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Analysis of a Kind of Stochastic Dynamics Model with Nonlinear Function*

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Abstract In this paper, we establish stochastic differential equations on the basis of a nonlinear deterministic model and study the global dynamics. For the deterministic model, we show that the basic reproduction number $\Re_0$ determines whether there is an endemic outbreak or not: if $\Re_0 < 1$, the disease dies out; while if $\Re_0 > 1$, the disease persists. For the stochastic model, we provide analytic results regarding the stochastic boundedness, perturbation, permanence and extinction. Finally, some numerical examples are carried out to confirm the analytical results. One of the most interesting findings is that stochastic fluctuations introduced in our stochastic model can suppress disease outbreak, which can provide us some useful control strategies to regulate disease dynamics.

Keywords Nonlinear incidence, Stochastic differential equation, Stationary distribution, Permanence, Extinction.

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1. Introduction

Bilinear and standard incidences have been frequently used in many epidemic models [21]. Several different forms of incidences have been proposed by some researchers. Let $S(t)$ and $I(t)$ be the numbers of susceptible and infective individuals at time t, respectively. Capasso and Serio [4] introduced a saturated incidence $Sf(I)$ into epidemic models to study of the cholera epidemic spread in Bari in 1973. The nonlinear incidences of the forms $\beta I^p S^q$ and $\beta I^p S/(1+aI^q)$ were proposed by Liu et al. [19]. Epidemic models with the incidence $\beta I^p S^q$ had also been studied in [14]. An SEIRS epidemic model with the saturation incidence $\beta SI/(1+aS)$ was examined in [8]. Epidemic models with the incidence $\beta I^p S/(1+aI^q)$ had been investigated in [28]. The nonlinear incidences of the form $\beta(I+\upsilon I^p)S$ proposed by van den Driessche and Watmough [7] was used in [29]. The more general forms of nonlinear incidence were considered in [27, 30]. In view of the fact that the transmission mechanism of many infectious diseases is not fully known, increasing attention has been paid to infectious disease models with nonlinear incidence in recent years. In [13], the global stability of a class of nonlinear epidemic models is considered.

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Stochastic models could be a more appropriate way of modeling epidemics in many circumstances [3, 11, 12, 15–18, 22, 25, 26, 31, 32]. For example, stochastic models are able to take care of randomness of infectious contacts occurring in the latent and infectious periods [17, 18, 26, 31]. It also has been showed that some stochastic epidemic models can provide an additional degree of realism in comparison with their deterministic counterparts [2, 6, 9]. Many realistic stochastic epidemic models can be derived based on their deterministic formulations. Allen [1] provided a great introduction to the methods of the methods of derivation for various types of stochastic models including stochastic differential equation (SDE) epidemic models. Liu et al. [16] established a deterministic model of nonlinear incidence rate, and studied the global stability of the model by the basic reproduction number of the model. Then, a stochastic model is formulated on the basis of the deterministic model, and the perturbation, persistence and extinction of the stochastic model in the deterministic model are studied. Britton [2] gave an excellent survey on SDE epidemic models which presented the exact and asymptotic properties of a simple stochastic epidemic model, and was illustrated by studying effects of vaccination and inference procedures for important parameters such as the basic reproduction number and the critical vaccination coverage. Gray [9] formulated a SDE SIS epidemic model, and proved that the model has a unique global positive solution and established conditions for extinction and persistence of infectious individuals.

There are different possible approaches to including random effects in the model, and both of which are from a biological and mathematical perspective [20]. The general stochastic differential equation SIRS model introduced in this manuscript adopts the approach by Mao et al. [23], which has been pursued in [3, 11, 12, 15–18, 25, 26, 31, 32], and assume that the parameters involved in the model always fluctuate around some average value due to continuous fluctuation in the environment. Following their approach, we will focus on a SDE SIR model with nonlinear incidence rate.

The rest of this paper is organized as follows: In Section 2, we find deterministic models from the literature and describe the results of their deterministic models. Considering the methods mentioned above, a stochastic model is formulated on the basis of deterministic model. In Section 3, we first prove the existence of global positive solutions for stochastic models. Secondly, we prove the extinction of diseases. Thirdly, we prove the perturbation of disease-free equilibrium points and the existence of stationary distribution for stochastic models. Finally, we prove the persistence in mean. In Section 4, the numerical examples are carried out to illustrate the main theoretical results. In Sections 5, we provide a brief discussion and the summary of the main results.

2. Model description

2.1. The deterministic SIR model

In [13], Li et al. formulated a nonlinear deterministic epidemiological model in which the nonlinear incidence $Sf(I)$. The authors assume that $f(I)$ is a real locally Lipschitz function at least on $[0, +\infty)$ which satisfies the following conditions:

(i) $f(0) = 0, f(I) > 0$ for $I > 0$;

(ii) $f(I)/I$ is continuous and monotone nonincreasing for $I > 0$, and $\lim\limits_{I \to 0^+} f(I)/I$

exists, denoted by $\beta(0 < \beta < +\infty)$;

(iii) $\int_{0^+}^{1} 1/f(u)\mathrm{d}u = +\infty$. Further, establish an SIR epidemic model with the nonlinear incidence $Sf(I)$ as follows:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu A - \mu S - Sf(I), \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = Sf(I) - (\mu + \gamma + \alpha)I, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R, \end{cases} \tag{2.1}$$

where function $f(I)$ satisfies above conditions. Moreover, $S = S(t)$, $I = I(t)$ and $R = R(t)$ represent the numbers of individuals in the susceptible, infected and removed compartments at time t, respectively. μ denotes the per capita natural death rate, μA denotes the recruitment of susceptible individuals, γ denotes the recovery rate of an infected individual and α denotes the per capita disease-induced mortality rate. The basic reproduction number is $\Re_0 = \frac{\beta A}{\mu+\gamma+\alpha}$. Its disease-free equilibrium point is $E_0 = (A, 0, 0)$ and endemic equilibrium point is $E_* = (S_*, I_*, R_*)$, where $S_* = \frac{(\mu+\gamma+\alpha)I_*}{f(I_*)}, R_* = \frac{\gamma I_*}{\mu}, I_* \in (0, A)$.

For the model, the authors obtain the following results (see [13]). The disease-free equilibrium is globally stable in the feasible region as the basic reproduction number is less than or equal to unity, and the endemic equilibrium is globally stable in the feasible region as the basic reproduction number is greater than unity.

2.2. Stochastic differential equation SIR model

Now, we turn to a continuous time SIR model, which takes random effects into account. In SIR models, the recovery rate of an infected individual γ is one of the key parameters to disease transmission. May [24] pointed out that all the parameters involved in the population model exhibit random fluctuation as the factors controlling them are not constant. Further, in the real situation, the recovery rate of an infected individual γ always fluctuate around some average value due to continuous fluctuation in the environment. Consequently, many researchers introduced stochastic perturbations into deterministic models to reveal the effects of environmental noise on the epidemic models [12, 17, 18, 26, 32]. As an extension of system (2.1), we introduce stochastic perturbations into system (2.1). Then, we obtain the following SIR epidemic model with nonlinear incidence:

$$\begin{cases} \mathrm{d}S = (\mu A - \mu S - Sf(I))\,\mathrm{d}t - \sigma_1 S\mathrm{d}B_1(t), \\ \mathrm{d}I = (Sf(I) - (\mu + \gamma + \alpha)I)\,\mathrm{d}t - \sigma_2 I\mathrm{d}B_2(t) - \sigma_4 I\mathrm{d}B_4(t), \\ \mathrm{d}R = (\gamma I - \mu R)\,\mathrm{d}t - \sigma_3 R\mathrm{d}B_3(t) + \sigma_4 I\mathrm{d}B_4(t), \end{cases} \tag{2.2}$$

where $B_i(t)$ represents a standard Brownian motion with $B_i(0) = 0$ and $\sigma_i > 0$ denotes the intensity of the white noise, $i = 1, 2, 3, 4$. Specifically, for the introduction of stochastic terms from (2.1), there are two cases, $x_i' = f(x_i) \to dx_i = f(x_i)dt - \sigma_i x_i dB_i(t), (i = 1, 2, 3)$ and $\gamma \to \gamma + \sigma_4 B_4'(t)$.

3. The stochastic model (2.2)

3.1. Preliminaries

First, we introduce some lemmas and notations, which will be used in the following parts. For the d-dimensional stochastic differential equation can be expressed as follows:

$$\mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}B(t), \tag{3.1}$$

where $f(t, X(t))$ is a function in $\mathbb{R}^d$ defined in $[t_0, +\infty] \times \mathbb{R}^d$ and $g(t, X(t))$ is a $d \times m$ matrix, f, g are locally Lipschitz functions in $X(t)$. $B(t)$ is an m-dimensional standard Brownian motion defined on the above probability space. The differential operator L of system (3.1) is defined by [23].

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(x,t)g(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.2}$$

If L acts on a function $V \in \mathbb{C}^{2,1}(\mathbb{R}^d \times [t_0, +\infty]; \mathbb{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[g^T(x,t)g(x,t)], \tag{3.3}$$

where $V_t(x,t) = \frac{\partial V}{\partial t}, V_x(x,t) = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}), V_{xx} = (\frac{\partial^2 V}{\partial x_1 \partial x_j})_{d\times d}$. In view of Itô's formula, if $x(t) \in \mathbb{R}^d$, then $\mathrm{d}V(x,t) = LV(x,t)\mathrm{d}t + V_x(x,t)g(x,t)\mathrm{d}B(t)$.

