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Oscillation of 2^{nd}-order Nonlinear Noncanonical Difference Equations with Deviating Argument

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Abstract The purpose of this paper is to establish some new criteria for the oscillation of the second-order nonlinear noncanonical difference equations of the form

$$\Delta\left(a\left(n\right)\Delta x\left(n\right)\right)+q(n)x^{\beta}\left(g(n)\right)=0,\quad n\geq n_0$$

under the assumption

$$\sum_{s=n}^{\infty}\frac{1}{a\left(s\right)}<\infty.$$

Corresponding difference equations of both retarded and advanced type are studied. A particular example of Euler type equation is provided in order to illustrate the significance of our main results.

Keywords Nonlinear difference equation, Retarded, Advanced, Noncanonical, Oscillation.

MSC(2010) 34N05, 39A10, 34A25.

1. Introduction

In this paper, we are concerned about some new criteria for the oscillation of the second-order nonlinear difference equation with deviating argument of the form

$$\Delta\left(a\left(n\right)\Delta x\left(n\right)\right)+q(n)x^{\beta}\left(g(n)\right)=0,\quad n\geq n_0\geq 0, \tag{1.1}$$

where $n\in\mathbb{N}\left(n_0\right)=\{n_0,n_0+1,\ldots\}$, β is the ratio of positive odd integers, $\left(a\left(n\right)\right)_{n\geq n_0}$ and $\left(g\left(n\right)\right)_{n\geq n_0}$ are sequences of positive real numbers, and $\left(g\left(n\right)\right)_{n\geq n_0}$ satisfies

$$g\left(n\right)\leq n-1\quad\forall n\in\mathbb{N}\left(n_0\right)\qquad\text{and}\qquad\lim_{n\to\infty}g(n)=\infty, \tag{1.2}$$

or

$$g\left(n\right)\geq n+1\quad\forall n\in\mathbb{N}\left(n_0\right). \tag{1.3}$$

We study (1.1) under the condition

$$\lim_{n\to\infty}R(n)<\infty\qquad\text{where}\qquad R(n):=\sum_{s=n}^{\infty}\frac{1}{a\left(s\right)}. \tag{1.4}$$

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By a solution of (1.1), we mean a real sequence $(x(n))_{n\geq n_0-m}$, $m = \inf_{n\in\mathbb{N}(n_0)}\{g(n)\}$, which satisfies (1.1) for all $n \geq n_0$. Such a solution is called "oscillatory", if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be "nonoscillatory". Equation (1.1) is "oscillatory", if all its solutions oscillate.

We also note that equation (1.1) is in canonical form if $R(n_0) = \infty$, and is in noncanonical form if $R(n_0) < \infty$. The structure of nonoscillatory (eventually positive) solutions $x(n)$ of (1.1) in the canonical form is of one sign and is eventually positive, while for the noncanonical form, we eventually find $\Delta x(n) > 0$ or $\Delta x(n) < 0$.

The problem of determining the oscillation and nonoscillation of solutions of difference equations has been a very active area of research in the last decade, and for the survey of recent results, we refer the reader to the monographs [1], [2], [5]. In recent years, there has been much research concerning the oscillation and asymptotic behavior of solutions of various classes of difference equations, and we mention $[1-9]$ and the references cited therein as example of some recent contributions in this area. There have been numerous studies on second-order difference equations due to their use in the natural sciences and as well as for theoretical interests. Recent results on the oscillatory and asymptotic behavior of solutions of second-order difference equations can be found, for example, in $[10-24]$. However, it appears that there are very few results regarding the oscillation of solutions of second-order difference equations of the form of equation (1.1) with (1.4) satisfied.

In view of this, our aim in this paper is to present some new sufficient conditions that ensure that all solutions of (1.1) are oscillatory. Contrary to the most existing results, oscillation of the studied equation is attained via only one condition. We also consider both retarded and advanced difference equations of type (1.1).

