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Traveling Wave Solutions in an Integrodifference Equation with Weak Compactness*

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Abstract This article studies the existence of traveling wave solutions in an integrodifference equation with weak compactness. Because of the special kernel function that may depend on the Dirac function, traveling wave maps have lower regularity such that it is difficult to directly look for a traveling wave solution in the uniformly continuous and bounded functional space. In this paper, by introducing a proper set of potential wave profiles, we can obtain the existence and precise asymptotic behavior of nontrivial traveling wave solutions, during which we do not require the monotonicity of this model.

Keywords Generalized upper and lower solutions, Traveling wave map, Minimal wave speed, Decay behavior.

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1. Introduction

When some species with non-overlapping generations are concerned, their spatial dispersal and growth often occur at different stages of the species, and many plants have the feature. In population dynamics, Mollison [14] and Weinberger [16] proposed some discrete time models equipping with spatial variables to describe these phenomena, which are integrodifference equations [11]. One typical integrodifference equation takes the following iterative form

$$w_{n+1}(x) = \int_{\mathbb{R}} b(w_n(y))k(x-y)dy,\ x \in \mathbb{R},\ n = 0, 1, 2, \cdots . \tag{1.1}$$

Regarding (1.1) as a model in population dynamics, then $w_n(x)$ often denotes the density of the species at location x of the nth generation, b is the birth function while k reflects the spatial movement law and may be a probability distribution that is also the so-called kernel function. Since Weinberger [16], the traveling wave solutions of (1.1) have been widely studied, see some results by Bourgeois et al. [1], Fang and Zhao [2], Hsu and Zhao [3], Kot [5], Li et al. [7], Liang and Zhao [9], Wang and Castillo-Chavez [15], Weinberger et al. [17], Yi et al. [18]. Here, a traveling wave

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solution of (1.1) is a special solution taking the form

$$w_n(x) = \varphi(t),\ t = x + cn \in \mathbb{R},$$

in which φ is the wave profile that propagates in the spatial media at the constant speed c, and $t = x + cn$ is the traveling wave coordinate. That is, a traveling wave solution must satisfy

$$\varphi(t) = \int_{\mathbb{R}} b(\varphi(y))k(t - c - y)dy := B(\varphi)(t),\ t \in \mathbb{R}.$$

Clearly, a traveling wave solution is a fixed point of B in proper functional space.

When b is continuous and k is Lebesgue measurable and integrable, then we see that the traveling wave map $B(\varphi)(t)$ is equicontinuous if $\varphi(t)$ belongs to proper bounded continuous functional set. Based on such a property of the map B, it is possible to obtain necessary smoothness in continuous functional space, which implies that fixed point theorem can be applied to study the existence of traveling wave solutions [3,10]. With the help of fixed point theorem, the existence of traveling wave solutions may be obtained by the existence of proper generalized upper and lower solutions [3, 10]. Of course, some other methods were also utilized to study the existence of nonconstant traveling wave solutions in [7, 9, 15, 17, 18], in which proper smoothness or monotone conditions of traveling wave maps are necessary.

However, in many examples, the kernel function is not Lebesgue measurable and integrable such that the smoothness of $B(\varphi)(t)$ encounters difficulty. In particular, considering a nondispersing (sessile) component, Lutscher [11, Section 12.4] proposed and studied some mathematical models, in which the kernel function may depend on the Dirac function. One example in [11, Section 12.4] and Li [6] takes the following form

$$u_{n+1}(x) = ru_n(x) + \int_{\mathbb{R}} f(u_n(y))k(x - y)dy,\ x \in \mathbb{R},\ n = 0, 1, 2, \cdots, \tag{1.2}$$

in which $r > 0$ is a constant formulating the dormant behavior of seeds, f is a function reflecting the newborn viable seeds, k is a probability distribution denoting the dispersal of seeds. Consider the traveling wave map

$$F_1(\varphi)(t) = r\varphi(t - c) + \int_{\mathbb{R}} f(\varphi(y))k(t - c - y)dy,$$

then we see that $F_1(\varphi)(t)$ may be not equicontinuous if $\varphi(t)$ belongs to proper bounded continuous functional set. Due to the deficiency of higher regularity of F_1, we can not directly utilize fixed point theorem as that in [3, 10].

