Journal of Nonlinear Modeling and Analysis
Volume 3, Number 4, December 2021, 603–616
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.603

Existence and Multiplicity of Positive Solutions for a Singular Nonlinear High Order Fractional Differential Problem with Multi-point Boundary Conditions*

Di Wang[1] and Jiqiang Jiang[1,†]

Abstract In this paper, a singular nonlinear high order fractional differential problem involving multi-point boundary conditions is solved by means of the fixed point index theory. Some properties of the first eigenvalue corresponding to relevant operator and some new height functions are also used to prove the existence and multiplicity of positive solutions. The nonlinearity depends on arbitrary fractional derivative.

Keywords Positive solutions, Fractional differential problem, Fixed point index, First eigenvalue, Multi-point boundary conditions.

MSC(2010) 26A33, 34A08, 34B18.

1. Introduction

We strive to investigate the existence and multiplicity of positive solutions for the following fractional equation problem:

$$\begin{cases} D_{0+}^{\alpha} y(t) + g(t) f(t, y(t), D_{0+}^{w} y(t)) = 0, \quad 0 < t < 1, \\ D_{0+}^{w} y(0) = D_{0+}^{w+1} y(0) = \cdots = D_{0+}^{w+n-2} y(0) = 0, \\ D_{0+}^{\beta} y(1) = \sum_{i=1}^{m} \eta_i D_{0+}^{\gamma} y(\zeta_i), \end{cases} \tag{1.1}$$

where $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n$, $n > 2$, $\eta_i \geq 0$, $i = 1, 2, \cdots, m$ $(m \in \mathbb{N}^+)$, $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_m < 1$, $\beta, \gamma \in \mathbb{R}$, $1 \leq \beta - w$, $\beta \leq n-2$ and $0 \leq \gamma \leq \beta$ with $(\Gamma(\alpha)/\Gamma(\alpha-\gamma)) \sum_{i=1}^{m} \eta_i \zeta_i^{\alpha-\gamma-1} < \Gamma(\alpha)/\Gamma(\alpha-\beta)$, $0 \leq w \leq 1$, D_{0+}^{α} is the α-order Riemann-Liouville derivative, $f(t, u, v) \in C([0,1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$, $g(t)$ is continuous and may have singularities at the points $t = 0, 1$. Under certain conditions, by using some properties of the first eigenvalue corresponding to relevant operator, the different height functions of the nonlinear term of the equation defined on the special bounded set and theory of the fixed point index, we obtain the existence and multiplicity results of positive solutions.

†the corresponding author.
Email address: qfjjq@163.com (J. Jiang)
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China
*The authors were supported by National Natural Science Foundation of China (No. 11871302).

In recent years, a large number of fractional differential equations with various boundary conditions have been paid attention to people in many fields such as science and engineering. This is mainly because in most cases we can use such a mathematical model to accurately and quickly solve many complex problems in various fields, such as biology physics, chemistry, control theory, engineering, mechanics, aerodynamics and other fields. For details, see [3, 4, 13, 21, 23, 29]. Recently, extensive research on differential equations has promoted the development of boundary value problems (BVP) of differential equations. They include singular BVP [5, 10, 17, 19, 22, 30], semipositone BVP [1, 9, 16, 24–26] and nonlocal BVP [2, 8, 11, 12, 15, 18, 27] as special cases. The existence, uniqueness and multiplicity of solutions to these problems are obtained by using nonlinear analysis techniques such as the nonlinear alternative technique, fixed point theorems, the method of monotone iterative, upper and lower solutions method. Now, we give some examples. In [19], Jiang et al. explored the following two-term fractional equation problem with two point boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha}x(t)+ax(t)=b(t)f(t,x(t)), \quad 0<t<1, \\ x(0)=0, \quad x(1)=0, \end{cases}$$

where $1<\alpha\le 2$, $a>0$ and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous, $b(t)$ is continuous and its singularities are $t=0,1$, by virtue of u_0-positive operator and theory of the fixed point index, at least one positive solution has been found. In [28], Zhang *et al.* studied the following differential equation, which is an integral boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t))=0, \quad 0<t<1, \\ x^{(\beta)}(0)=0, \quad 0\le\beta\le n-2, \\ [D_{0+}^{\gamma}x(t)]_{t=1}=\lambda\displaystyle\int_0^{\eta}g(t)D_{0+}^{\gamma}x(t)dt, \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative, $n-1<\alpha\le n$, $n\ge 3$, $\gamma\ge 1$, $\alpha-\gamma-1>0$, $\eta\in(0,1]$, $0\le\lambda\int_0^{\eta}g(t)t^{\alpha-\gamma-1}dt<1$, $g\in L^1[0,1]$ is nonnegative, the singularities of $f(t,x)$ are $t=0,1$ and $x=0$. By using Leggett-Williams fixed point theory, the authors demonstrated that the equation at least has three positive solutions. In [12], He et al. explored the following differential problem with the Riemann-Stieltjes integral and with any derivative in the integral:

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t),D_{0+}^{\beta}x(t))=0, \quad 0<t<1, \\ D_{0+}^{\beta}x(0)=D_{0+}^{\beta+1}x(0)=0, \\ [D_{0+}^{\gamma}x(t)]_{t=1}=\displaystyle\int_0^1 D_{0+}^{\gamma}x(s)dA(s), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville derivative, $2<\alpha\le 3$, $0<\beta\le\gamma<\alpha-2$, $\int_0^1 D_{0+}^{\gamma}x(s)dA(s)$ is a linear functional with the Riemann-Liouville integrals, the singularities of $f(t,u,v)$ are $t=0,1$ and $u=v=0$, by applying suitable upper and lower solutions and Schauder's fixed point theorem, the authors proved that the problem at least has one positive solution.