2. Main results

2.1. Equation (1.1) with retarded argument

Theorem 2.1. *Assume that* (1.2) *and* (1.4) *hold. If*

$$\limsup_{n\to\infty}\begin{pmatrix} R\left(g(n)+1\right)\sum_{s=n_0}^{g(n)-1} q(s) \\ +\sum_{s=g(n)}^{n-1} R(s+1)q(s) \\ +R^{-\beta}(g(n))\sum_{s=n}^{\infty} R(s+1)q(s)R^{\beta}(g(s)) \end{pmatrix} > \begin{cases} 1, \text{ if } \beta = 1 \\ 0, \text{ if } \beta\in(0,1) \end{cases}, \tag{2.1}$$

then all solutions of (1.1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq n_0-m}$ is a nonoscillatory solution of (1.1). Then, it is either eventually positive or eventually negative. As $(-x(n))_{n\geq n_0-m}$ is also a solution of (1.1), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq n_0 - m$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_2 \geq n_1$ such that $x(g(n)) > 0$, $\forall n \geq n_2$. In view of this, equation (1.1) becomes

$$\Delta\left(a(n)\Delta x(n)\right) = -q(n)x^{\beta}\left(g(n)\right) \leq 0, \quad n \geq n_2, \tag{2.2}$$

which means that $a(n)\Delta x(n)$ is nonincreasing and of one sign. Therefore, there exists a $n_3 \geq n_2$ such that

$$\Delta x(n) > 0, \quad n \geq n_3, \quad \text{Case (I)}$$

or

$$\Delta x(n) < 0 \quad \text{and} \quad \Delta(a(n)\Delta x(n)) \leq 0, \quad n \geq n_3, \quad \text{Case (II).}$$

Case (I): Clearly, there exists a real constant $k > 0$ such that $x(g(n)) > k$ for every $n \geq n_3$. From (1.1), we get

$$\Delta(a(n)\Delta x(n)) + q(n)k^\beta < 0.$$

Summing up this inequality from n_3 to $n-1$, we obtain

$$-a(n)\Delta x(n) + a(n_3)\Delta x(n_3) > k^\beta \sum_{s=n_3}^{n-1} q(s) \geq k^\beta \sum_{s=n_3}^{n-1} R(s)q(s). \tag{2.3}$$

Claim. (2.1) *guarantees that* $\sum_{s=n_3}^\infty R(s)q(s) = \infty$.

Indeed, it follows from (2.1) that there exists a positive constant c such that

$$R(g(n)+1)\sum_{s=n_0}^{g(n)-1} q(s) + \sum_{s=g(n)}^{n-1} R(s+1)q(s) + R^{-\beta}(g(n))\sum_{s=n}^{\infty} R(s+1)q(s)R^\beta(g(s)) \geq c. \tag{2.4}$$

Assume that $\sum_{s=n_3}^\infty R(s)q(s) < \infty$. Then, there exists $N \geq n_2$ such that

$$\sum_{s=n_3}^{\infty} R(s)q(s) < c/7.$$

Thus, for $n \geq N$, we have

$$\begin{aligned} R(g(n)+1)\sum_{s=n_3}^{g(n)-1} q(s) &= R(g(n)+1)\sum_{s=n_3}^{N-1} q(s) + R(g(n)+1)\sum_{s=N}^{g(n)-1} q(s) \\ &\leq R(g(n)+1)\sum_{s=n_3}^{N-1} q(s) + \sum_{s=N}^{g(n)-1} R(s)q(s) \leq R(g(n)+1)\sum_{s=n_3}^{N-1} q(s) + \frac{c}{7} \end{aligned}$$

and

$$R^{-\beta}(g(n))\sum_{s=n}^{\infty} R(s+1)q(s)R^\beta(g(s)) \leq \sum_{s=n}^{\infty} R(s+1)q(s) \leq \frac{c}{7}.$$

Therefore,

$$R(g(n)+1)\sum_{s=n_0}^{g(n)-1} q(s) + \sum_{s=g(n)}^{n-1} R(s+1)q(s) + R^{-\beta}(g(n))\sum_{s=n}^{\infty} R(s+1)q(s)R^\beta(g(s)) \leq \frac{3c}{7},$$

which contradicts (2.4). Consequently, $\sum_{s=n_3}^{\infty} R(s)q(s) = \infty$. Our claim has been proved.

In view of this, (2.3) gives

$$0 \le \lim_{n\to\infty} a\,(n)\,\Delta x\,(n) = -\infty,$$

which is a contradiction.

Case (II): First, we will show that $\lim_{n\to\infty} x\,(n) = 0$. Indeed, on the contrary, there exists a constant $b > 0$ such that $x\,(n) \ge b > 0$. Summing up (2.2) from n_3 to $n-1$, we get

$$-a\,(n)\,\Delta x\,(n) \ge b^{\beta} \sum_{s=n_3}^{n-1} q(s).$$

Summing up the last inequality from n_3 to ∞, we obtain

$$x\,(n_3) \ge b^{\beta} \sum_{u=n_3}^{\infty} \frac{1}{a\,(u)} \sum_{s=u}^{\infty} q(s) = b^{\beta} \sum_{u=n_3}^{\infty} R(s)q(s) = \infty,$$

which is a contradiction. Therefore, $\lim_{n\to\infty} x\,(n) = 0$.