Lemma 3.1. *[23] Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^n_+$ described by the following stochastic differential equation:*

$$dX(t) = b(X)dt + \sum_{r=1}^{k}\sigma_r(X)dB_r(t). \tag{3.4}$$

The diffusion matrix is defined as follows:

$$A(X) = (a_{ij}(x)), (a_{ij}(x)) = \sum_{r=1}^{k}\sigma_r^i(x)\sigma_r^j(x). \tag{3.5}$$

Lemma 3.2. *[10] The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$, if there exists a bounded domain $U \subset E_d$ with regular boundary Γ and (i) there is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant M|\xi|^2, x \in U, \xi \in \mathbb{R}^d$. (ii) there exists a nonnegative $\mathbb{C}^2$-function V such that LV is negative for any $E_d \backslash U$. Then,*

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_d} f(x)\pi(dx)\right\} = 1 \tag{3.6}$$

for all $x \in E_d$, where $f(\cdot)$ is a function integrable with respect to the measure π.

Lemma 3.3. *[33] Let $A(t)$ and $U(t)$ be two continuous adapted increasing process on $t \geqslant 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let $X(0)$ be a nonnegative F_0-measurable random variable such that $E(X(0)) < \infty$. Define $X(t) = X(0) + A(t) - U(t) + M(t)$ for all $t \geqslant 0$. If $X(t)$ is nonnegative, then $\lim_{t\to\infty} A(t) < \infty$ implies $\lim_{t\to\infty} U(t) < \infty, \lim_{t\to\infty} X(t) < \infty$ and $-\infty < \lim_{t\to\infty} M(t) < \infty$ hold with probability one.*

Lemma 3.4. *[5, 33] Let $M(t),\ t \geqslant 0$ be a local martingale vanishing at time 0 and define*

$$\rho_M(t) := \int_o^t \frac{d\,[M,M]\,(s)}{(1+s)^2}, t \geqslant 0, \tag{3.7}$$

where $[M,M]\,(t)$ is Meyers angle bracket process. Then, $\lim\limits_{t\to\infty} \frac{M(t)}{t} = 0$ a.s. provided that $\lim\limits_{t\to\infty} \rho_M(t) < \infty$ a.s.

Lemma 3.5. *For the solution $(S(t), I(t), R(t))$ of system (2.2) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, we have*

$$\limsup_{t\to\infty} \left(S(t) + I(t) + R(t)\right) < \infty \ a.s. \tag{3.8}$$

Moreover,

$$\begin{aligned} &\lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma_i S(\theta)\, dB_i(\theta) = 0,\ \lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma_i I(\theta)\, dB_i(\theta) = 0,\\ &\lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma_i R(\theta)\, dB_i(\theta) = 0,\ \lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma_i\, dB_i(\theta) = 0,\ (i = 1,2,3,4)\ a.s. \end{aligned} \tag{3.9}$$

Proof. From (2.2), we get

$$\mathrm{d}(S+I+R) = [\mu A - \mu(S+I+R) - \alpha I]\,\mathrm{d}t - \sigma_1 S \mathrm{d}B_1(t) - \sigma_2 I \mathrm{d}B_2(t) - \sigma_3 R \mathrm{d}B_3(t). \tag{3.10}$$

This equation has the solution of the form

$$\begin{aligned} &S(t) + I(t) + R(t)\\ &= A + \left[(S(0)+I(0)+R(0)) - A\right] e^{-\mu t} - \alpha \int_0^t e^{-\mu(t-s)} I(s)\mathrm{d}s + M(t)\\ &\leqslant A + \left[(S(0)+I(0)+R(0)) - A\right] e^{-\mu t} + M(t), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} M(t) = &-\sigma_1 \int_0^t e^{-\mu(t-s)} S(s) dB_1(s) - \sigma_2 \int_0^t e^{-\mu(t-s)} I(s) dB_2(s)\\ &-\sigma_3 \int_0^t e^{-\mu(t-s)} R(s) dB_3(s) \end{aligned}$$

is a continuous local martingale with $M(0) = 0$ a.s. Define

$$X(t) = X(0) + A(t) - U(t) + M(t), \tag{3.12}$$

with $X(0) = (S(0) + I(0) + R(0)),\ A(t) = A(1 - e^{-\mu t})$ and $U(t) = (S(0) + I(0) + R(0))(1 - e^{-\mu t})$ for all $t \geqslant 0$. Since the stochastic comparison theorem, $S(t) + I(t) + R(t) \leqslant X(t)$ a.s. It is easy to check that $A(t)$ and $U(t)$ are continuous adapted increasing processes on $t \geqslant 0$ with $A(0) = U(t) = 0$. By Lemma 3.3, we have that $\lim_{t\to\infty} X(t) < \infty$ a.s. Thus, we complete the proof of (3.8).

For the sake of convenience, we denote

$$
\begin{aligned}
&M_1(t)=\sigma_1\int_0^t S(s)\mathrm{d}B_1(s), M_2(t)=\sigma_2\int_0^t S(s)\mathrm{d}B_2(s), M_3(t)=\sigma_4\int_0^t S(s)\mathrm{d}B_4(s),\\
&M_4(t)=\sigma_2\int_0^t I(s)\mathrm{d}B_2(s), M_5(t)=\sigma_4\int_0^t I(s)\mathrm{d}B_4(s),\\
&M_6(t)=\sigma_2\int_0^t \mathrm{d}B_2(s), M_7(t)=\sigma_4\int_0^t \mathrm{d}B_4(s).
\end{aligned}
\tag{3.13}
$$

Compute that $[M,M](t)=\sigma_1^2\int_0^t S^2(s)\mathrm{d}s$ and by (3.8), we obtain

$$
\lim_{t\to\infty}\rho_1(t)=\lim_{t\to\infty}\int_0^t\frac{\sigma_1^2S^2(s)\mathrm{d}s}{(1+s)^2}\leqslant\sigma_1^2\sup_{t\geqslant 0}\left\{S^2(t)\right\}<\infty. \tag{3.14}
$$

Then, by Lemma 3.4, $\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_1 S(s)\mathrm{d}B_1(s)=0$. The left can be proved similarly. The proof is complete. □

Lemma 3.6. *For the solution $(S(t),I(t),R(t))$ of system (2.2) with any initial value $(S(0),I(0),R(0))\in\mathbb{R}_+^3$, we have*

$$
\limsup_{t\to\infty}\langle S(t)+I(t)+R(t)\rangle\leqslant A \ a.s. \tag{3.15}
$$

Proof. We set

$$
\begin{aligned}
M_a(t)&=\int_0^t S(s)\mathrm{d}B_1(s),\ M_a^*(t)=\int_0^t e^{-\mu(t-s)}S(s)\mathrm{d}B_1(s),\\
M_b(t)&=\int_0^t I(s)\mathrm{d}B_2(s),\ M_b^*(t)=\int_0^t e^{-\mu(t-s)}I(s)\mathrm{d}B_2(s),\\
M_c(t)&=\int_0^t R(s)\mathrm{d}B_3(s),\ M_c^*(t)=\int_0^t e^{-\mu(t-s)}R(s)\mathrm{d}B_3(s).
\end{aligned}
\tag{3.16}
$$

By Lemma 3.5, we have

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}M_a(t)=0,\ \lim_{t\to\infty}\frac{1}{t}M_a^*(t)=0,\\
&\lim_{t\to\infty}\frac{1}{t}M_b(t)=0,\ \lim_{t\to\infty}\frac{1}{t}M_b^*(t)=0,\\
&\lim_{t\to\infty}\frac{1}{t}M_c(t)=0,\ \lim_{t\to\infty}\frac{1}{t}M_c^*(t)=0 \text{ a.s.}
\end{aligned}
\tag{3.17}
$$

From (3.11), since

$$
\begin{aligned}
\langle M(t)\rangle =& -\frac{\sigma_1}{t}\int_0^t\int_0^s e^{-\mu(s-u)}S(u)\mathrm{d}B_1(u)\mathrm{d}s\\
&-\frac{\sigma_2}{t}\int_0^t\int_0^s e^{-\mu(s-u)}I(u)\mathrm{d}B_2(u)\mathrm{d}s\\
&-\frac{\sigma_3}{t}\int_0^t\int_0^s e^{-\mu(s-u)}R(u)\mathrm{d}B_3(u)\mathrm{d}s\\
=& -\frac{\sigma_1}{t}\left(\int_0^t S(u)\mathrm{d}B_1(u)-\int_0^t e^{-\mu(t-u)}S(u)\mathrm{d}B_1(u)\right)\\
&-\frac{\sigma_2}{t}\left(\int_0^t I(u)\mathrm{d}B_2(u)-\int_0^t e^{-\mu(t-u)}I(u)\mathrm{d}B_2(u)\right)\\
&-\frac{\sigma_3}{t}\left(\int_0^t R(u)\mathrm{d}B_3(u)-\int_0^t e^{-\mu(t-u)}R(u)\mathrm{d}B_3(u)\right).
\end{aligned}
\tag{3.18}
$$

By (3.17), we obtain $\lim_{t\to\infty}\langle M(t)\rangle=0$. Since

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\int_0^t [S(0)+I(0)+R(0)-A]e^{-\mu s}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{1}{\mu t}\left\{[S(0)+I(0)+R(0)-A]\left(1-e^{-\mu t}\right)\right\}=0,
\end{aligned}
\tag{3.19}
$$

from (3.11). By (3.11), (3.18) and (3.19), we obtain

$$\limsup_{t\to\infty}\langle S(t)+I(t)+R(t)\rangle\leqslant A \text{ a.s.}$$

This completes the proof. □

3.2. Existence of the global and positive solution

In this section, by using Lyapunov method, we show the solution of system (2.2) is positive and global.