Motivated by all the papers above, we discuss the existence and multiplicity of positive solutions of (1.1). Our article have various features. Firstly, the equation we study is a high order fractional differential equation. Compared with literature [12,19], our equation is more general. Secondly, the nonlinear term is dependent on any derivative and the boundary value conditions are also related to any derivative, this is different from [19]. Thirdly, we study (1.1) by virtue of some properties of the first eigenvalue corresponding to relevant operator and some new height functions, some sufficient conditions for the existence and multiplicity of positive solutions are established. The method used is different from [12,19,28]. Therefore, our conclusion are new and meaningful.

The remaining part of the paper is arranged as follows: We will state the basic definition and we will summarise the properties of Green's function in the next section. In addition, we will give key lammas. Section 3 is devoted to prove that (1.1) has at least one solution and at least three solutions.

2. Preliminaries

For the convenience of understanding, we first give some definitions and lemmas, which will play a crucial role in the process of proving our conclusion.

Definition 2.1 ([20]). The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $y : (0, \infty) \to \mathbb{R}$ is defined as

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

which provided the right-hand side of the equation is pointwise defined on $(0, \infty)$.

Definition 2.2 ([20]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a continuous function $y : (0, \infty) \to \mathbb{R}$ is defined as

$$\mathcal{D}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is the smallest integer greater than or equal to α, providing the right-hand side of the equation is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([25]). *(i) If $y \in L^1(0,1)$, $\mu > \sigma > 0$, then*

$$I_{0+}^{\mu} I_{0+}^{\sigma} y(t) = I_{0+}^{\mu+\sigma} y(t), \quad D_{0+}^{\sigma} I_{0+}^{\mu} y(t) = I_{0+}^{\mu-\sigma} y(t), \quad D_{0+}^{\sigma} I_{0+}^{\sigma} y(t) = y(t). \tag{2.1}$$

(ii) If $\mu > 0, \sigma > 0$, then

$$D_{0+}^{\mu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\mu)} t^{\sigma-\mu-1}. \tag{2.2}$$

Let $y(t) = I_{0+}^{w} z(t)$, where $z(t) \in C[0,1]$. By using Lemma 2.1 and the definition of Riemman-Liouville derivative, one has

$$\begin{aligned} &D_{0+}^{\alpha} y(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha} y(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha} I_{0+}^{w} z(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha+w} z(t) = D_{0+}^{\alpha-w} z(t), \\ &D_{0+}^{w} y(t) = D_{0+}^{w} I_{0+}^{w} z(t) = z(t), \\ &D_{0+}^{\mu+w} y(t) = D_{0+}^{\mu} z(t), \ D_{0+}^{\nu+w} y(t) = D_{0+}^{\nu} z(t). \end{aligned} \tag{2.3}$$

Let $\beta - w = \mu$, $\gamma - w = \nu$. Then, by (2.3), BVP (1.1) can be simplified to the following improved fractional equation:

$$\begin{cases} D_{0+}^{\alpha-w} z(t) + g(t) f(t, I_{0+}^{w} z(t), z(t)) = 0, \quad 0 < t < 1, \\ z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \\ D_{0+}^{\mu} z(1) = \sum_{i=1}^{m} \eta_i D_{0+}^{\nu} z(\zeta_i), \end{cases} \tag{2.4}$$

and conversely, by using (2.3) again, we can convert (2.4) into the form (1.1). Therefore, BVP (2.4) and BVP (1.1) are equivalent.

Lemma 2.2. *Let $z \in C[0,1]$. If $z > 0$ is a solution of BVP (2.4), then $I_{0+}^{w} z > 0$ is a solution of BVP (1.1).*

Proof. Let $z \in C[0,1]$ and $z > 0$ is a solution to the BVP (2.4). Now, for the function $y(t) = I_{0+}^{w} z(t)$, from Lemma 2.1, we have

$$\begin{aligned} D_{0+}^{\alpha} y(t) &= D_{0+}^{\alpha} I_{0+}^{w} z(t) = D_{0+}^{\alpha-w} z(t) \\ &= -g(t) f(t, I_{0+}^{w} z(t), z(t)) \\ &= -g(t) f(t, y(t), D_{0+}^{w} y(t)). \end{aligned} \tag{2.5}$$

In addition, by combining $y(t) = I_{0+}^{w} z(t)$, (2.3), and the boundary conditions of problem (2.4), we get

$$\begin{aligned} & D_{0+}^{w} y(0) = D_{0+}^{w+1} y(0) = \cdots = D_{0+}^{w+n-2} y(0) = 0, \\ & D_{0+}^{\beta} y(1) = \sum_{i=1}^{m} \eta_i D_{0+}^{\gamma} y(\zeta_i). \end{aligned} \tag{2.6}$$

Thus, we obtain $I_{0+}^{w} z(t) > 0$ is a solution of BVP (1.1). □

Remark 2.1. The expression can be obtained by applying the formula calculation,

$$\begin{aligned} I_{0+}^{w} t^{\alpha-w-1} &= \frac{1}{\Gamma(w)} \int_0^t (t-s)^{w-1} s^{\alpha-w-1} ds \\ &= \frac{B(w, \alpha - w)}{\Gamma(w)} t^{\alpha-1} = \frac{\Gamma(\alpha - w)}{\Gamma(\alpha)} t^{\alpha-1}. \end{aligned} \tag{2.7}$$

Next, we denote $\Delta = \frac{\Gamma(\alpha-w)}{\Gamma(\alpha-w-\beta)} - \frac{\Gamma(\alpha-w)}{\Gamma(\alpha-w-\gamma)} \sum_{i=1}^{m} \eta_i \zeta_i^{\alpha-w-\gamma-1}$.

