Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 3, Number 4, December 2021, 505–521 DOI:10.12150/jnma.2021.505

An Analogue-difference Method and Application to Induction Motor Models*

Xilan Liu[1,†], Jinrong Mu[2,3] and Wenxia Wang[4]

Abstract The paper established a so-called analogue-difference method (AD-M) to compute the numerical solutions for boundary value problems of higher-order differential equations, which can be a fundamental method and performs much better than the finite difference method (FDM), even for second-order boundary value problems. Numerical examples and results illustrate the simplicity, efficiency and applicability of the method, which also show that the proposed method has obvious advantages over the methods presented by recent state-of-the-art work for induction motor models.

Keywords Difference method, Analogue-difference method, Numerical solution.

MSC(2010) 65LXX, 34K15.

1. Introduction

The difference method (FDM) is a fundamental method to find numerical solutions for boundary value problems of ordinary differential equations. Since the simplicity and effectiveness, FDM has been applied to solve numerical solutions for second-order boundary value problems of ordinary differential equations and partial differential equations (see [3]- [11]), and this method can be found in many text books and papers related to numerical methods, see [2], [4], [7], [11] and references therein. In particular, [4] present the difference method for the classical second-order boundary value problem in details. However, this method is not satisfied, and it is even invalid in some situation for higher-order boundary value problems. On the other hand, biologically inspired intelligent computing approach, based on artificial neural networks (ANN) models, the authors of [1] established some methods by optimising efficient local search methods so-called sequential quadratic programming (SQP), interior point technique (IPT) and active set technique (AST), they applied

† the corresponding author.
Email address: doclanliu@163.com (X. Liu), mujie07070142@163.com (J. Mu), wwxgg@126.com (W. Wang)
1 School of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji, Shaanxi 721013, China
2 School of Optics and Photonics, Beijing Institute of Technology, Bejing 100081, China
3 Department of Mathematics and Statistics, Qinghai Minzu University, Xining, Qinghai 810007, China
4 Department of Mathematics, Taiyuan Normal University, Taiyuan, Shanxi 030012, China
* The authors were supported by National Natural Science Foundation of China (Nos. 11361047, 11561043) and the Natural Science Foundation of Qinghai Province (No. 2017-ZJ-908).

the methods to solve some fifth-order boundary value problems arisen in induction motor. The authors show that their proposed technique is state-of-the-art, which is good in accuracy. In [7] and [6], the authors developed the so-called new finite difference method to solve a second-order boundary value problem, and a eighth-order boundary value problem respectively. However, the methods are only fourth-order accurate. In this paper, we devote ourselves to improving the accuracy and efficiency of FDE by using the higher-order derivative substitution formulation based on Taylor expansion. Our new method can not only be applied to second-order boundary value problems, but also higher-order boundary value problems especially, and it also can be applied to all models of [1], which shows that the method has higher accuracy than those of [1].

To self completeness and clearness, we introduce some basic knowledge for FDM (see [4] and references therein). Consider the following second-order BVP

$$\begin{cases} Lu \equiv -u'' + q(t)u = f(t), a < t < b, \\ u(a) = \alpha, u(b) = \beta, \end{cases} \tag{1.1}$$

where $q(t), f(t) \in C[a,b], q(t) \geq 0$ for $t \in [a,b]$. Assumed that the problem (1.1) has a unique solution. The process of numerical solution using classical difference method as follows.

We divide the interval $[a,b]$ into N equal parts, and take the grid points as follows:

$$a = t_0 < t_1 < \cdots < t_i < \cdots < t_N = b, \tag{1.2}$$

where $t_i = a + ih$, step length $h = \frac{b-a}{N}$. Choose the second-order center difference quotient formula at node t_i

$$u''(t_i) \approx \frac{1}{h^2}[u(t_{i-1}) - 2u(t_i) + u(t_{i+1})], \tag{1.3}$$

then the following system obtained

$$\begin{cases} L_h u_i = -\frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) + q_i u_i = f_i, i = 1, 2, \cdots, N-1, \\ u_0 = \alpha, u_N = \beta, \end{cases} \tag{1.4}$$

where $q_i = q(t_i), f_i = f(t_i), i = 0, 1, \cdots, N$. The truncation error of this method is

$$R_i(u) = Lu(t_i) - L_h u(t_i) = \frac{h^2}{12} u^{(4)}(\xi_i), \quad \xi_i \in (t_{i-1}, t_{i+1}), \tag{1.5}$$

where L is the derivative operator defined by $Lu := u''$. The algebraic system (1.4) can be written in matrix form

$$\begin{pmatrix} -\frac{1}{h^2} & \frac{2}{h^2}+q_1 & -\frac{1}{h^2} & & & \\ & -\frac{1}{h^2} & \frac{2}{h^2}+q_2 & -\frac{1}{h^2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{h^2} & \frac{2}{h^2}+q_{N-1} & -\frac{1}{h^2} \\ 1 & 0 & \cdots\cdots\cdots & & 0 & 0 \\ 0 & 0 & \cdots\cdots\cdots & & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ \alpha \\ \beta \end{pmatrix}, \tag{1.6}$$

which can be simplified into

$$AU = F \tag{1.7}$$

with

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}, F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{1.8}$$

where

$$A_1 = \begin{pmatrix} -\frac{1}{h^2} & \frac{2}{h^2} + q_1 & -\frac{1}{h^2} & & & \\ & -\frac{1}{h^2} & \frac{2}{h^2} + q_2 & -\frac{1}{h^2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{h^2} & \frac{2}{h^2} + q_{N-1} & -\frac{1}{h^2} \end{pmatrix}, \tag{1.9}$$

$$A_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{1.10}$$

$$F_1 = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}, \qquad F_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{1.11}$$

This paper is organized as follows: In Section 2, we establish the general pth-order q-point method so-called analogue-difference method (ADM for short). In Section 3, ADM is used to solve second-order BVP, which shows that AMD behaviors are better than the classical finite difference method. Section 4 develops AMD for higher-order BVPs, and some numerical experiments are given to demonstrate the advantage of our method. The final section is a brief conclusion.