With the help of propagation theory of monotone semiflows [9, 17], Pan et al. [13] obtained the minimal wave speed of traveling wave solutions in (1.2) if f is monotone. Recently, Pan [12] has studied the existence and the asymptotic behavior of traveling wave solutions of (1.2) by constructing proper wave profile set. More precisely, the author first introduced a set Y by a pair of upper and lower solutions of wave equation of (1.2), in which upper and lower solutions are given by the conclusion in [13]. Based on such a set Y, they studied the set

$$\overline{Y} = \bigcap_{n>0} \overline{Co(F_1^n[Y])},$$

and obtained some properties of $\overline{Y}$ (see [12, Lemma 4.4]). With the help of fixed point theorem, the existence of fixed points in $F_1(\varphi)(t) = \varphi(t)$ is confirmed in $\overline{Y}$, which implies the existence of traveling wave solutions of (1.2). Moreover, in an earlier paper, Li [6] also studied the spreading speed and traveling wave solutions of (1.2).

The first aim of this paper is to study a more general model than (1.2). Our model takes the following form

$$u_{n+1}(x) = g(u_n(x)) + \int_{\mathbb{R}} f(u_n(y))k(x-y)dy,\ x \in \mathbb{R},\ n = 0, 1, 2, \cdots, \tag{1.3}$$

in which $g : \mathbb{R} \to \mathbb{R}$ is a continuous function clarified later. Evidently, (1.2) is a special case of (1.3), and we refer to Lutscher [11, Section 12.4]. Similar to that in (1.2), the corresponding traveling wave map is

$$F(\varphi)(t) = g(\varphi(t-c)) + \int_{\mathbb{R}} f(\varphi(y))k(t-c-y)dy.$$

Therefore, a fixed point of $F(\varphi)(t) = \varphi(t)$ is a traveling wave solution of (1.3), of which the existence will be studied in what follows. Our second aim is to directly show possible properties of traveling wave solutions by constructing upper and lower solutions that ensures the necessary regularity of the traveling wave mapping. For the purpose, we construct explicit upper and lower solutions that are exponentially decay, which coincides with the limit behavior in [6, Theorem 3.1] and [12, Theorem 2.1]. By adding the Lipschitz continuity of potential wave profiles, we find that F admits nice features such that we can apply the fixed point theorem. For both monotone and nonmotonotone g, f, we obtain the existence of traveling wave solutions of (1.2) by constructing upper and lower solutions, which provides an intuitionistic illustration of the results in [12, 13].

2. Main results

In this work, $C(\mathbb{R}, \mathbb{R})$ is the space of all uniformly continuous and bounded functions equipped with supremum norm. When $b > a$, we also denote

$$C_{[a,b]} = \{u \in C : a \le u(x) \le b, x \in \mathbb{R}\}.$$

That is, we study the existence of

$$F(\varphi)(t) = \varphi(t) \in C. \tag{2.1}$$

For the purpose, we introduce the following assumptions on (1.2).

(A1) $f(0) = g(0) = 0$, and there exists a constant $E > 0$ such that $g(u), f(u), g(u) + f(u) : [0, E] \to [0, E]$ are differentiable; there exists $r \in [0, 1)$ such that $g'(0) = r, f'(0) > 1 - r$ and

$$|g'(u)| \le r,\ \ \max\{0, g'(0)u - L_1 u^{1+\alpha}\} \le g(u) \le ru,$$
$$\max\{0, f'(0)u - L_1 u^{1+\alpha}\} < f(u) \le f'(0)u,\ \ u \in (0, E]$$

for some $L_1 > 0, \alpha \in (0, 1]$;

(A2) k is nonnegative, even, Lebesgue measurable and integrable such that $\int_{\mathbb{R}} k(x) dx = 1$; there exists $L_2 > 0$ such that $\int_{\mathbb{R}} |k(x-t) - k(y-t)|dt \le L_2|x-y|$ for any $x, y \in \mathbb{R}$ and there exists $\lambda > 0$ such that $\int_{\mathbb{R}} k(x)e^{\lambda x}dx < +\infty$.