Lemma 2.3 ([14]). *Suppose that $\Delta \neq 0$. Given $h \in C(0,1) \cap L^1(0,1)$, then the unique positive solution of the fractional equation:*

$$\begin{cases} D_{0+}^{\alpha-w} z(t) + h(t) = 0, \quad 0 < t < 1, \\ z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \\ D_{0+}^{\mu} z(1) = \sum_{i=1}^{m} \eta_i D_{0+}^{\nu} z(\zeta_i), \end{cases} \tag{2.8}$$

can be expressed as

$$z(t) = \int_0^1 G(t,s)h(s)ds, \quad t \in [0,1],$$

where

$$G(t,s) = K_1(t,s) + \frac{t^{\alpha-w-1}}{\Delta}\sum_{i=1}^{m}\eta_i K_2(\zeta_i, s),$$
$$K_1(t,s) = \begin{cases} \dfrac{t^{\alpha-w-1}(1-s)^{\alpha-w-\mu-1}}{\Gamma(\alpha-w)}, & 0 \le t \le s \le 1, \\ \dfrac{t^{\alpha-w-1}(1-s)^{\alpha-w-\mu-1} - (t-s)^{\alpha-w-1}}{\Gamma(\alpha-w)}, & 0 \le s \le t \le 1, \end{cases}$$
$$K_2(t,s) = \begin{cases} \dfrac{t^{\alpha-w-\nu-1}(1-s)^{\alpha-w-\mu-1}}{\Gamma(\alpha-w-\nu)}, & 0 \le t \le s \le 1, \\ \dfrac{t^{\alpha-w-\nu-1}(1-s)^{\alpha-w-\mu-1} - (t-s)^{\alpha-w-\nu-1}}{\Gamma(\alpha-w-\nu)}, & 0 \le s \le t \le 1. \end{cases} \tag{2.9}$$

Here, $G(t,s)$ is the Green function for problem (2.8)*. Obviously, for $t, s \in [0,1]$, $G(t,s)$ is continuous.*

Lemma 2.4 ([14]). *If $\eta_i > 0$, $i = 1, 2, \cdots, m$, and $\Delta > 0$, then the function $G(t,s)$ given by* (2.9) *has the following essential properties:*

(i) $G(t,s) \le D(s)$, $\forall\ t, s \in [0,1]$, *and here* $D(s) = a_1(s) + (1/\Delta)\sum_{i=1}^{m}\eta_i K_2(\zeta_i, s)$, $a_1(s) = (1-s)^{\alpha-w-\mu-1}(1-(1-s)^{\mu})/\Gamma(\alpha-w)$ *and* $s \in [0,1]$;

(ii) $G(t,s) \ge t^{\alpha-w-1}D(s)$, $\forall\ t, s \in [0,1]$;

(iii) $G(t,s) \le \delta t^{\alpha-w-1}$, $\forall\ t, s \in [0,1]$, *and* $\delta = 1/\Gamma(\alpha-w) + \sum_{i=1}^{m}\eta_i\zeta_i^{\alpha-w-\nu-1}/(\Delta\Gamma(\alpha-w-\nu))$.

For the sake of convenience, we present the following hypothesis for the full text.
(H_1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous.
(H_2) $g : (0,1) \to [0,+\infty)$ is continuous and $g(t) \not\equiv 0$ on any subinterval of $(0,1)$ satisfies $0 < \int_0^1 D(s)g(s)ds < +\infty$.

Let $E = C[0,1]$, $\|z\| = \max_{0\le t\le 1}|z(t)|$, then $(E, \|\cdot\|)$ be a Banach space. Set $P = \{z \in E : z(t) \ge 0, t \in [0,1]\}$ is a cone in E. We define a subcone X of P

$$X = \left\{z \in P : z(t) \ge t^{\alpha-w-1}\|z\|, t \in [0,1]\right\}.$$

For any $r > 0$, let $X_r = \{z \in X : \|z\| < r\}$, $\partial X_r = \{z \in X : \|z\| = r\}$, $\overline{X}_r = \{z \in X : \|z\| \le r\}$. Let $0 < a_1 < a_2 \le 1$, denote $b = \min_{t\in[a_1,a_2]} t^{\alpha-w-1}$ and $\xi(z) = \min_{t\in[a_1,a_2]} z(t)$, $z \in X$. $\forall\ R \ge r > 0$, set $X(\xi, r, R) = \{z \in X : r \le \xi(z), \|z\| \le R\}$, $\mathring{X}(\xi, r, R) = \{z \in X : r < \xi(z), \|z\| \le R\}$.

Define two operators T_0 and T_1 as follows:

$$(T_0 z)(t) = \int_0^1 G(t,s)g(s)z(s)ds, \quad t \in [0,1], \tag{2.10}$$

$$(T_1z)(t) = \int_0^1 G(t,s)g(s)f(s, I_{0+}^w z(s), z(s))ds, \quad t \in [0,1], \tag{2.11}$$

where T_0 is a linear operator and T_1 is a nonlinear operator. It is not hard to prove that $T_0 : E \to E$ is linear completely continuous and $T_0(P) \subset P$.

Lemma 2.5 (Krein-Rutmann [6]). *Assume that $T_0 : E \to E$ is linear completely continuous operator and $T_0(P) \subset P$. If there are $\psi \in E \setminus (-P)$ and a constant $c_1 > 0$ such that $c_1 T_0(\psi) \geq \psi$, then the spectral radius $r(T_0) \neq 0$ and T_0 has a positive eigenfunction $\varphi_0 > 0$ corresponding to its first eigenvalue $\lambda_1 = (r(T_0))^{-1}$. That is, $\varphi_0 = \lambda_1 T_0 \varphi_0$.*

From Lemma 2.4 and Lemma 2.5, we know the spectral radius $r(T_0) \neq 0$. In addition, T_0 has a eigenfunction $\varphi_0(t) > 0$ corresponding to its first eigenvalue $\lambda_1 = (r(T_0))^{-1}$.