2. Establishment of the method

Let $u(t)$ be a function sufficiently smooth on $[a,b]$. We take the partition (1.2), q (q is a positive integer) points near the point $t = t_i$ consecutively denoted by $t_j, t_{j+1}, \cdots, t_{j+q-1}$ and expand Taylor formula of $u(t_j), u(t_{j+1}), \cdots, u(t_{j+q-1})$ at point $t = t_i$ respectively. Then, the pth-order q-point analogue-difference formula obtained. The details shall be shown next.

To begin with, we shall choose the substitution formula of higher order derivative. The Taylor expansion of $u(t_i + nh)$ at the node t_i is

$$u(t_i + nh) = u(t_i) + nhu'(t_i) + \cdots + \frac{n^k}{k!} h^k u^{(k)}(t_i) + R_{k,n}, \tag{2.1}$$

where $R_{k,n} = o((nh)^k)$ or $R_{k,n} = \frac{(nh)^{k+1}}{(k+1)!} u^{(k+1)}(\xi), \xi \in (a,b)$. We take q points near the point $t = t_i$ consecutively and denoted by $t_j, t_{j+1}, \cdots, t_{j+q-1}$, where q and j are positive integer, $i = 1, 2, \cdots, N-1$ and obtain

$$u(t_j) = u(t_i) + (j-i)hu'(t_i) + \cdots + \frac{(j-i)^{q-1}}{(q-1)!} h^{q-1} u^{(q-1)}(t_i) + R_0,$$

$$u(t_{j+1}) = u(t_i) + (j-i+1)hu'(t_i) + \cdots + \frac{(j-i+1)^{q-1}}{(q-1)!} h^{q-1} u^{(q-1)}(t_i) + R_1, \tag{2.2}$$

$$\cdots$$

$$u(t_{j+q-1}) = u(t_i) + (j-i+q-1)hu'(t_i) + \cdots + \frac{(j-i+q-1)^{q-1}}{(q-1)!} h^{q-1} u^{(q-1)}(t_i) + \tag{2.3}$$

$$R_{q-1},$$

where $R_m = \frac{(j-i+m)^q}{q!} h^q u^{(q)}(\xi)$ $(\xi \in (a,b), m \in \mathbb{Z}, 0 \le m \le q-1)$.

Omit all residues in (2.2), one can choose some numbers C_k, $(k = 0, 1, 2, \cdots, q-1)$ such that for positive integer $p < q$, one has

$$C_0 u_j + C_1 u_{j+1} + C_2 u_{j+2} + \cdots + C_{q-1} u_{j+q-1} = h^p u_i^{(p)}, \tag{2.4}$$

and obtain the substitution formula

$$u_i^{(p)} = \frac{1}{h^p} \left(C_0 u_j + C_1 u_{j+1} + C_2 u_{j+2} + \cdots + C_{q-1} u_{j+q-1} \right), \tag{2.5}$$

where $u_i = u(t_i)$, $u_i^{(2)} = u''(t_i), \cdots, u_i^{(p)} = u^{(p)}(t_i)$, and C_k are real numbers, $p, k = 1, 2, \cdots, q-1$.

Next, we give the following definition for the formula (2.5).

Definition 2.1. Let $u(t)$ be a function sufficiently smooth on $[a,b]$. We call (2.5) a pth-order q-point analogue-difference formula. If $u_i \in \{u_k | k \in Z^+ \text{and } k = j, j+1, \cdots, j+q-1\}$, the formula (2.5) is called an inner analogue-difference formula, and if $u_i \notin \{u_k | k \in Z^+ \text{and } k = j, j+1, \cdots, j+q-1\}$, the formula (2.5) is called an outer analogue-difference formula.

Example 2.1. The first-order center difference quotient formula $u_i' = \frac{1}{2h}(u_{i+1} - u_{i-1})$ is a first-order three-point inner analogue-difference formula. Since $u_i \notin \{u_{i+1}, u_{i+2}\}$, the formula $u_i' = \frac{1}{h}(u_{i+2} - u_{i+1})$ is a first-order two-point outer analogue-difference formula.

For the case of $u_i \in \{u_k | k \in Z^+ \text{and } k = j, j+1, \cdots, j+q-1\}$ in Definition 1, that is, (2.5) is a inner analogue-difference formula, we have

Definition 2.2. When $j = i$, formula (2.5) is called a pth-order q-point forward analogue-difference formula. When $j = i - q + 1$, (2.5) is called pth-order q-point backward analogue-difference formula. In particular, when q is odd and $j = i - \frac{q-1}{2}$, (2.5) is called pth-order q-point center analogue-difference formula.

Example 2.2. $u_i' = \frac{1}{h}(u_i - u_{i-1})$ is a first-order two-point backward analogue-difference formula, $u_i' = \frac{1}{h}(u_{i+1} - u_i)$ is a first-order two-point forward analogue-difference formula, $u_i' = \frac{1}{2h}(u_{i+1} - u_{i-1})$ is a first-order three-point center analogue-difference formula, and $u_i^{(4)} = \frac{1}{h^4}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2})$ is a fourth-order five-point center analogue-difference formula.