Evidently, this model may be nonmonotone by (A1). Define

$$c^* = \inf_{\lambda > 0} \frac{\ln(r + f'(0)\int_{\mathbb{R}} e^{\lambda y}k(y)dy)}{\lambda}.$$

Then, $c^* < \infty$ such that the following fact holds.

Lemma 2.1. *Let*

$$\Lambda(\lambda, c) = re^{-\lambda c} + f'(0)\int_{\mathbb{R}} e^{\lambda y - \lambda c}k(y)dy = re^{-\lambda c} + f'(0)\int_{\mathbb{R}} e^{-\lambda y - \lambda c}k(y)dy.$$

If $c > c^*$, then $\Lambda(\lambda, c) = 1$ has two real roots $0 < \lambda_1^c < \lambda_2^c < \infty$ such that $\Lambda(\lambda, c) < 1$ with $\lambda \in (\lambda_1^c, \lambda_2^c)$. If $c = c^*$, then $\Lambda(\lambda, c) = 1$ has a unique positive root λ' such that

$$-cre^{-\lambda c} + f'(0)\int_{\mathbb{R}}(y - c)e^{\lambda y - \lambda c}k(y)dy = 0, \lambda = \lambda'$$

and

$$-cre^{-\lambda c} - f'(0)\int_{\mathbb{R}}(y + c)e^{-\lambda(y+c)}k(y)dy = 0, \lambda = \lambda'.$$

Proof. By direct calculation, we see that for each fixed $c > 0$, $\Lambda(\lambda, c)$ is bounded in a bounded interval or $\lambda \ge 0$, then $\Lambda(\lambda, c)$ is strictly convex. Moreover, if λ is fixed such that $\Lambda(\lambda, c)$ is bounded, then $\Lambda(\lambda, c)$ is strictly decreasing in $c > 0$. Since $\Lambda(0, 0) > 1$, and if $\Lambda(\lambda, 0)$ is bounded, then $\lim_{c\to\infty} \Lambda(\lambda, c) = 0$, and then the conclusion is true. The proof is complete. □

By these constants, we now state the main results.

Theorem 2.1. *Assume that $c > c^*$ holds. Then, there exists $\varphi(t) \in C_{[0,E]}$ satisfying (2.1) and the following properties:*

(1) $\varphi(t) > 0, t \in \mathbb{R}$ *is Lipschitz continuous;*

(2) $\lim_{t\to-\infty}[\varphi(t)e^{-\lambda_1^c t}] > 0, \lim_{t\to-\infty} \varphi(t) = 0, \liminf_{t\to\infty} \varphi(t) > 0;$

(3) *let $E_1 \in (0, E]$ such that $f(u) + g(u) > u, u \in (0, E_1), f(E_1) + g(E_1) = E_1$, if both $f(u), g(u)$ are nondecreasing in $u \in [0, E_1]$, then $\lim_{t\to-\infty} \varphi(t) = 0, \lim_{t\to\infty} \varphi(t) = E_1$.*

Theorem 2.2. *Assume that $c = c^*$ and $k(x) = 0$ if $|x|$ is large. Then, there exists $\varphi(t) \in C_{[0,E]}$ satisfying (2.1) and the following properties:*

(1) $\varphi(t) > 0, t \in \mathbb{R}$ *is Lipschitz continuous;*

(2) $\lim_{t\to-\infty}[\varphi(t)e^{-\lambda_1^c t}/t] < 0, \lim_{t\to-\infty} \varphi(t) = 0, \liminf_{t\to\infty} \varphi(t) > 0;$