Now, since

$$\begin{aligned} x\,(n) &\ge -\sum_{s=n}^{\infty} \frac{a\,(s)\,\Delta x(s)}{a\,(s)} \ge -\left(\sum_{s=n}^{\infty} \frac{1}{a\,(s)}\right) a\,(n)\,\Delta x(n) \\ &= -R(n)a\,(n)\,\Delta x(n), \end{aligned}$$

we have

$$x\,(n) + R(n)a\,(n)\,\Delta x(n) > 0. \tag{2.5}$$

Thus, it follows that

$$\Delta\left(\frac{x(n)}{R(n)}\right) = \frac{R(n)\Delta x(n) + \frac{x(n)}{a(n)}}{R(n+1)R(n)} \ge 0,$$

i.e. $x(n)/R(n)$ is eventually nondecreasing. Therefore,

$$\frac{x(n+1)}{R(n+1)} \ge \frac{x(n)}{R(n)}. \tag{2.6}$$

It is easy to see that equation (1.1) is equivalent to the equation

$$\Delta\left(a\,(n)\,\Delta x\,(n)\,R(n+1) + x(n+1)\right) + R(n+1)q(n)x^{\beta}\left(g(n)\right) = 0. \tag{2.7}$$

Now, in view of (2.6) and (2.5), we have

$$\begin{aligned} a\,(n)\,\Delta x\,(n)\,R(n+1) + x(n+1) &\ge a\,(n)\,\Delta x\,(n)\,R(n+1) + \frac{R(n+1)}{R(n)}x(n) \\ &= \left(\frac{a\,(n)\,\Delta x\,(n)\,R(n) + x(n)}{R(n)}\right) R(n+1) > 0. \end{aligned}$$

Summing up (2.7) from n to u and letting $u \to \infty$, we get

$$a\,(n)\,\Delta x\,(n)\,R(n+1) + x(n+1) \ge \sum_{s=n}^{\infty} R(s+1)q(s)x^{\beta}\left(g(s)\right). \tag{2.8}$$

Using the fact that x is nonincreasing, we have

$$a\left(n\right)\Delta x\left(n\right)R(n+1)+x(n)\geq\sum_{s=n}^{\infty}R(s+1)q(s)x^{\beta}\left(g(s)\right). \tag{2.9}$$

On the other hand, summation of (1.1) from n_2 to $n-1$ yields

$$-\,a\left(n\right)\Delta x\left(n\right)\geq\sum_{s=n_2}^{n-1}q(s)x^{\beta}\left(g(s)\right). \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$x(n)\geq R(n+1)\sum_{s=n_2}^{n-1}q(s)x^{\beta}\left(g(s)\right)+\sum_{s=n}^{\infty}R(s+1)q(s)x^{\beta}\left(g(s)\right).$$

Therefore,

$$\begin{aligned}x(g(n))\geq{}& R(g(n)+1)\sum_{s=n_2}^{g(n)-1}q(s)x^{\beta}\left(g(s)\right)+\sum_{s=g(n)}^{n-1}R(s+1)q(s)x^{\beta}\left(g(s)\right)\\&+\sum_{s=n}^{\infty}R(s+1)q(s)x^{\beta}\left(g(s)\right).\end{aligned} \tag{2.1}$$

Taking into account the fact that $x(n)/R(n)$ is nondecreasing, we obtain

$$\begin{aligned}\frac{x(g(n))}{x^{\beta}(g(n))}\geq{}& R(g(n)+1)\sum_{s=n_2}^{g(n)-1}q(s)+\sum_{s=g(n)}^{n-1}R(s+1)q(s)\\&+R^{-\beta}(g(n))\sum_{s=n}^{\infty}R(s+1)q(s)R^{\beta}(g(s)).\end{aligned}$$

Taking $\limsup$ as $n\to\infty$ on both sides of the above inequality, we are led to a contradiction with (2.1).