It is easy to see that the pth-order q-point analogue-difference formula (2.5) can be completely determined if p, q, j and i are given. In fact, if we omit the residues R_m $(m = 1, 2, \cdots, q-1)$ in (2.2), then we have

$$\begin{pmatrix} u_j \\ u_{j+1} \\ \vdots \\ u_{j+q-1} \end{pmatrix} = \begin{pmatrix} 1 & j-i & \cdots & \frac{(j-i)^{q-1}}{(q-1)!} \\ 1 & j-i+1 & \cdots & \frac{(j-i+1)^{q-1}}{(q-1)!} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & j-i+q-1 & \cdots & \frac{(j-i+q-1)^{q-1}}{(q-1)!} \end{pmatrix} \begin{pmatrix} u_i \\ hu_i' \\ \vdots \\ h^{q-1}u_i^{(q-1)} \end{pmatrix}. \tag{2.6}$$

(2.5) shows that

$$(C_0 \quad C_1 \quad \cdots \quad C_{q-1}) \begin{pmatrix} u_j \\ u_{j+1} \\ \vdots \\ u_{j+q-1} \end{pmatrix} = h^p u_i^{(p)}. \tag{2.7}$$

By (2.6) and (2.7), we get

$$\begin{aligned} &(C_0 \quad C_1 \quad \cdots \quad C_{q-1}) \begin{pmatrix} 1 & j-i & \cdots & \frac{(j-i)^{q-1}}{(q-1)!} \\ 1 & j-i+1 & \cdots & \frac{(j-i+1)^{q-1}}{(q-1)!} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & j-i+q-1 & \cdots & \frac{(j-i+q-1)^{q-1}}{(q-1)!} \end{pmatrix} \begin{pmatrix} u_i \\ \vdots \\ h^p u_i^{(p)} \\ \vdots \\ h^{q-1}u_i^{(q-1)} \end{pmatrix} \\ &= (0 \quad \cdots \quad 1 \quad \cdots \quad 0) \begin{pmatrix} u_i \\ \vdots \\ h^p u_i^{(p)} \\ \vdots \\ h^{q-1}u_i^{(q-1)} \end{pmatrix}. \end{aligned} \tag{2.8}$$

Hence,

$$(C_0 \quad C_1 \quad \cdots \quad C_{q-1}) = (0 \cdots 1 \cdots 0) \begin{pmatrix} 1 & j-i & \cdots & \frac{(j-i)^{q-1}}{(q-1)!} \\ 1 & j-i+1 & \cdots & \frac{(j-i+1)^{q-1}}{(q-1)!} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & j-i+q-1 & \cdots & \frac{(j-i+q-1)^{q-1}}{(q-1)!} \end{pmatrix}^{-1}, \tag{2.9}$$

which together with (2.5) may determine a pth-order q-point analogue-difference formula.

Remark 1. Note that in formula (2.5), $q \geq p+1$. That is to say, pth-order derivative needs at least $p+1$ points to be represented. In addition, for a given interval partition, the value of q is associated with the subscript of the starting point j. When some items are absent, the corresponding coefficients are zero by default, namely $C_i = 0$ for some i $(i = 0, 1, \cdots, q-1)$.

Theorem 2.1. *Assume that the function $u(t)$ is sufficiently smooth, r is the order of truncation error of the algorithm (2.5). Then,*

$$r = \begin{cases} q-p, & p \text{ is even}, q \text{ is odd and } j = i - \frac{q-1}{2}, \\ q-p-1, & \text{others}. \end{cases} \tag{2.10}$$

Proof. We know that by (2.2) and (2.4),

$$\begin{aligned} & C_0 u_j + C_1 u_{j+1} + C_2 u_{j+2} + \cdots + C_{q-1} u_{j+q-1} \\ = & D_0 \cdot u_i + D_1 \cdot h u_i' + \cdots + D_{q-1} \cdot h^{q-1} u_i^{(q-1)} + D_q \cdot h^q u_i^{(q)} \\ & + D_{q+1} \cdot h^{q+1} u_i^{(q+1)} + \cdots, \end{aligned} \tag{2.11}$$

where

$$D_k = (C_0 \quad C_1 \cdots \quad C_{q-1}) \begin{pmatrix} \frac{(j-i)^k}{k!} \\ \frac{(j-i+1)^k}{k!} \\ \vdots \\ \frac{(j-i+q-1)^k}{k!} \end{pmatrix}. \tag{2.12}$$

From (2.9), we have

$$D_1 = D_2 = \cdots = D_{p-1} = D_{p+1} = \cdots = D_{q-1} = 0, D_p = 1.$$

The residue of (2.4) is

$$R_q = D_q \cdot h^q u_i^{(q)} + D_{q+1} \cdot h^{q+1} u_i^{(q+1)} + \cdots. \tag{2.13}$$

Therefore, the truncation error of (2.5) is

$$\begin{aligned} R_q(u_i) &= u_i^{(p)} - \frac{1}{h^p}(C_0u_j + C_1u_{j+1} + \cdots + C_{q-1}u_{j+q-1}) \\ &= u_i^{(p)} - \frac{1}{h^p}(h^p u_i^{(p)} + R_q) \\ &= -\frac{R_q}{h^p} \\ &= -(D_q \cdot h^{q-p}u_i^{(q)} + D_{q+1} \cdot h^{q-p+1}u_i^{(q+1)} + \cdots) \end{aligned} \tag{2.14}$$

It is easy to know that $D_q = 0$ when p is even, q is odd and $j = i - \frac{q-1}{2}$. In this situation, the residue of (2.5) is

$$R_q(u_i) = D_{q+1} \cdot h^{q-p+1}u_i^{(q+1)} + o(h^{q-p+1}). \tag{2.15}$$

The above discussion finishes the proof.

□

Remark 2. Numerical simulation shows that for fixed p, q, i and j, an inner analogue-difference formula behaviors better than a outer analogue-difference formula, and among all inner analogue-difference formulas, a center analogue-difference formula behaviors best. Therefore, we adopt inner analogue-difference formulas.

