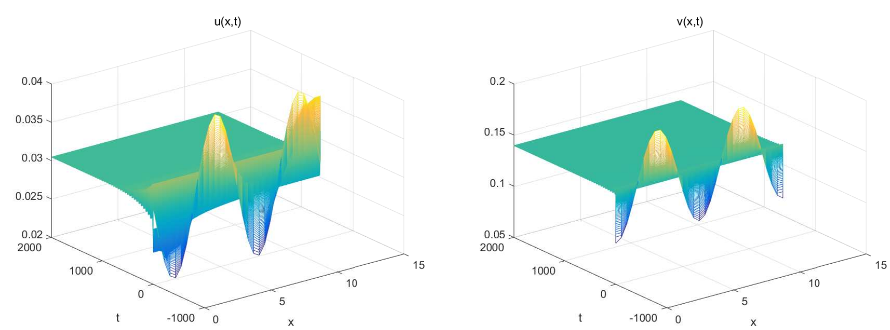

Figure 2. The numerical simulations of system (1.3) with $\tau = 1$, and the initial condition at $(0.03 - 0.01\sin(x), 0.14 - 0.05\cos(x))$. Left: component u (Locally asymptotically stable). Right: component v (Locally asymptotically stable).

5. Conclusion

In this paper, we study a host-generalist parasitoid model with diffusion term and time delay. We mainly analyze the diffusion induced Turing instability, and time delay induced Hopf bifurcation. Under the theory of center manifold and normal

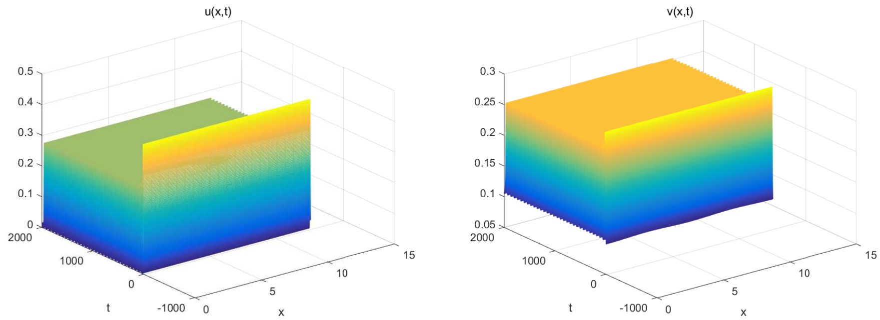

Figure 3. The numerical simulations of system (1.3) with $\tau = 2$, and the initial condition at $(0.03 - 0.001\sin(x), 0.1 - 0.001\cos(x))$. Left: component u (Stable). Right: component v (Stable).

form method, we give some parameters to determine the bifurcation direction and the stability of the bifurcating periodic solution. Our results suggest that diffusion and time delay are two important factors in the host-generalist parasitoid model. Diffusion may induce Turing instability and the non-homogeneous bifurcating periodic solutions. The hosts and generalist parasitoid will coexist in the form of periodic oscillations when time delay larger than the critical value.

References

[1] L. Cai, G. Chen and D. Xiao, *Multiparametric bifurcations of an epidemiological model with strong Allee effect*, Journal of Mathematical Biology, 2013, 67(2), 185–215.

[2] X. Chang and J. Wei, *Stability and Hopf bifurcation in a diffusuve predator-prey system incorporating a prey refuge*, Mathematical Biosciences and Engineering, 2013, 10, 979–996.

[3] M. Christelle, C. Chris, R. Shigui and J. Casas, *Control of invasive hosts by generalist parasitoids*, Mathematical Medicine and Biology, 2008, 25, 1–20.

[4] R. M. Etoua and C. Rousseau, *Bifurcation analysis of a generalized Gause model with prey harvesting and a generalized Holling response function of type III*, Journal of Differential Equations, 2010, 249(9), 2316–2356.

[5] M. Gilbert, A. Svato, M. Lehman and S. Bacher, *Spatial patterns and infestation processes in the horse chestnut leafminer Cameraria ohridella: a tale of two cities*, Entomologia Experimentalis Et Applicata, 2010, 107(1), 2797–2813.

[6] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.

[7] X. Li, W. Jiang and J. Shi, *Hopf bifurcation and Turing instability in the reaction-dffiusion Holling-Tanner predator-prey model*, IMA Journal of Applied Mathematics, 2013, 78, 287–306.

[8] M. R. Owen and M. A. Lewis, *How predation can slow, stop or reverse a prey invasion*, Bulletin of Mathematical Biology, 2001, 63(4), 655–684.

[9] S. Rana, *Bifurcation and complex dynamics of a discrete-time predator-prey system with simplified Monod-Haldane functional response*, Advances in Difference Equations, 2015, 1, 1–17.

[10] S. Ruan and J. Wei, *On the zeros of transcendental functions with applications to stability of delay differential equations with two delays*, Dynamics of Continuous, Discrete and Impulsive Systems. Series A. Mathematical Analysis, 2003, 10, 863–874.

[11] J. Wang, J. Shi and J. Wei, *Predator-prey system with strong Allee efect in prey*, Journal of Mathematical Biology, 2011, 62, 291–331.

[12] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Springer, New York, 1996.

[13] C. Xiang, J. Huang, S. Ruan and D. Xiao, *Bifurcation analysis in a host-generalist parasitoid model with Holling II functional response*, Journal of Differential Equations, 2019, 268(8), 618–4662.

[14] X. Yan and C. Zhang, *Stability and turing instability in a diffusive predator-prey system with Beddington-DeAngelis functional response*, Nonlinear Analysis: Real World Applications, 2014, 20, 1–13.

[15] R. Yang and J. Wei, *Stability and bifurcation analysis of a diffusive prey-predator system in Holling type III with a prey refuge*, Nonlinear Dynamics, 2015, 7, 631–646.

[16] F. Yi, J. Wei and J. Shi, *Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system*, Journal of Differential Equations, 2009, 246(5), 1944–1977.

[17] R. Yuan, W. Jiang and Y.Wang, *Saddle-node-Hopf bifurcation in a modified Leslie-Gower predator-prey model with time-delay and prey harvesting*, Journal of Mathematical Analysis and Applications, 2015, 422, 1072–1090.

[18] L. Zhou, Y. Tang and S. Hussein, *Stability and Hopf bifurcation for a delay competitive diffusion system*, Chaos, Solitons & Fractals, 2008, 37, 87–99.