Theorem 3.1. *For any given initial value $(S(0),I(0),R(0))\in\mathbb{R}_+^3$, there exists a unique positive solution $(S(t),I(t),R(t))$ to system (2.2) on $t\geqslant 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one. That is to say, $(S(t),I(t),R(t))\in\mathbb{R}_+^3$ for all $t\geqslant 0$ almost definite.*

Proof. Since the coefficients of system (2.2) are locally Lipschitz continuous, for any given initial value $(S(0),I(0),R(0))\in\mathbb{R}_+^3$, there exists a unique local solution $(S(t),I(t),R(t))$ on $t\in[0,\tau_e)$, where τ_e denotes the explosion time [6]. To verify that this solution is global, we only need to prove $\tau_e=+\infty$ a.s.

Let $k_0>0$ be enough large such that each component of $(S(0),I(0),R(0))$ is no large than k_0. For each integer $k>k_0$, define the stopping time

$$\tau_k=\inf\left\{t\in[0,\tau_e): S(t)\geqslant k, I(t)\geqslant k, R(t)\geqslant k\right\},$$

where throughout this paper we set $\inf\emptyset=+\infty$. Obviously, τ_k is increasing as $k\to\infty$. Set $\tau_\infty=\lim_{k\to\infty}\tau_k$then we can get $\tau_\infty\leqslant\tau_e$ a.s.

Define a $\mathbb{C}^2$-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$V(X) = S + I + R. \tag{3.20}$$

By Itô's formula, we get

$$\begin{aligned} \mathrm{d}V(X) &= (\mu A - \mu S - \mu I - \alpha I - \mu R)\,\mathrm{d}t - \sigma_1 S \mathrm{d}B_1(t) - \sigma_2 I \mathrm{d}B_2(t) - \sigma_3 R \mathrm{d}B_3(t) \\ &\triangleq LV \mathrm{d}t - \sigma_1 S \mathrm{d}B_1(t) - \sigma_2 I \mathrm{d}B_2(t) - \sigma_3 R \mathrm{d}B_3(t), \end{aligned} \tag{3.21}$$

where

$$LV(t) = \mu A - \mu S - \mu I - \alpha I - \mu R \leqslant \mu A \triangleq K.$$

For any $k \geqslant k_0$, there exists $T > 0$ such that $\tau_k \in (0, T \wedge \tau_k]$. By the generalized Itô's formula, for any $t \in (0, T \wedge \tau_k]$, we have

$$\begin{aligned} EV(X(T \wedge \tau_k)) &= EV(X(S(0), I(0), R(0))) + E\int_0^{T\wedge\tau_k} LV(X(s))ds \\ &\leqslant EV(X(S(0), I(0), R(0))) + KT. \end{aligned} \tag{3.22}$$

Let $k \to \infty$, then $t \to \infty$, it follows that $\lim_{k\to\infty} P(\tau_k \leqslant T) = 0$. Therefore, $P(\tau_\infty \leqslant T) = 0$. Since $T > 0$ is arbitrary, it results in

$$P(\tau_\infty < \infty) = 0,\ P(\tau_\infty = \infty) = 1. \tag{3.23}$$

Consequently, the proof of Theorem 3.1 is completed. □

3.3. Extinction of the disease

In deterministic model (2.1), the value of the basic reproduction number $\Re_0$ guarantees persistence or extinction of the disease. If $\Re_0 \leqslant 1$, then the disease will become extinct; if $\Re_0 > 1$, then the disease will be persistent. However, we will verify that if the white noise is large enough. Then, the disease will die out, although it may be persistent for deterministic case. The following theorem establishes a criterion for the extinction of a disease.

Define a parameter as follows:

$$\Re^* = \frac{2\beta A(\mu+\gamma+\alpha)^2}{\left\{\left[(\mu+\gamma+\alpha)^2\mu + \frac{1}{2}\sigma_3^2(\mu+\gamma+\alpha)^2\right] \wedge \left[\frac{1}{2}\sigma_2^2\gamma^2 + \frac{1}{2}\sigma_4^2(\mu+\alpha)^2\right]\right\}}. \tag{3.24}$$

Theorem 3.2. *Let $(S(t),\ I(t),\ R(t))$ be the solution of system (2.2) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. If $\Re^* < 1$, then the solution $(S(t), I(t), R(t))$ of system (2.2) has the following property*

$$\lim_{t\to\infty} \langle S \rangle_t = A \ a.s. \tag{3.25}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \ln[\gamma I + (\mu+\gamma+\alpha)R] \leqslant \beta A - \frac{1}{2(\mu+\gamma+\alpha)^2}\, Q < 0 \ a.s., \tag{3.26}$$

where

$$Q = \left\{\left[(\mu+\gamma+\alpha)^2\mu + \frac{1}{2}\sigma_3^2(\mu+\gamma+\alpha)^2\right] \wedge \left[\frac{1}{2}\sigma_2^2\gamma^2 + \frac{1}{2}\sigma_4^2(\mu+\alpha)^2\right]\right\}.$$

Proof. Let $Q = \gamma I + (\mu+\gamma+\alpha)R$. An application of the Itô's formula, we have

$$
\begin{aligned}
&\mathrm{d}\ln Q \\
&= \left[\frac{\gamma S f(I) - \mu(\mu+\gamma+\alpha)R}{\gamma I + (\mu+\gamma+\alpha)R} - \frac{1}{2}\frac{\sigma_2^2 I^2\gamma^2 + \sigma_3^2 R^2(\mu+\gamma+\alpha)^2 + \sigma_4^2 I^2(\mu+\alpha)^2}{(\gamma I + (\mu+\gamma+\alpha)R)^2}\right]\mathrm{d}t \\
&\quad - \frac{\sigma_2\gamma I}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_2(t) - \frac{\sigma_3 R(\mu+\gamma+\alpha)}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_3(t) + \frac{\sigma_4 I(\mu+\alpha)}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_4(t) \\
&\leqslant \frac{Sf(I)}{I}\mathrm{d}t - \frac{1}{(\gamma I + (\mu+\gamma+\alpha)R)^2}\left\{\left[(\mu+\gamma+\alpha)^2\mu + \frac{1}{2}\sigma_3^2(\mu+\gamma+\alpha)^2\right]R^2\right. \\
&\quad \left. + \left[\frac{1}{2}\sigma_2^2\gamma^2 + \frac{1}{2}\sigma_4^2(\mu+\alpha)^2\right]I^2\right\}\mathrm{d}t - \frac{\sigma_2\gamma I}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_2(t) \\
&\quad - \frac{\sigma_3 R(\mu+\gamma+\alpha)}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_3(t) + \frac{\sigma_4 I(\mu+\alpha)}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_4(t) \\
&\leqslant \beta S\mathrm{d}t - \frac{\sigma_2\gamma I}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_2(t) \\
&\quad - \frac{\sigma_3 R(\mu+\gamma+\alpha)}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_3(t) + \frac{\sigma_4 I(\mu+\alpha)}{\gamma I + (\mu+\gamma+\alpha)R}\mathrm{d}B_4(t) \\
&\quad - \frac{1}{2(\mu+\gamma+\alpha)^2}\left\{\left[(\mu+\gamma+\alpha)^2\mu + \frac{1}{2}\sigma_3^2(\mu+\gamma+\alpha)^2\right] \wedge \left[\frac{1}{2}\sigma_2^2\gamma^2 + \frac{1}{2}\sigma_4^2(\mu+\alpha)^2\right]\right\}\mathrm{d}t,
\end{aligned}
\tag{3.27}
$$

on account of

$$
\begin{aligned}
&Q - \frac{1}{(ax + (a+b)y)^2}\left\{cx^2 + dy^2\right\} \leqslant Q - \frac{1}{(a+b)^2(x+y)^2}(c \wedge d)\left(x^2+y^2\right) \\
&\leqslant Q - \frac{1}{2(a+b)^2}(c \wedge d),
\end{aligned}
$$

with

$$2(x+y)^2 \geqslant \left(x^2+y^2\right).$$

By system (2.2), one can see that

$$\mathrm{d}(S+I+R) = [\mu A - \mu(S+I+R) - \alpha I]\,\mathrm{d}t - \sigma_1 S\mathrm{d}B_1(t) - \sigma_2 I\mathrm{d}B_2(t) - \sigma_3 R\mathrm{d}B_3(t). \tag{3.28}$$

Integrating the both sides of (3.28) from 0 to t, by Lemma 3.6, we have

$$\limsup_{t\to\infty}\langle S(t) + I(t) + R(t)\rangle \leqslant A \text{ a.s.} \tag{3.29}$$

Integrating the both sides of (3.27) from 0 to t, together with (3.29), and noting that $\Re^* < 1$, one can get that

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln Q(t)}{t} \leqslant \beta A - \frac{1}{2(\mu+\gamma+\alpha)^2}&\left\{\left[(\mu+\gamma+\alpha)^2\mu + \frac{1}{2}\sigma_3^2(\mu+\gamma+\alpha)^2\right]\right. \\
&\left. \wedge \left[\frac{1}{2}\sigma_2^2\gamma^2 + \frac{1}{2}\sigma_4^2(\mu+\alpha)^2\right]\right\} < 0 \text{ a.s.},
\end{aligned}
\tag{3.30}
$$

which implies that

$$\lim_{t\to+\infty} I(t) = 0,\ \lim_{t\to+\infty} R(t) = 0 \text{ a.s.} \tag{3.31}$$