3. ADM for second-order BVP

In this section, we use 2th-order 5-point analogue-difference formula (say ADM formula) to find numerical solution of BVP(1.1).

Rearrange BVP(1.1) as follows

$$\begin{cases} -u'' + q(t)u = f(t), a \le t \le b \\ u(a) = \alpha, \\ u(b) = \beta, \\ u''(a) = \alpha q(a) - f(a), \\ u''(b) = \beta q(b) - f(b), \end{cases} \tag{3.1}$$

and choose the 2th-order 5-point ADM formula

$$u_i'' = \frac{1}{h^2}\sum_{k=0}^{4} C_k u_{i-2+k}, (2 \le i \le N-2) \tag{3.2}$$

where $[C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4] = \begin{bmatrix} -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \end{bmatrix}$.

In order to assure the higher accuracy, we choose the 2th-order 6-point or 2nd-order 7-point ADM formula at the boundary points $t = a$ and $t = b$ by

$$u_0^{(p)} = \frac{1}{h^p}\sum_{k=0}^{q-1} C_k u_k, (q = 6 \ or \ 7) \tag{3.3}$$

and

$$u_N^{(p)} = \frac{1}{h^p}\sum_{k=0}^{q-1} C_k u_{N-q+1+k}, (q = 6 \text{ } or \text{ } 7) \tag{3.4}$$

respectively. We say the method (3.2) together with (3.3)($q = 6$) and (3.4)($q = 6$) ADM2, and (3.2) together with (3.3)($q = 7$) and (3.4)($q = 7$) ADM3. The value of C_k is shown in the table below.

Table 1. Coefficient of ADM formula where $q = 6$

$u_i^{(p)}$	C_0	C_1	C_2	C_3	C_4	C_5
u_0'	$-\frac{137}{60}$	5	-5	$\frac{10}{3}$	$-\frac{5}{4}$	$\frac{1}{5}$
u_0''	$\frac{15}{4}$	$-\frac{77}{6}$	$\frac{107}{6}$	-13	$\frac{61}{12}$	$-\frac{5}{6}$
u_N'	$-\frac{1}{5}$	$\frac{5}{4}$	$-\frac{10}{3}$	5	-5	$\frac{137}{60}$
u_N''	$-\frac{5}{6}$	$\frac{61}{12}$	-13	$\frac{107}{6}$	$-\frac{77}{6}$	$\frac{15}{4}$

Table 2. Coefficient of ADM formula where $q = 7$

$u_i^{(p)}$	C_0	C_1	C_2	C_3	C_4	C_5	C_6
u_0'	$-\frac{49}{20}$	6	$-\frac{15}{2}$	$\frac{20}{3}$	$-\frac{15}{4}$	$\frac{6}{5}$	$-\frac{1}{6}$
u_0''	$\frac{203}{45}$	$-\frac{87}{5}$	$\frac{117}{4}$	$-\frac{254}{9}$	$\frac{33}{2}$	$-\frac{27}{5}$	$\frac{137}{180}$
u_N'	$\frac{1}{6}$	$-\frac{6}{5}$	$\frac{15}{4}$	$-\frac{20}{3}$	$\frac{15}{2}$	-6	$\frac{49}{20}$
u_N''	$\frac{137}{180}$	$-\frac{27}{5}$	$\frac{33}{2}$	$-\frac{254}{9}$	$\frac{117}{4}$	$-\frac{87}{5}$	$\frac{203}{45}$

Remark 1. Note that if we take $q = p + 1$ in analogue-difference formula, then the above methods are reduced to the classical finite difference methods. This is an important reason that the method named as an analogue-difference method. Throughout this article, C_k in ADM formula is just the coefficient of the corresponding formula, which does not mean that it has the same value in different formula.

Example 3.1. Consider the second-order boundary value problem

$$\begin{cases} u'' + u' = t, t \in [0,1], \\ u(0) = 1, u'(1) + u(1) = \frac{3}{2}. \end{cases} \tag{3.5}$$

Rewrite (3.5) into the following problem

$$\begin{cases} L_h u_i = u_i'' + u_i' = t_i, (0 \le i \le N) \\ u(0) = u_0 = 1, \\ u'(0) = u_0' = 0, \\ u''(0) = u_0'' = 0, \\ u'(1) + u(1) = u_N' + u_N = \frac{3}{2}, \\ u''(1) + u'(1) = u_N'' + u_N' = 1. \end{cases} \tag{3.6}$$

The analytical solution (AS for short) of (3.5) is $u(t) = \frac{1}{2}t^2 - t + 2 - e^{-t}$. We apply three methods FDM, ADM2 and ADM3 to obtain the numerical solutions $u_1(t)$, $u_2(t)$ and $u_3(t)$, respectively. Take the step length $h = 0.1$. The numerical simulation results shall be shown by Table 3, Table 4 and Figure 1.