The proof of the theorem is complete. □

Theorem 2.2. *Assume that* (1.2) *and* (1.4) *hold. If*

$$\limsup_{n\to\infty}\sum_{s=g(n)}^{n-1}R(s+1)q(s)>\begin{cases}1, & \text{if } \beta=1\\ 0, & \text{if } \beta\in(0,1)\end{cases}, \tag{2.12}$$

then all solutions of (1.1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x\left(n\right))_{t\geq t_0-m}$ is a nonoscillatory solution of (1.1). Then, it is either eventually positive or eventually negative. As $(-x\left(n\right))_{n\geq n_0-m}$ is also a solution of (1.1), we may restrict ourselves only to the case where $x(n)>0$ for all large n. Let $n_1\geq n_0-m$ be an integer such that $x(n)>0$ for all $n\geq n_1$. Then, there exists $n_2\geq n_1$ such that $x(g(n))>0$, $\forall n\geq n_2$. In view of this, equation (1.1) becomes

$$\Delta\left(a\left(n\right)\Delta x\left(n\right)\right)=-q(n)x^{\beta}\left(g(n)\right)\leq 0,\quad n\geq n_2,$$

which means that $a(n)\Delta x(n)$ is nonincreasing and of one sign. Therefore, there exists a $n_3 \geq n_2$ such that

$$\Delta x(n) > 0, \quad n \geq n_3, \quad \text{Case (I)}$$

or

$$\Delta x(n) < 0 \quad \text{and} \quad \Delta(a(n)\Delta x(n)) \leq 0, \quad n \geq n_3, \quad \text{Case (II)}.$$

Case (I): We are led to a contradiction (see the proof of Theorem 2.1).

Case (II): As in the proof of Theorem 2.1, (2.5) is satisfied, i.e.

$$x(n) + R(n)a(n)\Delta x(n) \geq 0.$$

Thus,

$$0 \leq x(g(n)) + R(g(n))a(g(n))\Delta x(g(n)) \leq x(g(n)).$$

Set

$$W(n) = x(n+1) + R(n+1)a(n)\Delta x(n) \leq x(n+1) \leq x(n).$$

Therefore,

$$\Delta W(n) + R(n+1)q(n)W^{\beta}(g(n)) \leq 0.$$

Summing up the last inequality from $g(n)$ to $n-1$, we get

$$W(g(n)) \geq \sum_{s=g(n)}^{n-1} R(s+1)q(s)W^{\beta}(g(s)) \geq W^{\beta}(g(n)) \sum_{s=g(n)}^{n-1} R(s+1)q(s),$$

or

$$\frac{W(g(n))}{W^{\beta}(g(n))} \geq \sum_{s=g(n)}^{n-1} R(s+1)q(s).$$

Taking lim sup as $n \to \infty$ on both sides of the above inequality, we are led to a contradiction with (2.12).

The proof of the theorem is complete. □

2.2. Equation (1.1) with advanced argument

Theorem 2.3. *Assume that* (1.3) *and* (1.4) *hold. If*

$$\limsup_{n\to\infty} \begin{pmatrix} R(g(n)+1)\sum_{s=n_0}^{n-1} q(s) \\ +R(g(n)+1)R^{-\beta}(g(n))\sum_{s=n}^{g(n)-1} q(s)R^{\beta}(g(s)) \\ +R^{-\beta}(g(n))\sum_{s=g(n)}^{\infty} R(s+1)q(s)R^{\beta}(g(s)) \end{pmatrix} > \begin{cases} 1, & \textit{if } \beta = 1 \\ 0, & \textit{if } \beta \in (0,1) \end{cases}, \tag{2.13}$$

then all solutions of (1.1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq n_0-m}$ is a nonoscillatory solution of (1.1). Then, it is either eventually positive or eventually negative. As $(-x(n))_{n\geq n_0-m}$ is also a solution of (1.1), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq n_0 - m$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_2 \geq n_1$ such that $x(g(n)) > 0$, $\forall n \geq n_2$. In view of this, equation (1.1) becomes

$$\Delta(a(n)\Delta x(n)) = -q(n)x^{\beta}(g(n)) \leq 0, \quad n \geq n_2,$$

which means that $a(n)\Delta x(n)$ is nonincreasing and of one sign. Therefore, there exists a $n_3 \geq n_2$ such that:

$$\Delta x(n) > 0, \quad n \geq n_3, \quad \text{Case (I)}$$

or

$$\Delta x(n) < 0 \quad \text{and} \quad \Delta(a(n)\Delta x(n)) \leq 0, \quad n \geq n_3, \quad \text{Case (II)}.$$

Case (I): We are led to a contradiction (see the proof of Theorem 2.1).