Lemma 2.6. *Assume that (H_1) and (H_2) hold, then $T_1 : \overline{X}_{r_2} \setminus X_{r_1} \to X$ is completely continuous.*

Proof. First, we prove $T_1(\overline{X}_{r_2} \setminus X_{r_1}) \subset X$. For any $z \in \overline{X}_{r_2} \setminus X_{r_1}$, $t \in [0,1]$, by the definition of T_1 and Lemma 2.4, it implies that

$$\begin{aligned}(T_1z)(t) &= \int_0^1 G(t,s)g(s)f(s, I_{0+}^w z(s), z(s))ds \\ &\geq t^{\alpha-w-1} \int_0^1 D(s)g(s)f(s, I_{0+}^w z(s), z(s))ds,\end{aligned}$$

and

$$(T_1z)(t) \leq \int_0^1 D(s)g(s)f(s, I_{0+}^w z(s), z(s))ds,$$

which yield

$$(T_1z)(t) \geq t^{\alpha-w-1} \|T_1 z\|.$$

Therefore, $T_1 z \in X$. That is, $T_1(\overline{X}_{r_2} \setminus X_{r_1}) \subset X$. Then, by the standard argument, we obtain that $T_1 : \overline{X}_{r_2} \setminus X_{r_1} \to X$ is completely continuous. □

Lemma 2.7 ([7]). *Let E be Banach space. $X \subset E$ is a cone. Assume that $T_1 : \overline{X}_{r_1} \to X$ is completely continuous.*

(i) If there is $z_0 \in X \setminus \{\theta\}$ such that $z - T_1 z \neq \mu z_0$, for $\mu \geq 0$, $z \in \partial X_{r_1}$, then $i(T_1, X_{r_1}, X) = 0$.

(ii) If $T_1 z \neq \mu z$, for $\mu \geq 1$, $z \in \partial X_{r_1}$, then $i(T_1, X_{r_1}, X) = 1$.

Lemma 2.8 ([7]). *Let $T_1 : \overline{X}_{r_3} \to X$ be completely continuous operator. If there is a concave functional $\xi > 0$ and $\xi(z) \leq \|z\|$ $(z \in X)$, for positive numbers $c_1 < c_2 \leq c_3$, satisfying the conditions:*

(i) $\mathring{X}(\xi, c_1, c_2) \neq \emptyset$ with $\xi(T_1 z) > c_1$ if $z \in X(\xi, c_1, c_2)$;

(ii) $T_1 z \in \overline{X}_{c_3}$ if $z \in X(\xi, c_1, c_3)$;

(iii) $\xi(T_1 z) > c_1$ for $z \in X(\xi, c_1, c_3)$ with $\|T_1 z\| > c_2$.

Then, $i(T_1, \mathring{X}(\xi, c_1, c_3), \overline{X}_{c_3}) = 1$.

Lemma 2.9 ([7], [28]). *Let X be a cone in the Banach space E, and operator $T_1 : X \to X$ is completely continuous. Let positive numbers be $c_1 < c_2 < c_3$.*

(i) If $\|T_1 z\| > \|z\|$ for any $z \in \partial X_{c_1}$, and $\|T_1 z\| \leq \|z\|$ for any $z \in \partial X_{c_2}$, then

$$i(T_1, \overline{X}_{c_2} \setminus \overline{X}_{c_1}, \overline{X}_{c_2}) = 1.$$

(ii) If $\|T_1 z\| > \|z\|$ for any $z \in \partial X_{c_1}$, and $\|T_1 z\| < \|z\|$ for any $z \in \partial X_{c_2}$, then

$$i(T_1, X_{c_2} \setminus \overline{X}_{c_1}, \overline{X}_{c_3}) = 1.$$

3. Main results

In the following, we can give the existence and multiplicity results of positive solutions in this paper.

Theorem 3.1. *Assume that (H_1) and (H_2) hold, and*

$$\liminf_{\substack{u\to 0^+\\ v\to 0^+}} \frac{f(t,u,v)}{u+v} > \lambda_1, \quad \limsup_{\substack{u+v\to +\infty\\ v\to +\infty}} \frac{f(t,u,v)}{v} < \lambda_1, \tag{3.1}$$

uniformly on $t \in [0, 1]$, where λ_1 is the first eigenvalue of T_0 and T_0 is defined by (2.10)*. Then, BVP* (1.1) *at least has a positive solution.*

Proof. According to (3.1), for $t \in [0, 1]$, there exists $r_1 > 0$ such that

$$f(t, u, v) \geq \lambda_1(u + v), \quad 0 \leq u \leq \frac{r_1}{\Gamma(w+1)},\ 0 \leq v \leq r_1, \tag{3.2}$$

and so for $z \in \partial X_{r_1}$, since

$$0 \leq I^w_{0+} z(s) \leq \frac{r_1}{\Gamma(w+1)},\ 0 \leq z(s) \leq r_1. \tag{3.3}$$

Then, by (3.2), (3.3) and Lemma 2.4, for $t \in [0, 1]$, we can get

$$\begin{aligned}(T_1 z)(t) =& \int_0^1 G(t,s)g(s)f(s, I^w_{0+}z(s), z(s))ds \\ \geq& \lambda_1 \int_0^1 G(t,s)g(s)(I^w_{0+}z(s) + z(s))ds \\ \geq& \lambda_1 (T_0 z)(t).\end{aligned} \tag{3.4}$$

