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Investigating the Turing Conditions for Diffusion-driven Instability in Predator-prey System with Hunting Cooperation Functional Response

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Abstract In this paper, we focus on stability analysis of steady-state solutions of a predator-prey system with hunting cooperation functional response. The results show that the Turing instability can be affected not only the existence of hunting cooperation, but also the diffusion coefficients: (1) in the absence of predator diffusion, diffusion-driven instability can be induced by hunting cooperation, but no stable patterns appear; (2) the system can occur diffusion-driven instability and Turing patterns, when both predator and prey have diffusion, and the diffusion coefficient of prey is greater than that of the predator. The numerical simulations of two cases are presented to verify the validity of our theoretical results.

Keywords Reaction-diffusion-ordinary differential equations, Steady state solutions, Hunting cooperation functional response, Carnivorous plants and insects model.

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1. Introduction

Predator-prey systems are classical mathematical models of biology, and have been widely concerned. In the dynamics of interacting predator-prey populations, the functional response is an important aspect in determining the different dynamical behaviors. Functional response is the number of prey successfully attacked per predator, and it represents the attack ability of the predator to the prey (see [4,19,21,22,32]). Hunting cooperation is a common factor in the predator-prey system. In fact, a direct consequence of hunting cooperation observed in many species (predators, birds, aquatic creatures, spiders) is the "Allee effect" on predators, that is, predators can continue to survive in the absence of hunting cooperation even if prey population is insufficient to sustain them. In [1,3,6,11,13,16–18,27,28,31], the authors showed that hunting cooperation can be beneficial to the predator population by increasing the attack rate. For example, Alves and Hilker [1] added a cooperation term to the attack rate of the predator population, and proposed the

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following functional response

$$\Phi(u,v) = (\lambda + av)\,u,$$

where $\lambda > 0$ is the attack rate of the per predator on the prey and $a \geq 0$ describes the predator cooperation in hunting. Then, the corresponding predator-prey model is

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{K}\right) - (\lambda + av)\,uv, \\ \dfrac{dv}{dt} = ev\,(\lambda + av) - mv, \end{cases} \tag{1.1}$$

where u and v are prey and predator densities respectively, r is the per capita intrinsic growth rate of prey, K is the carrying capacity of prey, e is the conversion efficiency, and m is the per capita mortality rate of predators. All the parameters involved in system (1.1) are assumed to be positive from the viewpoint of ecology. In [17], the authors introduced the self-diffusion and cross-diffusion into (1.1), and considered the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}\Delta u + ru\left(1 - \dfrac{u}{K}\right) - (\lambda + av)\,uv, \\ \dfrac{\partial v}{\partial t} = d_{22}\Delta v + ev\,(\lambda + av) - mv, \end{cases} \tag{1.2}$$

where the nonnegative constants $d_{11} \geq 0$ and $d_{22} \geq 0$ are the self-diffusion coefficients of the prey and predator populations. Δ is the usual Laplacian operator. Under the condition $0 \leq d_{11} \leq d_{22}$, there is no diffusion-driven Turing instability and the self-diffusion does not induce the Turing instability (see [17]).

Considering the predator-prey relationship between carnivorous plants and insects, scientists have found that carnivorous plants can benefit from insect capture through increased growth, earlier flowering and increased seed production. The ability of plants to feed on animals evolved independently in many plant lineages. The carnivorous habit has been described in about 600 species of angiosperms. If diffusion is introduced into the predator-prey relationship between carnivorous plants and insects, the diffusion rate of prey is much faster than that of predators. Then, we have

$$0 \leq d_{22} \leq d_{11}.$$

In this paper, we will investigate two cases: (1) $d_{22} = 0$; (2) $0 < d_{22} < d_{11}$. It is clear that in case (1), (1.2) is a system of a single reaction-diffusion equation coupled with the ordinary differential equation which has very different properties with the classical reaction-diffusion equation. Many researchers in this field produced impressive results (see [2, 5, 8–10, 23, 25]). As a case, the authors in [9] pointed out that the reaction-diffusion-ODE model exhibited diffusion-driven instability (Turing instability) under a condition of autocatalysis of non-diffusing components. However, the same mechanism, which destabilized constant solutions of such models, also destabilized all continuous spatially heterogeneous stationary solutions. As a result, there existed no stable Turing patterns in such reaction-diffusion-ODE systems. A reaction-diffusion equation coupled to an ordinary differential equation on convex domains was considered in [23]. However, under general cooperative or competitive interactions, the results did not support interesting patterns. To investigate the effect of diffusion on patterns, we also consider the case $d_{22} \neq 0$. There are many excellent works of diffusion-driven Turing patterns (see [7,12,14,15,20,24,26,29,30]). Contrary to the assumption in [17], under the new assumption $0 < d_{22} < d_{11}$, the existence of Turing patterns is considered.