(3) *(3) of Theorem 2.1 holds.*

3. Proof of main results: monotone case

In this section, we always assume that $E_1 > 0$ such that $f(u) + g(u) > u, u \in (0, E_1), f(E_1) + g(E_1) = E_1$ and both $f(u), g(u)$ are nondecreasing in $u \in [0, E_1]$. First, we prove the result for any given $c > c^*$. Define

$$\overline{\varphi}(t) = \min\{e^{\lambda_1^c t}, E_1\}, \underline{\varphi}(t) = \max\{0, e^{\lambda_1^c t} - qe^{(\lambda_1^c+\epsilon)t}\}, t \in \mathbb{R},$$

in which $\epsilon > 0$ such that $\lambda_1^c + \epsilon < \min\{(1+\alpha)\lambda_1^c, \lambda_2^c\}$ and $q > 1$ is large enough such that $\overline{\varphi}(t) > \underline{\varphi}(t), t \in \mathbb{R}$.

Lemma 3.1. *Assume that* $\varphi \in C^+$ *such that*

$$\underline{\varphi}(t) \leq \varphi(t) \leq \overline{\varphi}(t), t \in \mathbb{R}.$$

Then, there exists $q > 1$ *such that*

$$\underline{\varphi}(t) \leq F(\varphi)(t) \leq \overline{\varphi}(t), t \in \mathbb{R}. \tag{3.1}$$

Proof. If $\overline{\varphi}(t) = E_1$ or $\underline{\varphi}(t) = 0$, then the result is clear by (A1). If $\overline{\varphi}(t) = e^{\lambda_1^c t}$, then

$$\begin{aligned} F(\varphi)(t) &= g(\varphi(t-c)) + \int_{\mathbb{R}} f(\varphi(y))k(t-c-y)dy \\ &\leq r\varphi(t-c) + f'(0)\int_{\mathbb{R}} \varphi(y)k(t-c-y)dy \\ &\leq r\overline{\varphi}(t-c) + f'(0)\int_{\mathbb{R}} \overline{\varphi}(y)k(t-c-y)dy \\ &\leq re^{\lambda_1^c(t-c)} + f'(0)\int_{\mathbb{R}} e^{\lambda_1^c y}k(t-c-y)dy \\ &= e^{\lambda_1^c t}, \end{aligned}$$

which implies the left of (3.1).

Fix $L_3 > 0$ such that

$$[\overline{\varphi}(t)]^{1+\alpha} < L_3 e^{(\lambda_1^c+\epsilon)t}, t \in \mathbb{R},$$

which is admissible by the decay behavior. Then,

$$\begin{aligned} F(\varphi)(t) &= g(\varphi(t-c)) + \int_{\mathbb{R}} f(\varphi(y))k(t-c-y)dy \\ &\geq r\varphi(t-c) - L_1[\varphi(t-c)]^{1+\alpha} \\ &\quad + f'(0)\int_{\mathbb{R}} \varphi(y)k(t-c-y)dy - L_1\int_{\mathbb{R}} [\varphi(y)]^{1+\alpha} k(t-c-y)dy \\ &\geq r\underline{\varphi}(t-c) - L_1[\overline{\varphi}(t-c)]^{1+\alpha} \\ &\quad + f'(0)\int_{\mathbb{R}} \underline{\varphi}(y)k(t-c-y)dy - L_1\int_{\mathbb{R}} [\overline{\varphi}(y)]^{1+\alpha} k(t-c-y)dy \\ &\geq r\underline{\varphi}(t-c) - L_1L_3e^{(\lambda_1^c+\epsilon)(t-c)} \\ &\quad + f'(0)\int_{\mathbb{R}} \underline{\varphi}(y)k(t-c-y)dy - L_1L_3\int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)y}k(t-c-y)dy \end{aligned}$$