On the other hand, according to (3.28), we have

$$\begin{aligned}&\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}+\frac{R(t)-R(0)}{t}\\&=\mu A-\mu\langle S\rangle_t-\mu\langle I\rangle_t-\mu\langle R\rangle_t-\alpha\langle I\rangle_t\\&\quad-\frac{\sigma_1\int_0^t S(s)\mathrm{d}B_1(s)}{t}-\frac{\sigma_2\int_0^t I(s)\mathrm{d}B_2(s)}{t}-\frac{\sigma_3\int_0^t R(s)\mathrm{d}B_3(s)}{t}.\end{aligned}\tag{3.32}$$

By Lemma 3.5, Lemma 3.6 and (3.31), it implies that

$$\lim_{t\to+\infty}\langle S\rangle=A \text{ a.s.}$$

Thus, the proof of Theorem 3.2 is completed. □

3.4. Asymptotic behavior around the disease-free equilibrium E_0.

As mentioned in model (2.1), if $\Re_0\leqslant 1$, then the disease-free equilibrium E_0 of system (2.1) is always globally asymptotically stable, which indicates that the disease will go to extinction with the advancement of time. Noticing that E_0 is not an equilibrium of system (2.2), it is of great interest to study whether the disease will die out in the population. Now, we are in the position to investigate how the solution of system (2.2) spirals closely around E_0.

Theorem 3.3. *Suppose that $\Re_0\leqslant 1$. Then, for any solution $(S(t),I(t),R(t))$ of model (2.2) with initial value $(S(0),I(0),R(0))\in\mathbb{R}^3_+$, if $\sigma_1^2<\mu,\frac{1}{2}\sigma_2^2+\sigma_4^2<\mu+\gamma+\alpha-\frac{\gamma^2}{2\mu}$ and $\sigma_3^2<\mu$. Then, we obtain*

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t[(S(u)-A)^2+I^2(u)+R^2(u)]du\leqslant\frac{\sigma_1^2A^2}{m},$$

where $m=\min\left\{(\mu-\sigma_1^2),(\mu+\gamma+\alpha-\frac{1}{2}\sigma_2^2-\sigma_4^2-\frac{\gamma^2}{2\mu}),(\frac{\mu}{2}-\frac{1}{2}\sigma_3^2)\right\}$.

Proof. Define $V:\mathbb{R}^3_+\to\mathbb{R}_+$ by

$$V(t)=V_1(t)+\frac{2\mu+\gamma+\alpha}{\beta}V_2(t)+V_3(t),\tag{3.33}$$

where

$$V_1(t)=\frac{1}{2}(S(t)-A+I(t))^2,V_2(t)=I(t),V_3(t)=\frac{1}{2}(R(t))^2.\tag{3.34}$$

Along the trajectories of model (2.2), one can obtain that

$$\mathrm{d}V(t)=\mathrm{d}V_1(t)+\frac{2\mu+\gamma+\alpha}{\beta}\mathrm{d}V_2(t)+\mathrm{d}V_3(t).\tag{3.35}$$

In details,

$$\begin{aligned}\mathrm{d}V_1&=LV_1\mathrm{d}t-(S-A+I)(\sigma_1S\mathrm{d}B_1(t)+\sigma_2I\mathrm{d}B_2(t)+\sigma_4I\mathrm{d}B_4(t)),\\\mathrm{d}V_2&=LV_2\mathrm{d}t-\sigma_2I\mathrm{d}B_2(t)-\sigma_4I\mathrm{d}B_4(t),\end{aligned}\tag{3.36}$$

and

$$dV_3 = LV_3 dt - \sigma_3 R^2 dB_3(t) + \sigma_4 IR dB_4(t), \tag{3.37}$$

where

$$\begin{aligned} LV_1 &= (S-A+I)(\mu A - \mu S - (\mu+\gamma+\alpha)I) + \frac{1}{2}(\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_4^2 I^2), \\ LV_2 &= Sf(I) - (\mu+\gamma+\alpha)I, \\ LV_3 &= \gamma IR - \mu R^2 + \frac{1}{2}(\sigma_3^2 R^2 + \sigma_4^2 I^2). \end{aligned} \tag{3.38}$$

By (3.38), one can see that

$$\begin{aligned} LV_1 &= (S-A+I)(-\mu(S-A) - (\mu+\gamma+\alpha)I) \\ &\quad + \frac{1}{2}(\sigma_1^2(S-A+A)^2 + \sigma_2^2 I^2 + \sigma_4^2 I^2) \\ &= -\mu(S-A)^2 - (2\mu+\gamma+\alpha)(S-A)I - (\mu+\gamma+\alpha)I^2 \\ &\quad + \frac{1}{2}(\sigma_1^2(S-A+A)^2 + \sigma_2^2 I^2 + \sigma_4^2 I^2) \\ &= -\mu(S-A)^2 - (2\mu+\gamma+\alpha)(S-A)I - (\mu+\gamma+\alpha)I^2 \\ &\quad + \sigma_1^2(S-A)^2 + \sigma_1^2 A^2 + \frac{1}{2}(\sigma_2^2+\sigma_4^2)I^2 \\ &\leqslant -(\mu-\sigma_1^2)(S-A)^2 - (2\mu+\gamma+\alpha)(S-A)I \\ &\quad - (\mu+\gamma+\alpha - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_4^2)I^2 + \sigma_1^2 A^2, \end{aligned} \tag{3.39}$$

where in the last step we have used the elementary inequalities $(a+b)^2 \leqslant 2a^2 + 2b^2$. Analogously, by (3.38) and $2ab \leqslant a^2 + b^2$, $\Re_0 = \frac{\beta A}{\mu+\gamma+\alpha} < 1$, we get

$$\begin{aligned} LV_2 &= (S-A+A)f(I) - (\mu+\gamma+\alpha)I = (S-A)f(I) + Af(I) - (\mu+\gamma+\alpha)I \\ &\leqslant (S-A)\beta I + A\beta I - (\mu+\gamma+\alpha)I = (S-A)\beta I + (A\beta - (\mu+\gamma+\alpha))I \\ &\leqslant (S-A)\beta I, \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} LV_3 &= \gamma IR - \mu R^2 + \frac{1}{2}(\sigma_3^2 R^2 + \sigma_4^2 I^2) \\ &\leqslant \frac{\gamma^2}{2\mu} I^2 + \frac{\mu}{2} R^2 - \mu R^2 + \frac{1}{2}(\sigma_3^2 R^2 + \sigma_3^2 I^2) = -(\frac{\mu}{2} - \frac{1}{2}\sigma_3^2)R^2 + (\frac{\gamma^2}{2\mu} + \frac{1}{2}\sigma_4^2)I^2. \end{aligned} \tag{3.41}$$

By (3.39), (3.40) and (3.41), one can derive that

$$\begin{aligned} LV &\leqslant -(\mu-\sigma_1^2)(S-A)^2 - (2\mu+\gamma+\alpha)(S-A)I \\ &\quad - (\mu+\gamma+\alpha - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_4^2)I^2 + \sigma_1^2 A^2 \\ &\quad + \frac{2\mu+\gamma+\alpha}{\beta}(S-A)\beta I - (\frac{\mu}{2} - \frac{1}{2}\sigma_3^2)R^2 + (\frac{\gamma^2}{2\mu} + \frac{1}{2}\sigma_4^2)I^2 \\ &= -(\mu-\sigma_1^2)(S-A)^2 - (\mu+\gamma+\alpha - \frac{1}{2}\sigma_2^2 - \sigma_4^2 - \frac{\gamma^2}{2\mu})I^2 \\ &\quad - (\frac{\mu}{2} - \frac{1}{2}\sigma_3^2)R^2 + \sigma_1^2 A^2. \end{aligned} \tag{3.42}$$

Integrating (3.35) from 0 to t on both sides, then taking expectation and combining with (3.42) leads to

$$\begin{aligned} EV(t)-V(0) \leqslant & -(\mu-\sigma_1^2)E\int_0^t (S(u)-A)^2\mathrm{d}u \\ & -(\mu+\gamma+\alpha-\frac{1}{2}\sigma_2^2-\sigma_4^2-\frac{\gamma^2}{2\mu})E\int_0^t I^2(u)\mathrm{d}u \\ & -(\frac{\mu}{2}-\frac{1}{2}\sigma_3^2)E\int_0^t R^2(u)\mathrm{d}u+\sigma_1^2A^2t. \end{aligned} \tag{3.43}$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t Q\mathrm{d}u \leqslant \sigma_1^2A^2, \tag{3.44}$$

where
$Q=\left[(\mu-\sigma_1^2)(S(u)-A)^2+(\mu+\gamma+\alpha-\frac{1}{2}\sigma_2^2-\sigma_4^2-\frac{\gamma^2}{2\mu})I^2(u)+(\frac{\mu}{2}-\frac{1}{2}\sigma_3^2)R^2(u)\right]$.
Let $m=\min\left\{(\mu-\sigma_1^2),(\mu+\gamma+\alpha-\frac{1}{2}\sigma_2^2-\sigma_4^2-\frac{\gamma^2}{2\mu}),(\frac{\mu}{2}-\frac{1}{2}\sigma_3^2)\right\}$, then by (3.44), we can obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left[(S(u)-A)^2+I^2(u)+R^2(u)\right]\mathrm{d}u \leqslant \frac{\sigma_1^2A^2}{m}.$$

Thus, the proof of Theorem 3.3 is completed. □

Noting that when $\sigma_1=0$, E_0 is also the disease-free equilibrium of system (2.2) and (3.42) reduces to the following from

$$LV \leqslant -\mu(S-A)^2-(\mu+\gamma+\alpha-\frac{1}{2}\sigma_2^2-\sigma_4^2-\frac{\gamma^2}{2\mu})I^2-(\frac{\mu}{2}-\frac{1}{2}\sigma_3^2)R^2, \tag{3.45}$$

which shows that LV is negative definite as long as $\frac{1}{2}\sigma_2^2+\sigma_4^2<\mu+\gamma+\alpha-\frac{\gamma^2}{2\mu}$ and $\sigma_3^2<\mu$. Consequently, we have the following corollary.