Let $\varphi_0 > 0$ be the eigenfunction of T_0 corresponding to λ_1, so $\varphi_0 = \lambda_1 T_0 \varphi_0$. Then, we will prove the condition (i) of Lemma 2.7. That is,

$$z - T_1 z \neq \mu\varphi_0, \quad \mu \geq 0,\ z \in \partial X_{r_1}. \tag{3.5}$$

We use the contraction method. That is, there are $z_1 \in \partial X_{r_1}$ and $\mu_1 \geq 0$ such that $z_1 - T_1 z_1 = \mu_1 \varphi_0$. Therefore, when $\mu_1 > 0$, $z_1 = T_1 z_1 + \mu_1\varphi_0 \geq \mu_1\varphi_0$. Set $\overline{\mu} = \sup\{\mu | z_1 \geq \mu\varphi_0\}$, then $\overline{\mu} \geq \mu_0 > 0$, $z_1 \geq \overline{\mu}\varphi_0$. By (3.4)

$$z_1 = T_1 z_1 + \mu_1\varphi_0 \geq \lambda_1 T_0 z_1 + \mu_1\varphi_0 \geq \overline{\mu}\varphi_0 + \mu_1\varphi_0 = (\overline{\mu} + \mu_1)\varphi_0,$$

this contradicts the definition of $\overline{\mu}$. Therefore, (3.5) is established. Then, according to Lemma 2.7, we obtain

$$i(T_1, X_{r_1}, X) = 0. \tag{3.6}$$

Next, we will verify the condition (ii) of Lemma 2.7. According to (3.1), for $t \in [0,1]$, there exists $r_0 > r_1$ such that

$$f(t,u,v) \le \lambda_1 v, \quad u+v \ge r_0, v \ge r_0. \tag{3.7}$$

Now, let $M_1 = \max\{f(t,u,v) | t \in [0,1], 0 \le u+v \le r_0, 0 \le v \le r_0\} < +\infty$. Then, by (3.7) and for $t \in [0,1]$, we have

$$f(t,u,v) \le \lambda_1 v + M_1, \quad u+v \ge 0, v \ge 0. \tag{3.8}$$

Set

$$A = \{z \in X, z = \mu T_1 z, 0 \le \mu \le 1\}.$$

Next, we prove that A is bounded. For $\mu \in [0,1]$ and $z \in A$, from (3.8), we obtain

$$\begin{aligned} z(t) = \mu(T_1 z)(t) &\le \int_0^1 G(t,s)g(s)f(s, I_{0+}^w z(s), z(s))ds \\ &\le \int_0^1 G(t,s)g(s)[\lambda_1 z + M_1]ds \\ &\le \lambda_1 (T_0 z)(t) + N, \end{aligned} \tag{3.9}$$

where $N = M_1 \sup_{t\in[0,1]} \int_0^1 G(t,s)g(s)ds$. We know $r(\lambda_1 T_0) < 1$, so $I - \lambda_1 T_0$ is reversible. I is the identity operator. Then,

$$(I - \lambda_1 T_0)^{-1} = I + \lambda_1 T_0 + (\lambda_1 T_0)^2 + \cdots + (\lambda_1 T_0)^n + \cdots.$$

According to $T_0 : P \to P$, we have $(I - \lambda_1 T_0)^{-1}(P) \subset P$. Then, this together with (3.9) imply

$$z(t) \le (I - \lambda_1 T_0)^{-1} N, \quad t \in [0,1].$$

Thus, we get A is bounded.

We choose $r_2 > \max\{r_0, \sup A\}$, then we can easily get

$$z \ne \mu T_0 z, \quad z \in \partial X_{r_2}, 0 \le \mu \le 1.$$

Then, according to Lemma 2.7, we get

$$i(T_1, X_{r_2}, X) = 1. \tag{3.10}$$

By (3.6) and (3.10), we get

$$i(T_1, X_{r_2} \setminus \overline{X}_{r_1}, X) = i(T_1, X_{r_2}, X) - i(T_1, X_{r_1}, X) = 1.$$

Consequently, T_1 at least has a fixed point. That is, BVP (2.4) has a solution $z > 0$. From Lemma 2.2, then BVP (1.1) has a positive solution $y = I_{0+}^{w} z$. $\square$

Before giving the rest of the paper, we define some height functions.

$$\phi(t, r_1, r_2) = \max\left\{ f(t, u, v) : r_1 m_1(t) \le u \le \frac{r_2 t^w}{\Gamma(w+1)},\ r_1 m_2(t) \le v \le r_2 \right\},$$

$$\phi(t, r) = \max\left\{ f(t, u, v) : r m_1(t) \le u \le \frac{r t^w}{\Gamma(w+1)},\ r m_2(t) \le v \le r \right\},$$

$$\varphi_1(t, r) = \min\left\{ f(t, u, v) : r m_1(t) \le u \le \frac{r t^w}{\Gamma(w+1)},\ r m_2(t) \le v \le r \right\},$$

$$\varphi_2(t, r_1, r_2) = \min\left\{ f(t, u, v) : \frac{r_1 t^w}{\Gamma(w+1)} \le u \le \frac{r_2 t^w}{\Gamma(w+1)},\ r_1 \le v \le r_2 \right\},$$

where $m_1(t) = \frac{\Gamma(\alpha-w)}{\Gamma(\alpha)} t^{\alpha-1}$ and $m_2(t) = t^{\alpha-w-1}$.

Theorem 3.2. *Assume that* (H_1) *and* (H_2) *hold, and there exist constants* k_i, $i = 1, \cdots, 5$ *with* $0 < k_1 < k_2 < k_3 < k_4 < k_5$ *and* $k_3 < b k_4$ *such that*

(S_1) $\int_0^1 D(s) g(s) \varphi_1(s, k_1) ds \ge k_1$;

(S_2) $\int_0^1 D(s) g(s) \phi(s, k_2) ds < k_2$;

(S_3) $\int_{a_1}^{a_2} D(s) g(s) \varphi_2(s, k_3, k_4) ds > k_3 b^{-1}$;

(S_4) $\int_0^1 D(s) g(s) \phi(s, k_3, k_5) ds \le k_5$.