$$\geq r[e^{\lambda_1^c(t-c)} - qe^{(\lambda_1^c+\epsilon)(t-c)}] + f'(0)\int_{\mathbb{R}}[e^{\lambda_1^c y} - qe^{(\lambda_1^c+\epsilon)y}]k(t-c-y)dy$$
$$-L_1L_3e^{(\lambda_1^c+\epsilon)(t-c)} - L_1L_3\int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)y}k(t-c-y)dy$$
$$= e^{\lambda_1^c t} - q\Lambda(\lambda_1^c+\epsilon, c)e^{(\lambda_1^c+\epsilon)t}$$
$$-L_1L_3e^{(\lambda_1^c+\epsilon)(t-c)} - L_1L_3e^{(\lambda_1^c+\epsilon)t}\int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)(y-c)}k(y)dy$$
$$\geq e^{\lambda_1^c t} - qe^{(\lambda_1^c+\epsilon)t}$$

if

$$q > 1 + \frac{L_1L_3 + L_1L_3\int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)(y-c)}k(y)dy}{1-\Lambda(\lambda_1^c+\epsilon, c)}.$$

Now, we complete the proof of (3.1). □

Remark 3.1. We obtain a pair of upper and lower solutions of F.

Evidently, the upper and lower solutions of F satisfy the following properties.

Lemma 3.2. *Assume that Lemma 3.1 holds. Then, there exists $L > 0$ such that*

$$|\underline{\varphi}(t) - \underline{\varphi}(t+s)| \leq L|s|, |\overline{\varphi}(t) - \overline{\varphi}(t+s)| \leq L|s|, t, s \in \mathbb{R}$$

and $EL_2 < (1-r)L$.

Define a set

$$\Gamma\left(\underline{\varphi}, \overline{\varphi}\right) = \left\{ \varphi \in C : \begin{array}{c} (i)\ \underline{\varphi}(t) \leq \varphi(t) \leq \overline{\varphi}(t); \\ (ii)\ |\varphi(t_1) - \varphi(t_2)| \leq L|t_1 - t_2|, t_1, t_2 \in \mathbb{R}; \\ (iii)\ \varphi(t) \text{ is nondecreasing in } t \in \mathbb{R} \end{array} \right\}.$$

Evidently, we can fix $L > 0$ such that Γ is nonempty and convex by Lemmas 3.1-3.2.

Lemma 3.3. *Assume that Lemma 3.1 holds and $E_1L_2 < (1-r)L$ such that Γ is nonempty. Then, $F : \Gamma \to \Gamma$.*

Proof. By Lemma 3.1, it suffices to consider the Lipschitz continuity (ii). For any $\varphi \in F$, we have

$$|F(\varphi)(t_1) - F(\varphi)(t_2)|$$
$$= \left| g(\varphi(t_1 - c)) + \int_{\mathbb{R}} f(\varphi(y))k(t_1 - c - y)dyt \right.$$
$$\left. -g(\varphi(t_2 - c)) - \int_{\mathbb{R}} f(\varphi(y))k(t_2 - c - y)dy \right|$$
$$\leq |r\varphi(t_1 - c) - r\varphi(t_2 - c)|$$
$$+ \left| \int_{\mathbb{R}} f(\varphi(y))k(t_1 - c - y)dy - \int_{\mathbb{R}} f(\varphi(y))k(t_2 - c - y)dy \right|$$
$$\leq rL|t_1 - t_2| + \int_{\mathbb{R}} |f(\varphi(y))|\,|k(t_1 - c - y) - k(t_2 - c - y)|\,dy$$
$$\leq rL|t_1 - t_2| + E_1\int_{\mathbb{R}} |k(t_1 - c - y) - k(t_2 - c - y)|\,dy$$

$$\leq rL|t_1 - t_2| + E_1 L_2 |t_1 - t_2|$$
$$\leq L|t_1 - t_2|,$$

which completes the proof. □

Let $\mu > 0$ and

$$B_\mu = \left\{\varphi \in C : \sup_{t\in\mathbb{R}}\{|\varphi(t)|e^{-\mu|t|}\} < \infty\right\}, |\varphi|_\mu = \sup_{t\in\mathbb{R}}\{|\varphi(t)|e^{-\mu|t|}\}.$$

then $\left(B_\mu, |\cdot|_\mu\right)$ is a Banach space.