Corollary 3.1. *Suppose that $\Re_0\leqslant 1$ and $\sigma_1=0$. Then, the disease-free equilibrium E_0 of system (2.2) is stochastically asymptotically stable in the large provided $\frac{1}{2}\sigma_2^2+\sigma_4^2<\mu+\gamma+\alpha-\frac{\gamma^2}{2\mu}$ and $\sigma_3^2<\mu$.*

3.5. Existence of the stationary distribution

In this subsection, we consider the existence of a unique stationary distribution of system (2.2).

Theorem 3.4. *Assume that $\Re_0>1$, and the following conditions hold*

$$0<F<\min\left\{\eta_1S_*^2,\eta_2I_*^2,\eta_3R_*^2\right\}, \tag{3.46}$$

where

$$\begin{aligned} &\eta_1=\mu-\sigma_1^2, \eta_2=\mu+\gamma+\alpha-\sigma_2^2-2\sigma_4^2-\frac{\gamma^2}{2\mu}, \eta_3=\frac{\mu}{2}-\sigma_3^2, \\ &F=\sigma_3^2R_*^2+\sigma_4^2I_*^2+\frac{(2\mu+\gamma+\alpha)I_*^2}{2f(I_*)}(\sigma_2^2+\sigma_4^2). \end{aligned} \tag{3.47}$$

Then, for any solution $(S(t), I(t), R(t))$ of model (2.2) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, there is a stationary distribution π for system (2.2). Especially, we obtain

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [\eta_1(S(u)-S_*)^2 + \eta_2(I(u)-I_*)^2 + \eta_3(R(u)-R_*)^2]du \leqslant F, \tag{3.48}$$

where (S_*, I_*, R_*) is the unique endemic equilibrium of system (2.1).

Proof. Since $\Re_0 > 1$, then there is an equilibrium $E_* = (S_*, I_*, R_*)$ of system (2.2) and

$$\mu A = \mu S_* + S_* f(I_*), S_* f(I_*) = (\mu+\gamma+\alpha)I_*, \gamma I_* = \mu R_*. \tag{3.49}$$

Now, we define $\Phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$\Phi(S(t), I(t), R(t)) = \Phi_1(R(t)) + \frac{(2\mu+\gamma+\alpha)I_*}{f(I_*)}\Phi_2(I(t)) + \Phi_3(S(t), I(t)), \tag{3.50}$$

where

$$\begin{aligned}
&\Phi_1(R(t)) = \frac{1}{2}(R-R_*)^2, \Phi_2(I(t)) = I - I_* - I_* \ln\frac{I}{I_*},\\
&\Phi_3(S(t), I(t)) = \frac{1}{2}(S+I-S_*-I_*)^2.
\end{aligned} \tag{3.51}$$

By Itô's formula, one can obtain

$$\mathrm{d}\Phi(S(t), I(t), R(t)) = \mathrm{d}\Phi_1(R(t)) + \frac{(2\mu+\gamma+\alpha)I_*}{f(I_*)}\mathrm{d}\Phi_2(I(t)) + \mathrm{d}\Phi_3(S(t), I(t)), \tag{3.52}$$

In details,

$$\begin{aligned}
&\mathrm{d}\Phi_1(R(t)) = L\Phi_1 \mathrm{d}t - (R-R_*)(\sigma_3 R\mathrm{d}B_3(t) - \sigma_4 I\mathrm{d}B_4(t)),\\
&\mathrm{d}\Phi_2(I(t)) = L\Phi_2 \mathrm{d}t - (I-I_*)\sigma_2 \mathrm{d}B_2(t) - (I-I_*)\sigma_4 \mathrm{d}B_4(t),\\
&\mathrm{d}\Phi_3(S(t), I(t)) = L\Phi_3 \mathrm{d}t - (S+I-S_*-I_*)(\sigma_1 S\mathrm{d}B_1(t) + \sigma_2 I\mathrm{d}B_2(t) + \sigma_4 I\mathrm{d}B_4(t),
\end{aligned} \tag{3.53}$$

where

$$\begin{aligned}
L\Phi_1 &= (R-R_*)(\gamma I - \mu R) + \frac{1}{2}(\sigma_3^2 R^2 + \sigma_4^2 I^2),\\
L\Phi_2 &= (1-\frac{I_*}{I})(Sf(I) - (\mu+\gamma+\alpha)I) + \frac{I_*}{2}(\sigma_2^2 + \sigma_4^2),\\
L\Phi_3 &= (S+I-S_*-I_*)(\mu A - \mu S - (\mu+\gamma+\alpha)I) + \frac{1}{2}(\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_4^2 I^2).
\end{aligned} \tag{3.54}$$

Making use of (3.49) and the elementary inequalities $(a+b)^2 \leqslant 2a^2 + 2b^2, 2ab \leqslant$

$a^2 + b^2$ and by (3.54), one can obtain that

$$\begin{aligned} L\Phi_1 &= (R - R_*)(\gamma(I - I_*) - \mu(R - R_*)) + \frac{1}{2}\sigma_3^2(R - R_* + R_*)^2 + \frac{1}{2}\sigma_4^2(I - I_* + I_*)^2 \\ &\leqslant \gamma(R - R_*)(I - I_*) - \mu(R - R_*)^2 + \sigma_3^2(R - R_*)^2 + \sigma_3^2 R_*^2 + \sigma_4^2(I - I_*)^2 + \sigma_4^2 I_*^2 \\ &= \frac{\gamma}{\sqrt{\mu}}(R - R_*)\sqrt{\mu}(I - I_*) - \mu(R - R_*)^2 \\ &\quad + \sigma_3^2(R - R_*)^2 + \sigma_3^2 R_*^2 + \sigma_4^2(I - I_*)^2 + \sigma_4^2 I_*^2 \\ &\leqslant \frac{\gamma^2}{2\mu}(I - I_*)^2 - (\frac{\mu}{2} - \sigma_3^2)(R - R_*)^2 + \sigma_4^2(I - I_*)^2 + \sigma_3^2 R_*^2 + \sigma_4^2 I_*^2 \\ &= (\frac{\gamma^2}{2\mu} + \sigma_4^2)(I - I_*)^2 - (\frac{\mu}{2} - \sigma_3^2)(R - R_*)^2 + \sigma_3^2 R_*^2 + \sigma_4^2 I_*^2. \end{aligned} \tag{3.55}$$

Next, one can get that

$$\begin{aligned} L\Phi_2 &= (I - I_*)\left(\frac{Sf(I)}{I} - (\mu + \gamma + \alpha)\right) + \frac{I_*}{2}(\sigma_2^2 + \sigma_4^2) \\ &= (I - I_*)\left(\frac{Sf(I)}{I} - \frac{S_* f(I_*)}{I_*}\right) + \frac{I_*}{2}(\sigma_2^2 + \sigma_4^2) \\ &= (I - I_*)\left(S(\frac{f(I)}{I} - \frac{f(I_*)}{I_*}) + \frac{f(I_*)(S - S_*)}{I_*}\right) + \frac{I_*}{2}(\sigma_2^2 + \sigma_4^2). \end{aligned} \tag{3.56}$$

By condition (ii), we know that $f(I)/I$ is continuous and monotone nonincreasing for $I > 0$. Therefore, we obtain $(I - I_*)S\left(\frac{f(I)}{I} - \frac{f(I_*)}{I_*}\right) \leqslant 0$. Then, we have

$$L\Phi_2 \leqslant (I - I_*)\frac{f(I_*)(S - S_*)}{I_*} + \frac{I_*}{2}(\sigma_2^2 + \sigma_4^2) = \frac{f(I_*)}{I_*}(I - I_*)(S - S_*) + \frac{I_*}{2}(\sigma_2^2 + \sigma_4^2). \tag{3.57}$$

Now, we compute

$$\begin{aligned} L\Phi_3 &= ((S - S_*) + (I - I_*))\left(-\mu(S - S_*) - (\mu + \gamma + \alpha)(I - I_*)\right) \\ &\quad + \frac{1}{2}\sigma_1^2(S - S_* + S_*)^2 + \frac{1}{2}\sigma_2^2(I - I_* + I_*)^2 + \frac{1}{2}\sigma_4^2(I - I_* + I_*)^2 \\ &= -\mu(S - S_*)^2 - (\mu + \gamma + \alpha)(I - I_*)^2 - (2\mu + \gamma + \alpha)(I - I_*)(S - S_*) \\ &\quad + \sigma_1^2(S - S_*)^2 + \sigma_1^2 S_*^2 + \sigma_2^2(I - I_*)^2 + \sigma_2^2 I_*^2 + \sigma_4^2(I - I_*)^2 + \sigma_4^2 I_*^2 \\ &= -(\mu - \sigma_1^2)(S - S_*)^2 - (\mu + \gamma + \alpha - \sigma_2^2 - \sigma_4^2)(I - I_*)^2 \\ &\quad - (2\mu + \gamma + \alpha)(I - I_*)(S - S_*) + \sigma_1^2 S_*^2 + \sigma_2^2 I_*^2 + \sigma_4^2 I_*^2. \end{aligned} \tag{3.58}$$