Table 3. The value of numerical solutions and analytical solution

i	t_i	analytical solution	numerical solution		
		$u(t_i)$	$u_1(t_i)$	$u_2(t_i)$	$u_3(t_i)$
0	0	1	1	1	1
1	0.1	1.00016258	1.00310212	1.00016407	1.00016259
2	0.2	1.00126925	1.00686119	1.00127197	1.00126927
3	0.3	1.00418178	1.01216701	1.00418562	1.00418183
4	0.4	1.00967995	1.01982466	1.00968483	1.00968003
5	0.5	1.01846934	1.03056253	1.01847516	1.01846943
6	0.6	1.03118836	1.04503965	1.03119505	1.03118845
7	0.7	1.04841470	1.06385228	1.04842219	1.04841478
8	0.8	1.07067104	1.08753990	1.07067926	1.07067111
9	0.9	1.09843034	1.11659061	1.09843921	1.09843040
10	1	1.13212056	1.15144601	1.13212972	1.13212061

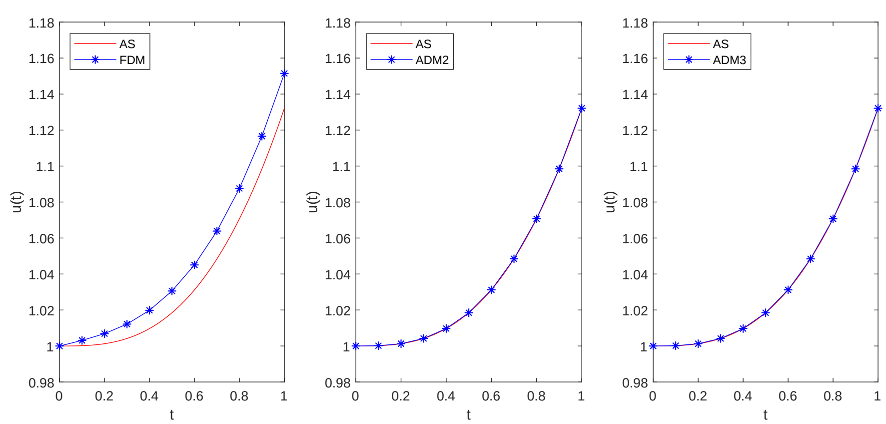

Figure 1. Numerical simulation image

Where AS (red curve) denotes the exact solution, and FDM, ADM2 and ADM3 (blue curves) denote the numerical solution curves corresponding to these three methods.

Table 4. Comparison result of three methods

i	t_i	truth value	absolute errors		
		AS	FDM	ADM2	ADM3
0	0	1	1.7764e-15	3.1086e-15	6.7724e-15
1	0.1	1.0002	2.9395e-03	1.4871e-06	3.7319e-09
2	0.2	1.0013	5.5919e-03	2.7190e-06	2.4887e-08
3	0.3	1.0042	7.9852e-03	3.8420e-06	5.5645e-08
4	0.4	1.0097	1.0145e-02	4.8732e-06	7.5791e-08
5	0.5	1.0185	1.2093e-02	5.8204e-06	8.5964e-08
6	0.6	1.0312	1.3851e-02	6.6903e-06	8.7757e-08
7	0.7	1.0484	1.5438e-02	7.4889e-06	8.2667e-08
8	0.8	1.0707	1.6869e-02	8.2203e-06	7.2031e-08
9	0.9	1.0984	1.8160e-02	8.8687e-06	5.7591e-08
10	1	1.1321	1.9325e-02	9.1590e-06	4.8905e-08

It is clear that, for second-order boundary value problem, ADM2 and ADM3 are better than FDM. In particular, ADM3 behaviours are much better than FDM. For higher-order boundary value problem, especially for those with odd order derivative, FDM has lower accuracy, and ADM can make up for this defect.

4. ADM for higher-order BVP

Consider pth-order linear boundary value problem

$$\begin{cases} Lu = u^{(p)} + \sum_{i=1}^{p} a_i(t)u^{(p-i)} = f(t), \\ B(u) = \eta, \end{cases} \tag{4.1}$$

where $a_i(t), f(t) \in C[a,b]$,

$$B(u) = \begin{pmatrix} B_1(u) \\ B_2(u) \\ \vdots \\ B_p(u) \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_p \end{pmatrix}. \tag{4.2}$$

$$B_i(u) = \sum_{k=0}^{m} \sum_{j=0}^{p-1} b_{ikj} u^{(j)}(\alpha_k), i = 1, 2, \cdots, p, \tag{4.3}$$

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \tag{4.4}$$

Theorem 4.1. *[3] If BVP(4.1) has a unique solution, then*

$$\det \begin{pmatrix} B_1(\alpha_1) & \cdots & B_1(\alpha_m) \\ \vdots & \ddots & \vdots \\ B_p(\alpha_1) & \cdots & B_p(\alpha_m) \end{pmatrix} \neq 0. \tag{4.5}$$

In this section, we always assume that problem (4.1) has a unique solution. We still use the partition

$$a = t_0 < t_1 < \cdots < t_i < \cdots < t_N = b, \tag{4.6}$$

where $t_i = a + ih$, step size $h = \frac{b-a}{N}$, and $\alpha_k \in \{t_i\}$, $k = 1, 2, \cdots, m$, $i = 1, 2, \cdots, N$.

Choosing the proper p and q in the formula (2.5), we can obtain difference equations corresponding to BVP (4.1) denoted by $A_1 U = F_1$ and the difference equations corresponding to boundary conditions $B(u) = \eta$ denoted by $A_2 U = F_2$, furthermore, the numerical solution of the problem (4.1) can be found by solving the following linear system:

$$AU = F, \tag{4.7}$$

where

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}, F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \tag{4.8}$$

Note that $rank(A) = rank(A_1) + rank(A_2)$, and $rank(A_1)$ is equals to the row number of matrix (A_1). It is easy to have the following results.

Theorem 4.2. *If the linear system (4.7) has a unique solution, then*

$$rank(A_1) + rank(A_2) = N + 1, \tag{4.9}$$

where $N = h(b - a)$.

Theorem 4.3. *If the highest derivative $u_i^{(p)}$ in equation (4.1) is expressed by pth-order q-point ADM formula, then $rank(A_1) = N - q + 1$.*

We take the following example to illustrate the performance of our method ADM.