Case (II): As in the proof of Theorem 2.1, (2.8 and (2.9) are satisfied. Combining (2.8) and (2.9), we have

$$x(g(n)) \geq R(g(n)+1)\sum_{s=n_2}^{n-1} q(s)x^{\beta}(g(s)) + R(g(n)+1)\sum_{s=n}^{g(n)-1} q(s)x^{\beta}(g(s)) + \sum_{s=g(n)}^{\infty} R(s+1)q(s)x^{\beta}(g(s)).$$

Taking into account the fact that $x(n)/R(n)$ is nondecreasing, we obtain

$$\frac{x(g(n))}{x^{\beta}(g(n))} \geq R(g(n)+1)\sum_{s=n_0}^{n-1} q(s) + R(g(t)+1)R^{-\beta}(g(n))\sum_{s=n}^{g(n)-1} q(s)R^{\beta}(g(s)) + R^{-\beta}(g(n))\sum_{s=g(n)}^{\infty} R(s+1)q(s)R^{\beta}(g(s)).$$

Taking $\limsup$ as $n \to \infty$ on both sides of the above inequality, we are led to a contradiction with (2.13).

The proof of the theorem is complete. □

3. Examples

In this section, we provide two examples to illustrate our main results.

Example 3.1. Consider the second-order retarded difference equation

$$\Delta(n(n+1)\Delta x(n)) + \frac{\alpha}{n}x^{\beta}(n-m+1) = 0, \quad n \geq n_0 > 1, \tag{3.1}$$

where α is a positive real number and $m > 1$ is a positive integer.

Here, $a(n) = n(n+1)$. Thus,

$$R(n) := \sum_{s=n}^{\infty} \frac{1}{a(s)} = \sum_{s=n}^{\infty} \frac{1}{s(s+1)} = \frac{1}{n}.$$

That is, (1.4) holds.

Hence, (2.1) takes the form

$$\limsup_{n\to\infty}\begin{pmatrix} \frac{\alpha}{n-m+2}\sum_{s=n_0}^{n-m}\frac{1}{s} + \sum_{s=n-m+1}^{n-1}\frac{\alpha}{s(s+1)} \\ +(n-m+1)^{\beta}\sum_{s=n}^{\infty}(s-m+1)^{-\beta}\frac{\alpha}{s(s+1)} \end{pmatrix} \geq \sum_{s=n_0}^{\infty}\frac{\alpha}{s(s+1)} = \frac{\alpha}{n_0}.$$

That is, all conditions of Theorem 2.1 are satisfied if $\alpha > n_0$. Consequently, if $\alpha > n_0$, then all solutions of (3.1) are oscillatory.

Example 3.2. Consider the second-order advanced difference equation

$$\Delta\left(n(n+1)\Delta x\left(n\right)\right)+\frac{\alpha}{n}x^{\beta}\left(n+m+1\right)=0,\quad n\geq n_{0}>1,\tag{3.2}$$

where α is a positive real number and $m>1$ is a positive integer.

Here, $a\left(n\right)=n(n+1)$. Therefore, $R(n)=1/n$, i.e. (1.4) holds.

Hence, (2.13) takes the form

$$\limsup_{n\to\infty}\begin{pmatrix}\frac{\alpha}{n+m+2}\sum_{s=n_0}^{n-1}\frac{1}{s}\\ +\frac{1}{n+m+2}\left(n+m+1\right)^{\beta}\sum_{s=n}^{n+m}\frac{\alpha}{s}\left(s+m+1\right)^{-\beta}\\ +\left(n+m+1\right)^{\beta}\sum_{s=n+m+1}^{\infty}\left(s+m+1\right)^{-\beta}\frac{\alpha}{s(s+1)}\end{pmatrix}$$
$$\geq\sum_{s=n_0}^{\infty}\frac{\alpha}{s(s+1)}=\frac{\alpha}{n_0}.$$

That is, all conditions of Theorem 2.3 are satisfied, if $\alpha>n_0$. Consequently, if $\alpha>n_0$, then all solutions of (3.2) are oscillatory.

4. Concluding remarks

The results of this paper is presented in a form, which is essentially new. Contrary to the most existing results for second-order difference equations in the noncanonical form, oscillation of the studied equation is attained via only one condition.

It will be of interest to investigate on the higher-order difference equations of the form

$$\Delta\left(a\left(n\right)\Delta^{m-1}x\left(n\right)\right)+q(n)x^{\beta}\left(g(n)\right)=0,\quad n\geq n_0\geq 0,\quad m>2,$$

where $\left(a\left(n\right)\right)_{n\geq n_0}$, $q(n)$ and $\left(g\left(n\right)\right)_{n\geq n_0}$ are as in equation (1.1).

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