From (3.52), we can derive

$$L\Phi(S(t), I(t), R(t)) = L\Phi_1(R(t)) + \frac{(2\mu + \gamma + \alpha)I_*}{f(I_*)}L\Phi_2(I(t)) + L\Phi_3(S(t), I(t)). \tag{3.59}$$

Substituting (3.55)-(3.58) into (3.59) leads to

$$
\begin{aligned}
&L\Phi(S(t),I(t),R(t))\\
&\leqslant (\frac{\gamma^2}{2\mu}+\sigma_4^2)(I-I_*)^2-(\frac{\mu}{2}-\sigma_3^2)(R-R_*)^2+\sigma_3^2R_*^2+\sigma_4^2I_*^2\\
&\quad+\frac{(2\mu+\gamma+\alpha)I_*}{f(I_*)}\left(\frac{f(I_*)}{I_*}(I-I_*)(S-S_*)+\frac{I_*}{2}(\sigma_2^2+\sigma_4^2)\right)\\
&\quad-(\mu-\sigma_1^2)(S-S_*)^2-(\mu+\gamma+\alpha-\sigma_2^2-\sigma_4^2)(I-I_*)^2\\
&\quad-(2\mu+\gamma+\alpha)(I-I_*)(S-S_*)+\sigma_1^2S_*^2+\sigma_2^2I_*^2+\sigma_4^2I_*^2\\
&=-(\mu-\sigma_1^2)(S-S_*)^2-(\mu+\gamma+\alpha-\sigma_2^2-2\sigma_4^2-\frac{\gamma^2}{2\mu})(I-I_*)^2\\
&\quad-(\frac{\mu}{2}-\sigma_3^2)(R-R_*)^2+\sigma_3^2R_*^2+\sigma_4^2I_*^2+\frac{(2\mu+\gamma+\alpha)I_*^2}{2f(I_*)}(\sigma_2^2+\sigma_4^2)\\
&\triangleq -\eta_1(S-S_*)^2-\eta_2(I-I_*)^2-\eta_3(R-R_*)^2+F,
\end{aligned}
\tag{3.60}
$$

where η_1,η_2,η_3 and F are defined respectively in Theorem 3.4.

Consequently, we have

$$
\begin{aligned}
\mathrm{d}\Phi(S(t),I(t),R(t))\leqslant\ &-\eta_1(S-S_*)^2-\eta_2(I-I_*)^2-\eta_3(R-R_*)^2+F\\
&-(R-R_*)(\sigma_3R\mathrm{d}B_3(t)-\sigma_4I\mathrm{d}B_4(t))\\
&-\frac{(2\mu+\gamma+\alpha)I_*}{f(I_*)}\left((I-I_*)\sigma_2\mathrm{d}B_2(t)+(I-I_*)\sigma_4\mathrm{d}B_4(t)\right)\\
&-(S+I-S_*-I_*)(\sigma_1S\mathrm{d}B_1(t)+\sigma_2I\mathrm{d}B_2(t)+\sigma_4I\mathrm{d}B_4(t).
\end{aligned}
\tag{3.61}
$$

Integrating (3.61) from 0 to t on both sides results in

$$
\begin{aligned}
&\Phi(S(t),I(t),R(t))-\Phi(S(0),I(0),R(0))\\
&\leqslant\int_0^t\left[-\eta_1(S(u)-S_*)^2-\eta_2(I(u)-I_*)^2-\eta_3(R(u)-R_*)^2\right]\mathrm{d}u+Ft+M(t),
\end{aligned}
\tag{3.62}
$$

where $M(t)$ is a local martingale defined by

$$
\begin{aligned}
&M(t)\\
&=-\int_0^t(S(u)+I(u)-S_*-I_*)\left(\sigma_1S(u)\mathrm{d}B_1(u)+\sigma_2I(u)\mathrm{d}B_2(u)+\sigma_4I(u)\mathrm{d}B_4(u)\right)\\
&\quad-\frac{(2\mu+\gamma+\alpha)I_*}{f(I_*)}\int_0^t(I(u)-I_*)\left(\sigma_2\mathrm{d}B_2(u)+\sigma_4\mathrm{d}B_4(u)\right)\\
&\quad-\int_0^t(R(u)-R_*)\left(\sigma_3R(u)\mathrm{d}B_3(u)-\sigma_4I(u)\mathrm{d}B_4(u)\right).
\end{aligned}
\tag{3.63}
$$

Taking expectation on both sides of (3.62) leads to

$$
\begin{aligned}
&E\Phi(S(t),I(t),R(t))-E\Phi(S(0),I(0),R(0))\\
&\leqslant E\int_0^t\left[-\eta_1(S(u)-S_*)^2-\eta_2(I(u)-I_*)^2-\eta_3(R(u)-R_*)^2\right]\mathrm{d}u+Ft.
\end{aligned}
\tag{3.64}
$$

Therefore, we have

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[\eta_1(S(u)-S_*)^2+\eta_2(I(u)-I_*)^2+\eta_3(R(u)-R_*)^2\right]\mathrm{d}u\leqslant F. \tag{3.65}$$

Noting that if $0<F<\min\left\{\eta_1S_*^2,\eta_2I_*^2,\eta_3R_*^2\right\}$, then the ellipsoid

$$-\eta_1(S(t)-S_*)^2-\eta_2(I(t)-I_*)^2-\eta_3(R(t)-R_*)^2+F=0 \tag{3.66}$$

lies entirely in $\mathbb{R}_+^3$. We can take U to be any neighborhood of the ellipsoid such $\bar{U}\subset\mathbb{R}_+^3$, where $\bar{U}$ denotes the closure of U. Thereby, we can get $L\Phi(S,I,R)<0$ for $(S,I,R)\in\mathbb{R}_+^3\backslash U$, which shows that condition in Lemma 3.2 holds.

On the other hand, we can rewrite system (2.2) as the form of (3.1):

$$\begin{aligned}\mathrm{d}\begin{bmatrix}S(t)\\ I(t)\\ R(t)\end{bmatrix}&=\begin{bmatrix}\mu A-\mu S-Sf(I)\\ Sf(I)-(\mu+\gamma+\alpha)I\\ \gamma I-\mu R\end{bmatrix}\mathrm{d}t+\begin{bmatrix}-\sigma_1S\\ 0\\ 0\end{bmatrix}\mathrm{d}B_1(t)+\begin{bmatrix}0\\ -\sigma_2I\\ 0\end{bmatrix}\mathrm{d}B_2(t)\\ &\quad+\begin{bmatrix}0\\ 0\\ -\sigma_3R\end{bmatrix}\mathrm{d}B_3(t)+\begin{bmatrix}0\\ -\sigma_4I\\ +\sigma_4I\end{bmatrix}\mathrm{d}B_4(t).\end{aligned} \tag{3.67}$$

By Lemma 3.1, we obtain the diffusion matrix is

$$A=\begin{bmatrix}\sigma_1^2S_*^2 & 0 & 0\\ 0 & \sigma_2^2I_*^2+\sigma_4^2I_*^2 & -\sigma_4^2I_*^2\\ 0 & -\sigma_4^2I_*^2 & \sigma_3^2R_*^2+\sigma_4^2I_*^2\end{bmatrix}. \tag{3.68}$$

There exists a positive number $M=\min\left\{\sigma_1^2S^2,\sigma_2^2I^2,\sigma_3^2R^2\right\}$ such that for all $(x_1,x_2,x_3)\in\bar{U}$ and $\xi\in\mathbb{R}_+^3$, we have

$$\begin{aligned}\sum_{i,j=1}^3a_{ij}\xi_i\xi_j&=\left(\sigma_1^2S^2\right)\xi_1^2+\left(\sigma_2^2I^2+\sigma_4^2I^2\right)\xi_2^2-2\sigma_4^2I^2\xi_2\xi_3+\left(\sigma_3^2R^2+\sigma_4^2R^2\right)\xi_3^2\\ &=\sigma_1^2S^2\xi_1^2+\sigma_2^2I^2\xi_2^2+\sigma_3^2R^2\xi_3^2+\sigma_4^2I^2\left(\xi_2-\xi_3\right)^2\\ &\geqslant\sigma_1^2S^2\xi_1^2+\sigma_2^2I^2\xi_2^2+\sigma_3^2R^2\xi_3^2\\ &\geqslant\min\left\{\sigma_1^2S^2,\sigma_2^2I^2,\sigma_3^2R^2\right\}|\xi|^2\\ &=M|\xi|^2,\end{aligned} \tag{3.69}$$

which implies that condition in Lemma 3.2 also holds. Consequently, we can conclude that system (2.2) has a stationary distribution $\pi(\cdot)$. Thus, the proof of Theorem 3.4 is completed. □

3.6. Persistence in mean

Considering that the first two equations of the system (2.2) do not contain variable R, we obtain the equivalent system as follows:

$$\begin{cases}\mathrm{d}S=(\mu A-\mu S-Sf(I))\,\mathrm{d}t-\sigma_1S\mathrm{d}B_1(t),\\ \mathrm{d}I=(Sf(I)-(\mu+\gamma+\alpha)I)\,\mathrm{d}t-\sigma_2I\mathrm{d}B_2(t)-\sigma_4I\mathrm{d}B_4(t),\end{cases} \tag{3.70}$$

The conditions of the extinction for system (3.70) have been obtained. In this subsection, we investigate the conditions which lead to the persistence by means of epidemic system (3.70) under the stochastic disturbances, which implies the infectious disease is prevalent.