Example 4.1. Consider the third-order BVP

$$\begin{cases} u''' + u' = 2e^t, t \in [0, \pi], \\ u(0) = 3, u'(0) = 2, u(\pi) + u''(\pi) = 2e^\pi + 1. \end{cases} \tag{4.10}$$

The problem (4.10) has an analytical solution

$$u(t) = \sqrt{2}\sin(t + \frac{\pi}{4}) + e^t + 1. \tag{4.11}$$

Rewrite the problem (4.10) as follows:

$$\begin{cases} u_i''' + u_i' = 2e^{t_i}, \\ u_0 = u(0) = 3, \\ u_0' = u'(0) = 2, \\ u_N + u_N'' = u(\pi) + u''(\pi) = 2e^{\pi} + 1, \\ u_0'' = u''(0) = 0, \\ u_0''' = u'''(0) = 0. \end{cases} \tag{4.12}$$

By Theorem 1, we choose the ADM formula

$$u_i''' = \frac{1}{h^3} \sum_{k=0}^{5} C_k u_{i-3+k} (3 \le i \le N-2), \tag{4.13}$$

where $[C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5] = \begin{bmatrix} \frac{1}{4} & -\frac{7}{4} & \frac{7}{2} & -\frac{5}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$. Similar to the second-order case, we take the formula at the boundary points as follows:

$$\begin{cases} u_i' = \frac{1}{h} \sum_{k=0}^{3} C_k u_{i-2+k}, (2 \le i \le N-1,), \\ u_0^{(p)} = \frac{1}{h^p} \sum_{k=0}^{5} C_k u_k, \\ u_N^{(p)} = \frac{1}{h^p} \sum_{k=0}^{5} C_k u_{N-q+1+k} \end{cases} \tag{4.14}$$

The value of each C_k is shown in the table below.

Table 5. Coefficient of difference formula $q = 6$

$u_i^{(p)}$	C_0	C_1	C_2	C_3	C_4	C_5
u_i'	$\frac{1}{6}$	-1	$\frac{1}{2}$	$\frac{1}{3}$	-	-
u_0'	$-\frac{11}{6}$	3	$-\frac{3}{2}$	$\frac{1}{3}$	-	-
u_0''	$\frac{35}{12}$	$-\frac{26}{3}$	$\frac{19}{2}$	$-\frac{14}{3}$	$\frac{11}{12}$	-
u_N''	$\frac{11}{12}$	$-\frac{14}{3}$	$\frac{19}{2}$	$-\frac{26}{3}$	$\frac{35}{12}$	-
u_0'''	$-\frac{17}{4}$	$\frac{71}{4}$	$-\frac{59}{2}$	$\frac{49}{2}$	$-\frac{41}{4}$	$\frac{7}{4}$

or

$$\begin{cases} u_i' = \frac{1}{h} \sum_{k=0}^{4} C_k u_{i-2+k}, (2 \le i \le N-2), \\ u_0^{(p)} = \frac{1}{h^p} \sum_{k=0}^{6} C_k u_k, \\ u_N^{(p)} = \frac{1}{h^p} \sum_{k=0}^{6} C_k u_{N-q+1+k}. \end{cases} \tag{4.15}$$

Table 6. Coefficient of difference formula $q = 7$

$u_i^{(p)}$	C_0	C_1	C_2	C_3	C_4	C_5	C_6
u_i'	$\frac{1}{12}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{1}{12}$	-	-
u_0'	$-\frac{25}{12}$	4	-3	$\frac{4}{3}$	$-\frac{1}{4}$	-	-
u_0''	$\frac{15}{4}$	$-\frac{77}{6}$	$\frac{107}{6}$	-13	$\frac{61}{12}$	$-\frac{5}{6}$	-
u_N''	$-\frac{5}{6}$	$\frac{61}{12}$	-13	$\frac{107}{6}$	$-\frac{77}{6}$	$\frac{15}{4}$	-
u_0'''	$-\frac{49}{8}$	29	$-\frac{461}{8}$	62	$-\frac{307}{8}$	13	$-\frac{15}{8}$

The method (4.13) with (4.14) is denoted by ADM4, and (4.13) with (4.15) is denoted by ADM5. Numerical simulation results are show as follows.

Table 7. The absolute error of FDM and ADM

i	t_i	truth value $u(t_i)$	absolute errors FDM	ADM4	ADM5
0	0	3	0	0	0
1	0.31416	3.62918	0.00086	0.00952	0.00167
2	0.62832	4.27126	0.82021	0.02413	0.00442
3	0.94248	4.96313	2.40801	0.04306	0.00831
4	1.25664	5.77366	4.62523	0.06487	0.01300
5	1.57080	6.81048	7.26145	0.08795	0.01793
6	1.88496	8.22810	10.04939	0.11066	0.02236
7	2.19911	10.23826	12.68681	0.13197	0.02539
8	2.51327	13.12405	14.86217	0.15018	0.02621
9	2.82743	17.25998	16.28102	0.16935	0.02305
10	3.14159	23.14069	16.69040	0.17322	0.01789

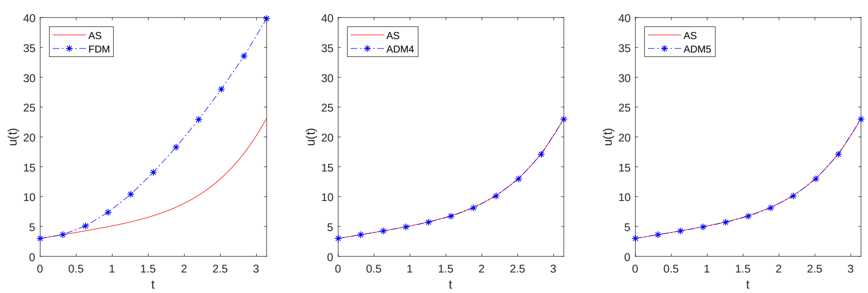

Figure 2. Numerical simulation for (4.10)

Where AS, FDM, ADM3 and ADM4 have the similar meanings as in Figure 1. Note that FDM behaviors so bad that we need to improve the accuracy of it by partition refinement (see Figure 3). In Table 8, we apply the method FDM by choosing the nodes $t = \frac{k}{5}\pi$ $(k = 1, 2, 3, 4)$ and different values of N. We can see clearly that the error of the ADM (taking $N = 10$, see last column of Table 7) is less than the error of the method FDM (taking $N = 1000$, see Table 8).