Then, BVP (1.1) *at least has three positive solutions.*

Proof. By Lemma 2.6, when $r_1 = k_1$ and $r_2 = k_5$, for any $z \in \overline{X}_{k_5} \setminus X_{k_1}$, $t \in [0, 1]$, we have $k_1 m_2(t) \le z(t) \le k_5$ and $k_1 m_1(t) \le I_{0+}^{w} z(t) \le \frac{k_5 t^w}{\Gamma(w+1)}$. We know $T_1 : \overline{X}_{k_5} \setminus X_{k_1} \to X$ is completely continuous. According to the extension theorem, $\widetilde{T}_1 : X \to X$ is a completely continuous operator extended from T_1. We will still use T_1 to represent it below. First, we confirm that all the conditions of Lemma 2.8 are satisfied. Then, $i(T_1, \mathring{X}(\xi, k_3, k_5), \overline{X}_{k_5}) = 1$.

(i) Obviously, $\mathring{X}(\xi, k_3, k_4) \ne \emptyset$. For $z \in X(\xi, k_3, k_4)$ and $t \in [a_1, a_2] \subset (0, 1]$, we have $k_3 \le z(t) \le k_4$ and $\frac{k_3 t^w}{\Gamma(w+1)} \le I_{0+}^{w} z(t) \le \frac{k_4 t^w}{\Gamma(w+1)}$. According to Lemma 2.4 and assumption (S_3),

$$\begin{aligned} \xi(T_1 z) &\ge \min_{t \in [a_1, a_2]} t^{\alpha-w-1} \int_{a_1}^{a_2} D(s) g(s) f(s, I_{0+}^{w} z(s), z(s)) ds \\ &\ge b \int_{a_1}^{a_2} D(s) g(s) \varphi_2(s, k_3, k_4) ds > k_3. \end{aligned} \tag{3.11}$$

(ii) For $z \in X(\xi, k_3, k_5)$ and $t \in [0, 1]$, we have

$$k_3 m_2(t) \le z(t) \le k_5, \quad k_3 m_1(t) \le I_{0+}^{w} z(t) \le \frac{k_5 t^w}{\Gamma(w+1)}. \tag{3.12}$$

According to Lemma 2.4 and assumption (S_4),

$$\begin{aligned} (T_1 z)(t) &\le \int_0^1 D(s) g(s) f(s, I_{0+}^{w} z(s), z(s)) ds \\ &\le \int_0^1 D(s) g(s) \phi(s, k_3, k_5) ds \le k_5. \end{aligned} \tag{3.13}$$

Consequently, we obtain $T_1z \in \overline{X}_{k_5}$.
(iii) For $z \in X(\xi, k_3, k_5)$, $t \in [a_1, a_2]$ and $\|T_1z\| > k_4$, obviously, $bk_4 \geq k_3$. Then, we have

$$\begin{aligned}\xi(T_1z) = \min_{t\in[a_1,a_2]}(T_1z)(t) &\geq \min_{t\in[a_1,a_2]} t^{\alpha-w-1}\|T_1z\| \\ &= b\|T_1z\| > bk_4 \geq k_3.\end{aligned} \tag{3.14}$$

Therefore, from Lemma 2.8, when $c_1 = k_3$, $c_2 = k_4$, $c_3 = k_5$, we get the conclusion that

$$i(T_1, \mathring{X}(\xi, k_3, k_5), \overline{X}_{k_5}) = 1. \tag{3.15}$$

Next, for $z \in \partial X_{k_5}$, obviously, $\|z\| = k_5$, and for $t \in [0,1]$, we have $k_3m_2(t) \leq k_5m_2(t) \leq z(t) \leq k_5$ and $k_3m_1(t) \leq k_5m_1(t) \leq I^w_{0+}z(t) \leq \frac{k_5t^w}{\Gamma(w+1)}$. Therefore, (3.12) holds. Then, by (3.12), Lemma 2.4 and assumption (S_4), in the same way that we proved (3.13), we have

$$\|T_1z\| \leq k_5, \quad \forall z \in \partial X_{k_5}. \tag{3.16}$$

For $z \in \partial X_{k_2}$ and $t \in [0,1]$, we have $k_2m_2(t) \leq z(t) \leq k_2$ and $k_2m_1(t) \leq I^w_{0+}z(t) \leq \frac{k_2t^w}{\Gamma(w+1)}$. According to Lemma 2.4 and assumption (S_2), we can obtain

$$\begin{aligned}(T_1z)(t) &\leq \int_0^1 D(s)g(s)f(s, I^w_{0+}z(s), z(s))ds \\ &\leq \int_0^1 D(s)g(s)\phi(s, k_2)ds < k_2.\end{aligned}$$

Then,

$$\|T_1z\| < k_2, \quad \forall z \in \partial X_{k_2}. \tag{3.17}$$

For $z \in \partial X_{k_1}$ and $t \in [0,1]$, we obtain $k_1m_2(t) \leq z(t) \leq k_1$ and $k_1m_1(t) \leq I^w_{0+}z(t) \leq \frac{k_1t^w}{\Gamma(w+1)}$. According to Lemma 2.4 and assumption (S_1), we have

$$\begin{aligned}(T_1z)(t) &\geq t^{\alpha-w-1}\int_0^1 D(s)g(s)f(s, I^w_{0+}z(s), z(s))ds \\ &\geq t^{\alpha-w-1}\int_0^1 D(s)g(s)\varphi_1(s, k_1)ds,\end{aligned}$$

which yields

$$\|T_1z\| \geq \max_{t\in[0,1]} t^{\alpha-w-1}\int_0^1 D(s)g(s)\varphi_1(s,k_1)ds = \int_0^1 D(s)g(s)\varphi_1(s,k_1)ds \geq k_1, \quad \forall z \in \partial X_{k_1}. \tag{3.18}$$