Motivated by the aforementioned discussion, in this paper, we explore a mechanism of pattern formation arising in the processes described by a system of a predator-prey system with hunting cooperation functional response and diffusions. Based on the availability of hunting cooperation functional response and diffusions, we divide it into two cases. The contributions of this paper can be highlighted as follows:

- In the absence of predator diffusion, diffusion-driven instability can occur, if hunting cooperation exists. However, all the non-constant stationary solutions are unstable.
- Under the condition in which the diffusion coefficient of prey is greater than that of the predator, we analyze the existence of Turing instability and Turing pattern at positive equilibrium, when both predator and prey with diffusion.

The rest of this paper is arranged as follows: When predator diffusion $d_{22} = 0$, the existence and instability of stationary solutions and the stability of non-constant stationary solutions are given in Section 2. When predator diffusion $d_{22} \neq 0$, the condition of Turing instability is derived in Section 3. Some numerical simulations are provided to illustrate the feasibility of the theoretical results in Section 4.

2. Diffusion-driven Turing instability when $d_{22} = 0$

In this section, we mainly discuss the diffusion-driven instability in the absence of predator diffusion and hunting cooperation or only in the absence of predator diffusion. For simplicity of the exposition, we assume that spatial spread occupies the interval $x \in [0,1]$. Introducing the dimensionless parameters (see [17]), the system (1.2) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}\Delta u + \sigma u\left(1 - \dfrac{u}{\beta}\right) - (1+\alpha v)\,uv, \\ \dfrac{\partial v}{\partial t} = d_{22}\Delta v + v\,(1+\alpha v)\,u - v. \end{cases} \tag{2.1}$$

It is usually supplemented with the homogeneous Neumann (zero flux) boundary conditions and with nonnegative initial conditions

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0, t \geq 0, \\ v_x(0,t) = v_x(1,t) = 0, t \geq 0, \\ u(x,0) = u_0 \geq 0, v(x,0) = u_0 \geq 0. \end{cases} \tag{2.2}$$

System (2.1) without diffusion becomes the following ordinary differential system

$$\begin{cases} \dfrac{du}{dt} = \sigma u\left(1 - \dfrac{u}{\beta}\right) - (1+\alpha v)\,uv, \\ \dfrac{dv}{dt} = v\,(1+\alpha v)\,u - v. \end{cases} \tag{2.3}$$

In the following, the system (2.3) discussed in this paper is illustrated in details. To deduce main results, two hypotheses and a lemma are given, which will be used in main results.

(H_1) $1 < \beta < \sqrt{\frac{1}{|\sigma\alpha|}}$.

(H_2) $0 < \alpha < \alpha_* = \dfrac{(\beta+\sigma)^2}{\beta^2(\beta+\sigma-1)}$.

Lemma 2.1. *Under the hypotheses* $(H_1) - (H_2)$*, system (2.3) has two unstable boundary equilibria* $E_0 = (0,0)$*,* $E_1 = (\beta, 0)$ *and a unique stable positive equilibrium* $E_* = (u_*, v_*)$*, where* $u_* = 1/(1+\alpha v_*)$ *and* v_* *is the positive root of the following cubic polynomial equation*

$$\beta\alpha^2 v^3 + 2\beta\alpha v^2 + \beta(1-\alpha\sigma)v + \sigma(1-\beta) = 0.$$

Furthermore, for the positive equilibrium $E_* = (u_*, v_*)$, the characteristic equation is

$$\lambda^2 + T_0\lambda + D_0 = 0,$$

where $T_0 = u_*\left(\dfrac{\sigma}{\beta} - \alpha v_*\right) > 0$ and $D_0 = u_*^2 v_*\left(-\dfrac{\alpha\sigma}{\beta} - \dfrac{u_* - 2}{u_*^3}\right) > 0$. For the proof of the above results, please refer to [17].