Table 8. The absolute error of refined FDM

N	$t = \frac{1}{5}\pi$	$t = \frac{2}{5}\pi$	$t = \frac{3}{5}\pi$	$t = \frac{4}{5}\pi$
100	0.18401	0.72040	1.39287	1.92217
200	0.09987	0.37358	0.70978	0.96876
300	0.06835	0.25199	0.47602	0.64735
400	0.05193	0.19008	0.35806	0.48605
500	0.04187	0.15259	0.28695	0.38910
600	0.03507	0.12745	0.23940	0.32438
700	0.03017	0.10942	0.20536	0.27813
800	0.02647	0.09586	0.17980	0.24341
900	0.02358	0.08529	0.15990	0.21641
1000	0.02126	0.07681	0.14396	0.19479

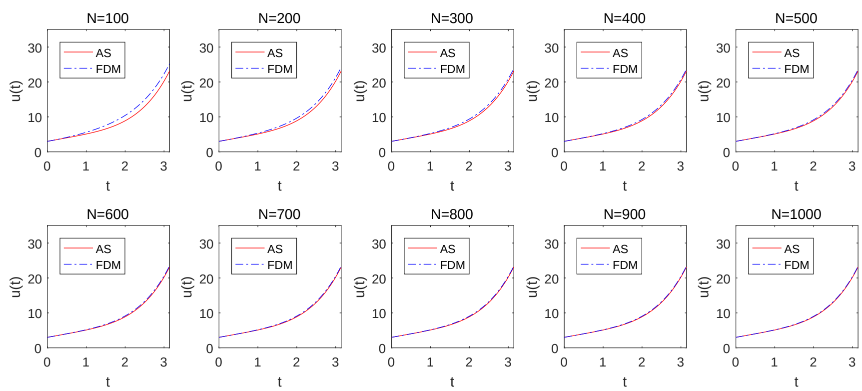

Figure 3. The simulation of refined FDM for different N

Next, we apply our proposed method ADM to the models of [1] and compare ADM with the methods of [1].

Example 4.2. Consider fifth-order BVP,

$$\begin{cases} u^{(5)} + u = 4e^t cos(t) + 2e^t(1 - sin(t)) + 5e^t sin(t), 0 \le t \le 1 \\ u(0) = 1, \\ u(1) = e(1 - sin(1)), \\ u'(0) = 0, \\ u'(1) = -e(cos(1) + sin(1) - 1), \\ u''(0) = -1. \end{cases} \tag{4.16}$$

The analytical solution of (4.16) is $u(t) = e^t(1 - sin(t))$.

For this fifth-order boundary value problem, we take the $5th$-order 8-point ADM formula to deal with the equation, and 5th-order 10-point ADM formula to deal with boundary points. We choose the ADM formula

$$u_i^{(5)} = \frac{1}{h^5}\sum_{k=0}^{7} C_k u_{i-3+k}, (3 \le i \le N-4), \tag{4.17}$$

where
$[C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_6 \quad C_7]$
$= \left[-\frac{1}{6} \quad -\frac{1}{3} \quad \frac{9}{2} \quad -\frac{35}{3} \quad \frac{85}{6} \quad -9 \quad \frac{17}{6} \quad -\frac{1}{3}\right].$
Specific information of coefficient selection in ADM formula for boundary points and comparison results of the methods presented in Table 9-Table 11 and Figure 4.

Table 9. Coefficient of ADM formula for (4.16)

$u_i^{(p)}$	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
u_0'	$-\frac{1836}{649}$	9	-18	28	$-\frac{63}{2}$	$\frac{126}{5}$	-14	$\frac{36}{7}$	$-\frac{9}{8}$	$\frac{1}{9}$
u_0''	$\frac{2553}{395}$	$-\frac{4609}{140}$	$\frac{5869}{70}$	$-\frac{6289}{45}$	$\frac{6499}{40}$	$-\frac{265}{2}$	$\frac{6709}{90}$	$-\frac{967}{35}$	$\frac{3407}{560}$	$-\frac{761}{1260}$
$u_0^{(5)}$	$-\frac{3013}{144}$	$\frac{7807}{48}$	$-\frac{6787}{12}$	$\frac{13873}{12}$	$-\frac{36769}{24}$	$\frac{32773}{24}$	$-\frac{9823}{12}$	$\frac{3817}{12}$	$-\frac{3487}{48}$	$\frac{1069}{144}$
u_N'	$-\frac{1}{9}$	$\frac{9}{8}$	$-\frac{36}{7}$	14	$-\frac{126}{5}$	$\frac{63}{2}$	-28	18	-9	$\frac{1836}{649}$
$u_N^{(5)}$	$-\frac{1069}{144}$	$\frac{3487}{48}$	$-\frac{3817}{12}$	$\frac{9823}{12}$	$-\frac{32773}{24}$	$\frac{36769}{24}$	$-\frac{13873}{12}$	$\frac{6787}{12}$	$-\frac{7807}{48}$	$\frac{3013}{144}$