Theorem 3.5. *Let $(S(t), I(t), R(t))$ be the solution of system (3.70) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. When both the conditions $\frac{f(A)}{A} \geqslant \mu$ and*

$$\tilde{\Re} = \frac{2\left(f(A) - (\mu+\gamma+\alpha)\right)}{\sigma_2^2 + \sigma_4^2} > 1 \tag{3.71}$$

hold, then the epidemic disease $I(t)$ is persistent in mean. In other words,

$$0 < \frac{A\mu\left(\sigma_2^2+\sigma_4^2\right)}{2f(A)(\mu+\gamma+\alpha)}\left(\tilde{\Re}-1\right) \leqslant \liminf_{t\to\infty}\langle I(t)\rangle \leqslant \limsup_{t\to\infty}\langle I(t)\rangle \leqslant A. \tag{3.72}$$

Proof. For system (3.70), computing the sum of two equations yields

$$\begin{aligned} d\left(S(t)+I(t)\right) = {}& [\mu A - \mu S(t) - (\mu+\gamma+\alpha)I(t)]\,dt \\ & -\sigma_1 S(t)dB_1(t) - \sigma_2 I(t)dB_2(t) - \sigma_4 I(t)dB_4(t). \end{aligned} \tag{3.73}$$

Integrating both sides of inequality (3.73) from 0 to t and dividing both sides by t, we obtain

$$\begin{aligned} &\frac{1}{t}\left[S(t)-S(0)+I(t)-I(0)\right] \\ &= \mu A - \mu\langle S(t)\rangle - (\mu+\gamma+\alpha)\langle I(t)\rangle - \frac{M_1(t)}{t} - \frac{M_4(t)}{t} - \frac{M_5(t)}{t}, \end{aligned} \tag{3.74}$$

where

$$M_1(t) = \int_0^t \sigma_1 S(\theta)\mathrm{d}B_1(\theta), M_4(t) = \int_0^t \sigma_2 I(\theta)\mathrm{d}B_2(\theta), M_5(t) = \int_0^t \sigma_4 I(\theta)\mathrm{d}B_4(\theta). \tag{3.75}$$

Taking the limit of both sides of inequality (3.74) and by Lemma 3.5, we can get

$$0 = \mu A - \mu\lim_{t\to+\infty}\langle S(t)\rangle - (\mu+\gamma+\alpha)\lim_{t\to+\infty}\langle I(t)\rangle, \tag{3.76}$$

i.e.,

$$\lim_{t\to+\infty}\langle S(t)\rangle = A - \frac{(\mu+\gamma+\alpha)}{\mu}\lim_{t\to+\infty}\langle I(t)\rangle. \tag{3.77}$$

Next, we define $\mathbb{C}^2$-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$V(S(t), I(t)) = \ln I(t) + I(t) + S(t). \tag{3.78}$$

Applying Itô's formula results in

$$\mathrm{d}V(S(t), I(t)) = LV\mathrm{d}t - \sigma_2\mathrm{d}B_2(t) - \sigma_4\mathrm{d}B_4(t) - \sigma_1 S\mathrm{d}B_1(t) - \sigma_2 I\mathrm{d}B_2(t) - \sigma_4 I\mathrm{d}B_4(t), \tag{3.79}$$

where

$$\begin{aligned} LV &= \left[\frac{Sf(I)}{I} - (\mu+\gamma+\alpha) - \frac{1}{2}(\sigma_2^2+\sigma_4^2)\right] + \mu A - \mu S - (\mu+\gamma+\alpha)I \\ &\geqslant \left[\frac{f(A)}{A}S - (\mu+\gamma+\alpha) - \frac{1}{2}(\sigma_2^2+\sigma_4^2)\right] + \mu A - \mu S - (\mu+\gamma+\alpha)I \\ &= \left[\mu A - (\mu+\gamma+\alpha) - \frac{1}{2}(\sigma_2^2+\sigma_4^2)\right] + \left(\frac{f(A)}{A} - \mu\right)S - (\mu+\gamma+\alpha)I. \end{aligned} \tag{3.80}$$

Integrating both sides of inequality (3.79) from 0 to t and dividing both sides by t, we get

$$
\begin{aligned}
&\frac{1}{t}\left(\ln I(t)+I(t)+S(t)\right)\\
&\geqslant\left[\mu A-(\mu+\gamma+\alpha)-\frac{1}{2}(\sigma_2^2+\sigma_4^2)\right]+\left(\frac{f(A)}{A}-\mu\right)\langle S(t)\rangle\\
&\quad-(\mu+\gamma+\alpha)\langle I(t)\rangle-\frac{M_1(t)}{t}-\frac{M_4(t)}{t}-\frac{M_5(t)}{t}-\frac{M_6(t)}{t}-\frac{M_7(t)}{t}\\
&\quad+\frac{1}{t}\left(\ln I(0)+I(0)+S(0)\right),
\end{aligned}
\tag{3.81}
$$

i.e.

$$
\begin{aligned}
&(\mu+\gamma+\alpha)\langle I(t)\rangle\\
&\geqslant\left[\mu A-(\mu+\gamma+\alpha)-\frac{1}{2}(\sigma_2^2+\sigma_4^2)\right]+\left(\frac{f(A)}{A}-\mu\right)\langle S(t)\rangle-\frac{M_1(t)}{t}-\frac{M_4(t)}{t}\\
&\quad-\frac{M_5(t)}{t}-\frac{M_6(t)}{t}-\frac{M_7(t)}{t}-\frac{\ln I(t)-\ln I(0)}{t}-\frac{I(t)-I(0)}{t}-\frac{S(t)-S(0)}{t},
\end{aligned}
\tag{3.82}
$$

where

$$
\begin{aligned}
&M_1(t)=\int_0^t\sigma_1 S(\theta)\mathrm{d}B_1(\theta), M_4(t)=\int_0^t\sigma_2 I(\theta)\mathrm{d}B_2(\theta), M_5(t)=\int_0^t\sigma_4 I(\theta)\mathrm{d}B_4(\theta),\\
&M_6(t)=\int_0^t\sigma_2\mathrm{d}B_2(\theta), M_7(t)=\int_0^t\sigma_4\mathrm{d}B_4(\theta).
\end{aligned}
\tag{3.83}
$$

Taking the limit of both sides of inequality (3.82), by Lemma 3.5, we have

$$
\begin{aligned}
&(\mu+\gamma+\alpha)\lim_{t\to+\infty}\langle I(t)\rangle\\
&\geqslant\left[\mu A-(\mu+\gamma+\alpha)-\frac{1}{2}(\sigma_2^2+\sigma_4^2)\right]+\left(\frac{f(A)}{A}-\mu\right)\lim_{t\to+\infty}\langle S(t)\rangle.
\end{aligned}
\tag{3.84}
$$

From inequality (3.77) and inequality (3.84), and assuming that the condition $\frac{f(A)}{A}\geqslant\mu$ holds, then we can obtain

$$
\begin{aligned}
(\mu+\gamma+\alpha)\lim_{t\to+\infty}\langle I(t)\rangle&\geqslant\left[\mu A-(\mu+\gamma+\alpha)-\frac{1}{2}(\sigma_2^2+\sigma_4^2)\right]\\
&\quad+\left(\frac{f(A)}{A}-\mu\right)\left(A-\frac{\mu+\gamma+\alpha}{\mu}\lim_{t\to+\infty}\langle I(t)\rangle\right)\\
&=-(\mu+\gamma+\alpha)-\frac{1}{2}(\sigma_2^2+\sigma_4^2)+f(A)\\
&\quad-\left[\frac{f(A)}{A}\frac{\mu+\gamma+\alpha}{\mu}-(\mu+\gamma+\alpha)\right]\lim_{t\to+\infty}\langle I(t)\rangle,
\end{aligned}
\tag{3.85}
$$

i.e.

$$
\begin{aligned}
\lim_{t\to+\infty} \langle I(t)\rangle &\geqslant \frac{\mu A}{f(A)(\mu+\gamma+\alpha)}\left[f(A)-(\mu+\gamma+\alpha)-\frac{1}{2}(\sigma_2^2+\sigma_4^2)\right] \\
&= \frac{\mu A(\sigma_2^2+\sigma_4^2)}{2f(A)(\mu+\gamma+\alpha)}\left[\frac{2\left(f(A)-(\mu+\gamma+\alpha)\right)}{(\sigma_2^2+\sigma_4^2)}-1\right] \\
&\triangleq \frac{\mu A(\sigma_2^2+\sigma_4^2)}{2f(A)(\mu+\gamma+\alpha)}\left(\tilde{\Re}-1\right).
\end{aligned}
\tag{3.86}
$$

When the condition $\tilde{\Re} = \frac{2(f(A)-(\mu+\gamma+\alpha))}{(\sigma_2^2+\sigma_4^2)} > 1$ holds, taking the inferior limit of both sides of (3.86) yields

$$
\lim_{t\to+\infty} \inf \langle I(t)\rangle \geqslant \frac{\mu A(\sigma_2^2+\sigma_4^2)}{2f(A)(\mu+\gamma+\alpha)}\left(\tilde{\Re}-1\right) > 0. \tag{3.87}
$$

From Lemma 3.6, we know $\limsup_{t\to\infty} \langle I(t)\rangle \leqslant A$. Therefore, we get

$$
0 < \frac{A\mu\left(\sigma_2^2+\sigma_4^2\right)}{2f(A)(\mu+\gamma+\alpha)}\left(\tilde{\Re}-1\right) \leqslant \liminf_{t\to\infty} \langle I(t)\rangle \leqslant \limsup_{t\to\infty} \langle I(t)\rangle \leqslant A. \tag{3.88}
$$

This completes the proof of Theorem 3.5. □

4. Numerical examples

In this section, we further analyze the stochastic model (2.2) by means of the numerical examples.