Lemma 3.4. *Assume that Lemma 3.1 holds and $E_1L_2 < (1-r)L$ such that Γ is nonempty. Then, Γ is bounded and closed in $\left(B_\mu, |\cdot|_\mu\right)$. Moreover, $F : \Gamma \to \Gamma$ is completely continuous in the sense of $|\cdot|_\mu$ if $\mu > 0$ is small.*

Proof. Since the first statement is similar to Huang and Zou [4, Lemma 3.1], we only consider the complete continuity. Select $\Phi, \Psi \in \Gamma$, then

$$\begin{aligned}
&|F(\Phi)(t) - F(\Psi)(t)| \\
=& \left|g(\Phi(t)) - g(\Psi(t)) + \int_{\mathbb{R}} \left[f(\Phi(y)) - f(\Psi(y))\right] k(t-y-c)dy\right| \\
\leq& r\left|\Phi(t) - \Psi(t)\right| + L_2E_1 \int_{\mathbb{R}} \left|\Phi(y)) - \Psi(y)\right| k(t-y-c)dy
\end{aligned}$$

and

$$\begin{aligned}
&\left|F(\Phi)(t) - F(\Psi)(t)\right| e^{-\mu|t|} \\
\leq& r\left|\Phi(t) - \Psi(t)\right| e^{-\mu|t|} + L_2E_1e^{-\mu|t|} \int_{\mathbb{R}} \left|\Phi(y)) - \Psi(y)\right| k(t-y-c)dy \\
\leq& r\left|\Phi - \Psi\right|_\mu + L_2E_1e^{-\mu|t|} \int_{\mathbb{R}} \left|\Phi(y)) - \Psi(y)\right| e^{-\mu|y|}e^{\mu|y|}k(t-y-c)dy \\
\leq& r\left|\Phi - \Psi\right|_\mu + L_2E_1\left|\Phi - \Psi\right|_\mu \int_{\mathbb{R}} e^{\mu|y|}e^{-\mu|t|}k(t-y-c)dy \\
\leq& r\left|\Phi - \Psi\right|_\mu + L_2E_1\left|\Phi - \Psi\right|_\mu \int_{\mathbb{R}} e^{\mu|t-y|}k(t-y-c)dy,
\end{aligned}$$

which implies the continuity if $\int_{\mathbb{R}} e^{\mu|s|}k(s-c)ds < \infty$.

Moreover, if $\varphi \in \Gamma$, then the equicontinuous of $F(\varphi)(t)$ is true by the definition of Γ. For any $\epsilon > 0$, let $N > 0$ such that

$$E_1e^{-\mu|t|} < \epsilon, |t| > N. \tag{3.2}$$

Using Ascoli-Arzela lemma if $t \in [-N, N]$, there exist a constant $T \in \mathbb{N}$ and

$$\{\varphi^i(t)\}_{i=1}^T \subset \{F(\varphi)(t) : \varphi(t) \in \Gamma\}$$

such that $\{\varphi^i(t)\}_{i=1}^T$ is a finite $\epsilon-$net of $\{F(\varphi)(t) : \varphi(t) \in \Gamma\}$ when $|t| \leq N$. From (3.2), $\{\varphi^i(t)\}_{i=1}^T$ is a finite $\epsilon-$net of $\{F(\varphi)(t) : \varphi(t) \in \Gamma\}$ for $t \in \mathbb{R}$, which implies the compactness. The proof is complete. □

Lemma 3.5. *Theorem 2.1 holds.*

Proof. Fix $q > 1$ such that Lemma 3.1 holds. Select $L > 0$ such that $E_1 L_2 < (1-r)L$ and Γ is nonempty, which is admissible. By Schauder's fixed point theorem, F has a fixed point $\varphi \in \Gamma$, and the limit behavior of φ when $t \to -\infty$ is clear by Γ. Further applying the monotonicity, we see that $\lim_{t\to\infty} \varphi(t) = E_1$. The proof is complete. □