Table 10. Analytical solution versus numerical solutions

N	t	AS	ADM	AST	IPT	SPQ
0	0	1	1	0.99023494	0.98016020	0.98322052
1	0.1	0.99483793	0.99483792	0.98453007	0.97395398	0.97709352
2	0.2	0.97874749	0.97874743	0.96831848	0.95767165	0.96076648
3	0.3	0.95094825	0.95094809	0.94111319	0.93112183	0.93396683
4	0.4	0.91088080	0.91088049	0.90249353	0.89402221	0.89637594
5	0.5	0.85828219	0.85828174	0.85218070	0.84607518	0.84770479
6	0.6	0.79327313	0.79327261	0.79012504	0.78705526	0.78778124
7	0.7	0.71645760	0.71645710	0.71660571	0.71690878	0.71664979
8	0.8	0.62903559	0.62903525	0.63234352	0.63586679	0.63468444
9	0.9	0.53292981	0.53292968	0.53862756	0.54457175	0.54271540
10	1	0.43092654	0.43092654	0.43745614	0.44421835	0.44216991

Table 11. Analytical solution and the absolute errors

N	t	AS	ADM	AST	IPT	SPQ
0	0	1	7.8493e-14	9.7651e-03	1.9840e-02	1.6779e-02
1	0.1	0.99483793	8.9059e-09	1.0308e-02	2.0884e-02	1.7744e-02
2	0.2	0.97874749	6.0595e-08	1.0429e-02	2.1076e-02	1.7981e-02
3	0.3	0.95094825	1.6763e-07	9.8351e-03	1.9826e-02	1.6981e-02
4	0.4	0.91088080	3.1085e-07	8.3873e-03	1.6859e-02	1.4505e-02
5	0.5	0.85828219	4.4701e-07	6.1015e-03	1.2207e-02	1.0577e-02
6	0.6	0.79327313	5.2242e-07	3.1481e-03	6.2179e-03	5.4919e-03
7	0.7	0.71645760	4.9177e-07	1.4812e-04	4.5118e-04	1.9219e-04
8	0.8	0.62903559	3.4286e-07	3.3079e-03	6.8312e-03	5.6488e-03
9	0.9	0.53292981	1.2780e-07	5.6977e-03	1.1642e-02	9.7856e-03
10	1	0.43092654	0	6.5296e-03	1.3292e-02	1.1243e-02

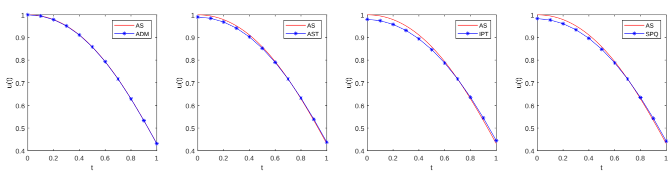

Figure 4. Numerical simulation image of four methods

Based on the above simulations, the image deviation of all methods in [1] caused by the error is obvious. The accuracy of all methods of [1] has been improved significantly. Similar to Example 4.2, we may use ADM to obtain the numerical solutions of another two models of [1], which also show that our proposed method ADM is more accurate than all three methods of [1]. Here, we omit the detailed information.

5. Conclusion

The paper established a method so called an analogue-difference method (ADM) to compute the numerical solutions for boundary value problems of higher-order differential equations, which performs much better than the classical finite difference method (FDM), especially FDE for the problems with odd-order derivative. This method can be a supplementary one to FDM. Numerical examples and results illustrated the efficiency and applicability of the method, which also shows that the proposed method has higher accuracy than those of [1] for induction motor models. From the formula (2.5) and Theorem 1, one can see that the number q is bigger, the truncation error is less. However, one can not use ADM to improve the order of truncation error unlimitedly. This problem and the stability of ADM shall be considered latter.

References

[1] I. Ahmad, F. Ahmad and M. A. Z. Raja, *Intelligent computing to solve fifth-order boundary value problem arising in induction motor models*, Neural Computing and Applications, 2018, 29, 49–466.

[2] D. Arslan, *A new second-order difference approximation for nonlocal boundary value problem with boundary layers*, Mathematical Modelling and Analysis, 2020, 25(2), 257–270.

[3] W. Ge, *Nonlinear Boundary Value Problems for Nonlinear Ordinary Differential Equations*, Science Press, Beijing, 2007.

[4] R. Li and Z. He, *Numerical Method for Differential Equations*, East China University of Science and Technology Press, Shanghai, 2005.

[5] J. Lu and Z. Guan, *Numerical Methods for Partial Differential Equations* (Second Edition), Qinghua University Publishing House, Beijing, 2004.

[6] P. K. Pandey, *A numerical technique for the solution of general eighth order boundary value problems: a finite difference method*, Ural Mathematical Journal, 2018, 4(1), 56–62.

[7] R. D. Richtmyer and K.W. Morton, *Difference Methods for Initial Value Problems*, Interscience Publishers, New York, 1967.

[8] M. Shalini and M. Sathyavathy, *Solving graphically sine wave second order boundary value problem using Laplace transform and finite difference method*, International Journal of Mathematics Trends and Technology, 2019, 65(1), 57–61.

[9] J. W. Thomas, *Numerical Partial Differential Equations* (TAM, Volume 22), Springer-Verlag, New York, 1995.

[10] Y. Xu, J. Mu and X. Liu, *Numerical solution for second-order periodic-integrable boundary value problems*, Far East Journal of Applied Mathematics, 2015, 92(2), 153–161.

[11] J. Zeng, Y. Yang and L. Guan, *Numerical Solutions of Differential Equations*, Chinese Science Publishing House, Beijing, 2011.

[12] L. Zhou, W. Jiang and Q. Li, *Eigenvalues of fourth-order singular Sturm-Liouville boundary value problems*, Journal of Nonlinear Modeling and Analysis, 2020, 2(4), 485–493.