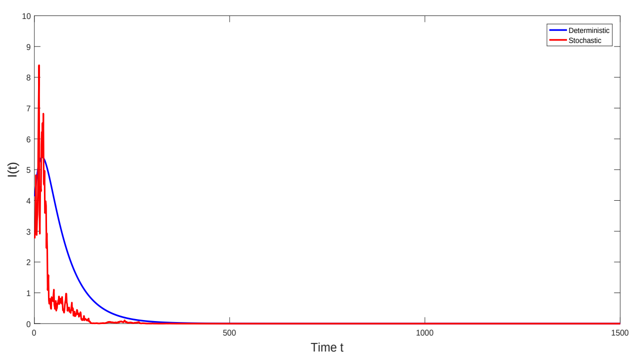

Figure 1. When $\Re^* < 1$, the infective $I(t)$ will be extinct.

Example 4.1. In model (2.2), we define function $f(I) = \beta\big/(1+aI^{0.5})$, which obviously satisfies condition (i-iii). Then, we take the parameters $a = 0.01$, $\mu = 0.15$, $A = 0.5$, $\beta = 0.2$, $\gamma = 0.1$, $\alpha = 0.15$ and $\sigma_1 = 0.82$, $\sigma_2 = 0.82$, $\sigma_3 = 0.82$, $\sigma_4 = 0.82$, we obtain $\Re^* = 0.9518 < 1$. Here, we choose initial value $S(0) = 6$, $I(0) = 4$, $R(0) = 3$. From the numerical simulation given in Figure 1, we can see that the infective $I(t)$ will be extinct (see Theorem 3.2).

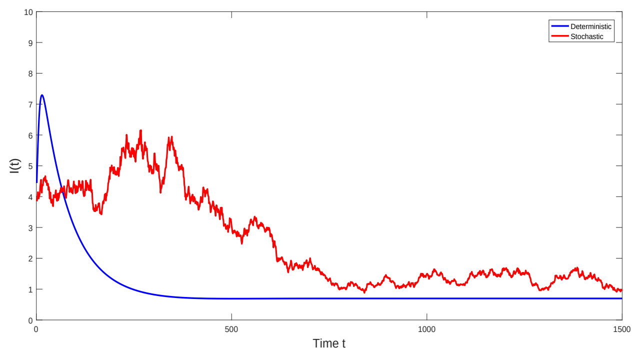

Figure 2. When $\tilde{\Re} > 1$ and $\frac{f(A)}{A} \geqslant \mu$, the infective $I(t)$ will be persistent in mean.

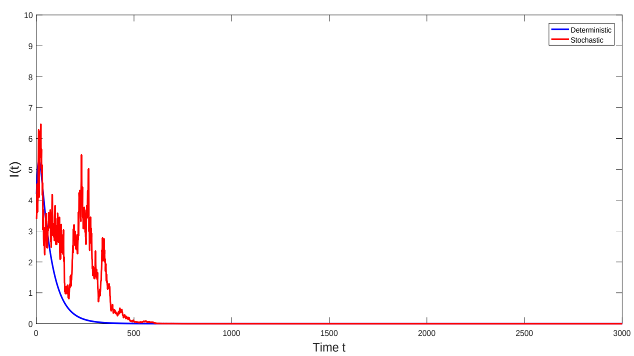

Figure 3. The infected $I(t)$ of deterministic differential equation model (2.1) and stochastic differential equation model (2.2) are extinct, and the solution of (2.2) fluctuates near the solution of (2.1).

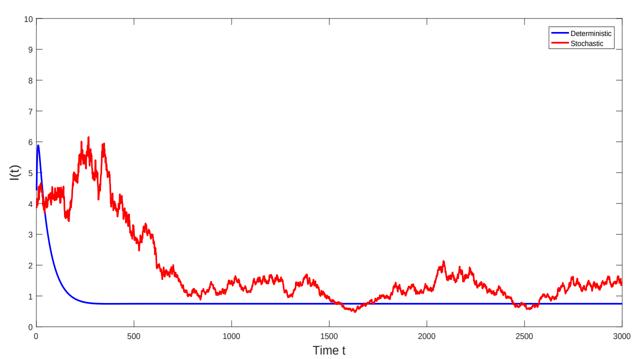

Figure 4. The infected $I(t)$ of deterministic differential equation model (2.1) and stochastic differential equation model (2.2) are persistent in mean, and the solution of (2.2) fluctuates near the solution of (2.1).

Example 4.2. In model (2.2), we take the parameters $a = 0.01,\ \mu = 0.15,\ A = 2,\ \beta = 0.5,\ \gamma = 0.05,\ \alpha = 0.1$ and $\sigma_1 = 0.1,\ \sigma_2 = 0.1,\ \sigma_3 = 0.1,\ \sigma_4 = 0.1$, we have $\tilde{\Re} > 1$ and $\frac{f(A)}{A} \geqslant \mu$. Here, we choose initial value $S(0) = 6,\ I(0) = 4,\ R(0) = 3$. From the numerical simulation given in Figure 2, we can see that the infective $I(t)$ will be persistent in mean (see Theorem 3.5).

Example 4.3. In model (2.2), we take the parameters $a = 0.01,\ \mu = 0.18,\ A = 0.3,\ \beta = 0.4,\ \gamma = 0.1,\ \alpha = 0.1$ and $\sigma_1 = 0.4,\ \sigma_2 = 0.4,\ \sigma_3 = 0.4,\ \sigma_4 = 0.4$. Then, $\Re_0 = 0.3158 < 1$ and the parameters satisfy the condition of Theorem 3.3. Here, we choose initial value $S(0) = 6,\ I(0) = 4,\ R(0) = 3$. From the numerical simulation given in Figure 3, we can see that the infected $I(t)$ of deterministic differential equation model (2.1) and stochastic differential equation model (2.2) will be extinct, and the solution of stochastic differential equation model (2.2) will fluctuate near the solution of deterministic model (2.1) (see Theorem 3.3). On the other hand, we take the parameters $a = 0.01,\ \mu = 0.2,\ A = 2,\ \beta = 0.8,\ \gamma = 0.1,\ \alpha = 0.1$ and $\sigma_1 = 0.1,\ \sigma_2 = 0.1,\ \sigma_3 = 0.08,\ \sigma_4 = 0.1$, then $\Re_0 = 4 > 1$. Here, we choose initial value $S(0) = 6,\ I(0) = 4,\ R(0) = 3$, we have $S_* = 0.5043,\ I_* = 0.7478,\ R_* = 0.3739$ and $\eta_1 S_*^2 = 0.0483,\ \eta_2 I_*^2 = 0.1929,\ \eta_3 R_*^2 = 0.0131,\ F = 0.0101$, which satisfies the condition $\Re_0 > 1$ and $0 < F < \min\left\{\eta_1 S_*^2, \eta_2 I_*^2, \eta_3 R_*^2\right\}$ of Theorem 3.4. Thus, we can conclude that model (2.2) has a unique endemic stationary distribution. From the numerical simulation given in Figure 4, we can see that the infected $I(t)$ of deterministic differential equation model (2.1) and stochastic differential equation model (2.2) will be persistent in mean, and the solution of stochastic differential equation model (2.2) will fluctuate near the solution of deterministic model (2.1) (see Theorem 3.4).

5. Discussions

In terms of human diseases, the nature of epidemic spread and growth is inherently random due to the unpredictability of person-to-person contacts [22]. Hence, the variability and randomness of the environment is fed through to the state of the epidemic. In this paper, we establish stochastic differential equations (2.2) on the basis of a nonlinear deterministic model (2.1) and study the global dynamics. In our model (2.2), we suppose that stochastic environmental factor acts simultaneously on each individual in the population, and assume that the stochastic perturbation is a white noise type that is influenced on the recovery rate γ. The value of our study is to provide the analytic results on the existence of global positive solution, stochastic boundedness, permanence, extinction, asymptotic stability and ergodic property of the solution of the SDE model (2.2). We summarize our main findings as follows:

(1) Noise can restrain the disease outbreak, as Theorem 3.2 indicates that the extinction of disease in the stochastic model (2.2) occurs if $\Re^* < 1$ holds. It is easy to see that $\Re^*$ is decreasing in σ_2^2, σ_3^2 and σ_4^2. Hence, the disease will die out exponentially as long as σ_2^2, σ_3^2 and σ_4^2 are sufficiently large such that $\Re^* < 1$. This implies that large environment fluctuations in I-class and R-class can suppress the outbreak of disease.

(2) The epidemic disease is persistent in mean, as Theorem 3.5 indicates that the epidemic disease $I(t)$ is persistent in mean in the stochastic model (2.2) occurs, if $\frac{f(A)}{A} \geqslant \mu$ and $\tilde{\Re} = \frac{2(f(A)-(\mu+\gamma+\alpha))}{\sigma_2^2+\sigma_4^2} > 1$ hold.

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