Lemma 3.6. *Theorem 2.2 holds.*

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1. We only show the generalized upper and lower solutions here, and denote $c^* = c$ for simplicity. Fix $S > 1$ such that

$$k(x) = 0, |x| > S - c.$$

Let $K > 0$ be a constant such that $(-t+K)e^{\lambda' t}$ is monotone increasing if $t < 0$ and $(S+K)e^{-\lambda' S} > E_1$. Moreover, select $q > S^2$ large enough such that

$$(-t - q\sqrt{-t})e^{\lambda' t} < (-t+K)e^{\lambda' t}, t \le -q^2.$$

Define

$$\overline{\varphi}(\xi) = \begin{cases} \min\{(-t+K)e^{\lambda' t}, E_1\}, t \le 0, \\ E_1, t \ge 0, \end{cases} \qquad \underline{\varphi}(t) = \begin{cases} (-t - q\sqrt{-t})e^{\lambda' t}, t \le -q^2, \\ 0, t \ge -q^2. \end{cases}$$

Now, we prove that (3.1) still holds if $q > 1$ is large enough.

If $\overline{\varphi}(t) = E_1$, then the result is clear. Otherwise, $t < -S - c$ such that

$$\begin{aligned} F(\varphi)(t) &= g(\varphi(t-c)) + \int_{\mathbb{R}} f(\varphi(y))k(t-c-y)dy \\ &\le r\varphi(t-c) + f'(0)\int_{\mathbb{R}} \varphi(y)k(t-c-y)dy \\ &\le r\overline{\varphi}(t-c) + f'(0)\int_{\mathbb{R}} \overline{\varphi}(y)k(t-c-y)dy \\ &\le (-(t-c)+K)e^{\lambda'(t-c)} + f'(0)\int_{\mathbb{R}} (-y+K)e^{\lambda' y}k(t-c-y)dy \\ &= (-t+K)e^{\lambda' t}, \end{aligned}$$

which implies the right of (3.1).

For the left of (3.1), the result is clear if $\underline{\varphi}(t) = 0$. Otherwise, $\underline{\varphi}(t) > 0$ such that $t + S + c^* < 0$. Moreover, we can fix $\lambda_1 \in (\lambda', (1+\alpha)\lambda'), L' > 0$ such that

$$L_1 \left[\overline{\varphi}(y)\right]^{1+\alpha} \le L' e^{\lambda_1 y}, y < -c,$$

Similar to that in Li et al. [8], let $q > 1$ large enough such that $t < -q^2$ implies

$$\begin{aligned} q\sqrt{-t}e^{\lambda' t} \ge\ & L' e^{\lambda_1 (t-c)} + L' \int_{-S}^{S} e^{\lambda_1 y} k(t-c-y)dy \\ & + rq\sqrt{-(t-c)}e^{\lambda'(t-c)} + qf'(0)\int_{-S}^{S} \sqrt{-(t-c-y)}e^{\lambda'(t-c-y)}k(y)dy, \end{aligned}$$

which further indicates the left of (3.1). The proof is complete. □

4. Proof of main results without monotonicity

In this section, we study that f, g only satisfy (A1) based on the conclusion in Section 3. Define two continuous functions

$$\underline{f}(u) = \inf_{v\in[u,E]} f(v), \underline{g}(u) = \inf_{v\in[u,E]} g(v), u \in [0, E].$$

Then, both $\underline{f}, \underline{g}$ are nondecreasing if $u \in [0, E]$. Moreover, from (A2), there exists $\underline{E} > 0$ such that

$$\underline{f}(u) + \underline{g}(u) > u, u \in (0, \underline{E}), \underline{f}(\underline{E}) + \underline{g}(\underline{E}) = \underline{E}.$$

Consider the following operator

$$\underline{F}(\Phi)(t) = \underline{g}(\Phi(t-c)) + \int_{\mathbb{R}} \underline{f}(\Phi(y))k(t-c-y)dy, \Phi \in C_{[0,E]},$$

then the existence of nontrivial fixed points of $\underline{F}(\Phi)(t) = \Phi(t)$ can be obtained by results in Section 3.