Therefore, by (3.16), (3.17), (3.18) and Lemma 2.9, we obtain

$$i(T_1, \overline{X}_{k_5} \setminus \overline{X}_{k_1}, \overline{X}_{k_5}) = 1, \tag{3.19}$$

$$i(T_1, X_{k_2} \setminus \overline{X}_{k_1}, \overline{X}_{k_5}) = 1. \tag{3.20}$$

From (3.18), it is obvious that T_1 has no fixed point on ∂X_{k_2}. Moreover, for $z \in X(\xi, k_3, k_4)$, by (3.11), we can get $\xi(T_1 z) > k_3$. Then, for $z \in X(\xi, k_3, k_5)$ with $\|T_1 z\| > k_4$, we also get $\xi(T_1 z) > k_3$. Therefore, T_1 has no fixed point on $X(\xi, k_3, k_5) \setminus \mathring{X}(\xi, k_3, k_5)$.

It follows from (3.15), (3.19) and (3.20) that

$$\begin{aligned} & i(T_1, \overline{X}_{k_5} \setminus (X(\xi, k_3, k_5) \cup \overline{X}_{k_2}), \overline{X}_{k_5}) \\ & \quad = i(T_1, \overline{X}_{k_5} \setminus \overline{X}_{k_1}, \overline{X}_{k_5}) - i(T_1, X_{k_2} \setminus \overline{X}_{k_1}, \overline{X}_{k_5}) \\ & \quad - i(T_1, \mathring{X}(\xi, k_3, k_5), \overline{X}_{k_5}) = -1. \end{aligned} \tag{3.21}$$

Consequently, by (3.15), (3.20) and (3.21), we conclude that T_1 at least has three fixed points $z_1 \in X_{k_2} \setminus \overline{X}_{k_1}$, $z_2 \in \mathring{X}(\xi, k_3, k_5)$ and $z_3 \in \overline{X}_{k_5} \setminus (X(\xi, k_3, k_5) \cup \overline{X}_{k_2})$. Clearly, $y_i = I_{0+}^{w} z_i$, $i = 1, 2, 3$, are three positive solutions of BVP (1.1). $\square$

4. An example

Example 4.1. Consider the following equation

$$\begin{cases} -D_{0+}^{\frac{7}{2}} y(t) = g(t) f(t, y(t), D_{0+}^{\frac{1}{4}} y(t)), \quad 0 < t < 1, \\ D_{0+}^{\frac{1}{4}} y(0) = D_{0+}^{\frac{5}{4}} y(0) = D_{0+}^{\frac{9}{4}} y(0) = 0, \\ D_{0+}^{\frac{7}{4}} y(1) = \dfrac{2}{3} D_{0+}^{\frac{3}{2}} y(0.8). \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{2}$, $n = 4$, $w = \frac{1}{4}$, $\beta = \frac{7}{4}$, $\gamma = \frac{3}{2}$, $g(t) = \frac{5}{\sqrt[4]{t(1-t)}}$, $f(t, y(t), D_{0+}^{\frac{1}{4}} y(t)) = (D_{0+}^{\frac{1}{4}} y(t))^{-\frac{1}{8}} + [y(t) + D_{0+}^{\frac{1}{4}} y(t)]^{-\frac{1}{4}}$. By direct computation, we have $(\Gamma(\alpha)/\Gamma(\alpha - \gamma)) \sum_{i=1}^{m} \eta_i \zeta_i^{\alpha-\gamma-1} = (\Gamma(\frac{7}{2})/\Gamma(2)) \times \frac{2}{3} \times 0.8 \approx 1.7728 < \Gamma(\frac{7}{2})/\Gamma(\frac{7}{4}) = 3.6142$ and $\Delta \approx 1.3129 > 0$. Obviously, $f : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous, $g : (0,1) \to [0, +\infty)$ is continuous and $g(t)$ is singular at the points $t = 0$ and $t = 1$ with $0 < \int_0^1 D(s) g(s) ds \leq 2.7427 < +\infty$. In addition, we can easily obtain that

$$\liminf_{\substack{u \to 0^+ \\ v \to 0^+}} \frac{f(t,u,v)}{u+v} = \liminf_{\substack{u \to 0^+ \\ v \to 0^+}} \frac{(u+v)^{-\frac{1}{4}} + v^{-\frac{1}{8}}}{u+v} = +\infty,$$

$$\limsup_{\substack{u+v \to +\infty \\ v \to +\infty}} \frac{f(t,u,v)}{v} = \limsup_{\substack{u+v \to +\infty \\ v \to +\infty}} \frac{(u+v)^{-\frac{1}{4}} + v^{-\frac{1}{8}}}{v} = 0,$$

where $u(t) = y(t)$, $v(t) = D_{0+}^{\frac{1}{4}} y(t)$, then $u(t) = I_{0+}^{\frac{1}{4}} v(t)$. This means that

$$\limsup_{\substack{u+v \to +\infty \\ v \to +\infty}} \frac{f(t,u,v)}{v} < \lambda_1 < \liminf_{\substack{u \to 0^+ \\ v \to 0^+}} \frac{f(t,u,v)}{u+v}.$$

Therefore, we prove that the problem satisfies all the assumptions of Theorem 3.1. Consequently, Theorem 3.1 guarantees that equation (4.1) has at least a solution.