Lemma 4.1. *For any given $c > c^*$, there exists a nondecreasing function $\Phi \in C_{[0,\underline{E}]}$ such that*

(1) $\underline{F}(\Phi)(t) = \Phi(t)$ *and* $\Phi(t) > 0, t \in \mathbb{R}$ *is Lipschitz continuous;*

(2) $\lim_{t\to-\infty}[\Phi(t)e^{-\lambda_1^c t}] > 0, \lim_{t\to-\infty} \Phi(t) = 0, \lim_{t\to\infty} \Phi(t) = \underline{E}, \Phi(t) \le e^{\lambda_1^c t}, t \in \mathbb{R}$.

For any given $c > c^*$, we define

$$\overline{\Psi}(t) = \min\{e^{\lambda_1^c t}, E\}, \underline{\Psi}(t) = \Phi(t), t \in \mathbb{R},$$

in which $\Phi(t)$ is defined by Lemma 4.1. By direct calculation, we have the following conclusion.

Lemma 4.2. *Assume that $\Psi \in C^+$ such that*

$$\underline{\Psi}(t) \le \Psi(t) \le \overline{\Psi}(t), t \in \mathbb{R}.$$

Then,

$$\underline{\Psi}(t) \le F(\Psi)(t) \le \overline{\Psi}(t), t \in \mathbb{R}.$$

Define a set

$$\Gamma^*\left(\underline{\Psi}, \overline{\Psi}\right) = \left\{ \varphi \in C : \begin{array}{l} (i)\ \underline{\Psi}(t) \le \Psi(t) \le \overline{\Psi}(t); \\ (ii)\ |\Psi(t_1) - \Psi(t_2)| \le L|t_1 - t_2|, t_1, t_2 \in \mathbb{R} \end{array} \right\}.$$

Evidently, we can fix $L > 0$ such that Γ^* is nonempty and convex. Similar to the discussion in Section 3, we give the following main result in this section.

Lemma 4.3. *Assume that $c > c^*$. Then, we can fix $L > 0$ large enough such that there exists $\Psi(t) \in \Gamma^*$ satisfying (1) and (2) of Theorem 2.1.*

Now, we consider the case $c = c^*$.

Lemma 4.4. *When $c = c^*$, if $k(x) = 0$ for large $|x|$, then there exist a constant $K_1 > 0$ and a nondecreasing function $\Phi \in C_{[0,\underline{E}]}$ such that*

(1) *$\underline{F}(\Phi)(t) = \Phi(t)$ and $\Phi(t) > 0, t \in \mathbb{R}$ is Lipschitz continuous;*

(2) $\lim_{t\to-\infty}[\Phi(t)e^{-\lambda' t}/(-t)] > 0, \lim_{t\to-\infty}\Phi(t) = 0, \lim_{t\to\infty}\Phi(t) = \underline{E}, \Phi(t) \leq (-t + K_1)e^{\lambda' t}, t < 0.$

Define

$$\overline{\Psi}(t) = \min\{(|t| + K)e^{\lambda' t}, E\}, \underline{\Psi}(t) = \Phi(t), t \in \mathbb{R},$$

in which $K \geq K_1$ is a constant and $\Phi(t)$ is defined by Lemma 4.4. If K is large, then we can obtain generalized upper and lower solutions such that the following result holds.

Lemma 4.5. *Assume that $c = c^*$, $k(x) = 0$ if $|x|$ is large. Then, we can fix $L > 0$ large enough such that there exists $\Psi(t) \in \Gamma^*$ satisfying (1) and (2) of Theorem 2.2.*

By what we have done, we complete the proof of Theorems 2.1 and 2.2. Before ending this paper, we make the following remark.

Remark 4.1. By taking a limit process similar to that in Hsu and Zhao [3], we may obtain the existence of traveling wave solutions when $c = c^*$ even if $k(x)$ is not compactly supported. However, the limit process does not show the asymptotic behavior of traveling wave solutions precisely.

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