5. Conclusion

In this paper, we obtained several sufficient conditions for the existence of positive solutions for nonlinear fractional differential equation involving multi-point boundary conditions. Our results will be a useful contribution to the existing literature on the topic of fractional-order nonlocal differential equations. The results of the existence are demonstrated on a relevant example.

Acknowledgements

The authors would like to thank the reviewer(s) for their valuable suggestions to improve presentation of the paper.

References

[1] R. Agarwal and R. Luca, *Positive solutions for a semipositone singular Riemann-Liouville fractional differential problem*, International Journal of Nonlinear Sciences and Numerical Simulation, 2019, 20(7-8), 823–831.

[2] B. Ahmad, A. Alsaedi, S. Aljoudi and S. Ntouyas, *A six-point nonlocal boundary value problem of nonlinear coupled sequential fractional integro-differential equations and coupled integral boundary conditions*, Journal of Applied Mathematics and Computing, 2018, 56, 367–389.

[3] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Journal of Mathematical Analysis and Applications, 2005, 311(2), 495–505.

[4] A. Cabada and T. Kisela, *Existence of positive periodic solutions of some nonlinear fractional differential equations*, Communications in Nonlinear Science and Numerical Simulation, 2017, 50, 51–67.

[5] Y. Cui, *Uniqueness of solution for boundary value problems for fractional differential equations*, Applied Mathematics Letters, 2016, 51, 48–54.

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[7] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cone*, Academic Press, New York, 1988.

[8] L. Guo, L. Liu and Y. Wu, *Existence of positive solutions for singular higher-order fractional differential equations with infinite-points boundary conditions*, Boundary Value Problems, 2016, 114, 1–22.

[9] X. Hao, L. Liu and Y. Wu, *Positive solutions for nonlinear fractional semipositone differential equation with nonlocal boundary conditions*, Journal of Nonlinear Science and Applications, 2016, 9, 3992–4002.

[10] X. Hao and H. Wang, *Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions*, Open Mathematics, 2018, 16(1), 581–596.

[11] X. Hao, H. Wang and L. Liu, *Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator*, Boundary Value Problems, 2017, 182, 1–18.

[12] J. He, X. Zhang, L. Liu, Y. Wu and Y. Cui, *Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions*, Boundary Value Problems, 2018, 189, 1–17.

[13] J. Henderson and R. Luca, *On a system of second-order multi-point boundary value problems*, Applied Mathematics Letters, 2012, 25(12), 2089–2094.

[14] J. Henderson and R. Luca, *Existence of positive solutions for a singular fractional boundary value problem*, Nonlinear Analysis: Modelling and Control, 2017, 22(1), 99–114.

[15] J. Jiang and L. Liu, *Existence of solutions for a sequential fractional differential system with coupled boundary conditions*, Boundary Value Problems, 2016, 159, 1–15.

[16] J. Jiang, L. Liu and Y. Wu, *Multiple positive solutions of singular fractional differential system involving Stieltjes integral conditions*, Electronic Journal of Qualitative Theory of Differential Equations, 2012, 37(43), 1–18.

[17] J. Jiang, L. Liu and Y. Wu, *Positive solutions to singular fractional differential system with coupled boundary conditions*, Communications in Nonlinear Science and Numerical Simulation, 2013, 18(11), 3061–3074.

[18] J. Jiang, W. Liu and H. Wang, *Positive solutions for higher order nonlocal fractional differential equation with integral boundary conditions*, Journal of Function Spaces, 2018, Article ID 6598351.

[19] J. Jiang, W. Liu and H. Wang, *Positive solutions to singular Dirichlet-type boundary value problems of nonlinear fractional differential equations*, Advances in Difference Equations, 2018, 169, 1–14.

[20] A. A. Kilbas, H. Srivastava and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science B. V., Amsterdam, 2006.

[21] R. Leggett and L. Williams, *Multiple positive fixed point of nonlinear operator on ordered Banach spaces*, Indiana University Mathematics Journal, 1979, 28, 673–688.

[22] L. Liu, H. Li, C. Liu and Y. Wu, *Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary conditions*, Journal of Nonlinear Science and Applications, 2017, 10(1), 243–262.

[23] S. Reutskiy, *A new numerical method for solving high-order fractional eigenvalue problems*, Journal of Computational and Applied Mathematics, 2016, 317, 603–623.

[24] H. Wang and J. Jiang, *Multiple positive solutions to singular fractional differential equations with integral boundary conditions involving $p-q$ order derivatives*, Advances in Difference Equations, 2020, 2, 1–13.

[25] X. Zhang, L. Liu and Y. Wu, *The uniqueness of positive solution for a singular fractional differential system involving derivatives*, Communications in Nonlinear Science and Numerical Simulation, 2013, 18(6), 1400–1409.

[26] X. Zhang, Z. Shao, Q. Zhong and Z. Zhao, *Triple positive solutions for semipositone fractional differential equations m-point boundary value problems with singularities and $p-q$-order derivatives*, Nonlinear Analysis: Modelling and Control, 2018, 23(6), 889–903.

[27] X. Zhang and Q. Zhong, *Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions*, Fractional Calculus and Applied Analysis, 2017, 20(6), 1471–1484.

[28] X. Zhang and Q. Zhong, *Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables*, Applied Mathematics Letters, 2018, 80, 12–19.

[29] Q. Zhong, X. Zhang and Z. Shao, *Positive solutions for singular higher-order semipositone fractional differential equations with conjugate type integral conditions*, Journal of Nonlinear Sciences and Applications, 2017, 10, 4983–5001.

[30] Y. Zou and G. He, *On the uniqueness of solutions for a class of fractional differential equations*, Applied Mathematics Letters, 2017, 74, 68–73.