Now, we begin to consider the model with prey diffusion. In the case of $d_{11} \neq 0$, $d_{22} = 0$, the system (2.1) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}\Delta u + \sigma u\left(1 - \dfrac{u}{\beta}\right) - (1+\alpha v)uv, \\ \dfrac{\partial v}{\partial t} = v(1+\alpha v)u - v, \end{cases} \tag{2.4}$$

with Neumann boundary condition

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0, t \geq 0, \\ u(x,0) = u_0 \geq 0, v(x,0) = u_0 \geq 0. \end{cases} \tag{2.5}$$

First, we introduce the concept of a regular stationary solution of a reaction diffusion-ordinary differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + d\dfrac{\partial^2 u}{\partial x^2}, & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = g(u,v), & x \in \Omega, t > 0, \end{cases} \tag{2.6}$$

where x in a bounded domain $\Omega \subseteq \mathbb{R}^n$ for $n \geq 1$ with a $\mathbb{C}^2-$boundary $\partial\Omega$, which supplemented with the Neumann boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial n} = 0, \quad x \in \Omega, \quad t \geq 0, \\ u(x,0) = u_0 \geq 0, v(x,0) = v_0 \geq 0. \end{cases} \tag{2.7}$$

Definition 2.1. If there exists a solution (not necessarily unique) of the equation $f\left(u(x), v(x)\right) = 0$ such that the relation $v(x) = k(u(x))$ for all $x \in \Omega$ is given with a $\mathbb{C}^1$ - function $k = k(u)$, then we call (u, v) is a regular stationary solution of (2.6)$-$(2.7).

It is clearly that the systems (2.4)$-$(2.5) have a regular stationary solution (u_*, v_*), where $v_* = \frac{1}{\alpha}(\frac{1}{u_*} - 1)$. Then, the conclusions about the constant stationary solution are given in the following.

Theorem 2.1. *Under the hypotheses* $(H_1) - (H_2)$*, there exists diffusion-driven Turing instability. In this case, the constant positive stationary solution* (u_*, v_*) *of systems (2.4)$-$(2.5) is unstable.*

Proof. For the positive equilibrium (u_*, v_*), the corresponding Jacobi matrix of systems (2.4)$-$(2.5) is as follows:

$$A = \begin{pmatrix} -\dfrac{\sigma u_*}{\beta} & -\left(1 + 2\alpha v_*\right) u_* \\ v_* + \alpha v_*^2 & \alpha u_* v_* \end{pmatrix}.$$

A straightforward analysis indicates that the eigenvalues of linearized operator can be derived by discussing roots of following characteristic equation

$$\lambda^2 + T_k \lambda + D_k = 0,$$

where

$$T_k = T_0 + d_{11} k^2 \pi^2 > 0, \quad D_k = D_0 - d_{11} k^2 \pi^2 \alpha u_* v_*.$$

From above characteristic equation, one sees that $D_k < 0$ for $k \geq k_0$, where $k_0 = \left[\dfrac{D_0}{d_{11}\pi^2}\right] + 1$ and "[ ]" represents the integral function. Therefore, the constant stationary solution (u_*, v_*) of systems (2.4)$-$(2.5) is unstable, the Turing instability can occur for some $k \geq k_0$. This completes the proof. □

Remark 2.1. If there is no hunting cooperation in systems (2.4)$-$(2.5), we have $\alpha < 0$. Then, $D_k = D_0 - d_{11} k^2 \pi^2 \alpha u_* v_* > 0$. Consequently, the positive equilibrium (u_*, v_*) is still stable. Hence, there is no diffusion-driven Turing instability.

In the following, we consider the existence and instability of stationary solutions. Let $\tilde{x} = \sqrt{\dfrac{1}{d_{11}}} x$ and $T = \sqrt{\dfrac{1}{d_{11}}}$. Then, the systems (2.4)$-$(2.5) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \tilde{x}^2} + \sigma u \left(1 - \dfrac{u}{\beta}\right) - \left(1 + \alpha v\right) uv, & \tilde{x} \in [0, T], t > 0, \\ \dfrac{\partial v}{\partial t} = v\left(1 + \alpha v\right) u - v, & \tilde{x} \in [0, T], t > 0, \end{cases} \tag{2.8}$$

with Neumann boundary condition

$$\begin{cases} u_{\tilde{x}}\left(0, t\right) = u_{\tilde{x}}\left(T, t\right) = 0, t \geq 0, \\ u\left(\tilde{x}, 0\right) = u_0 \geq 0, v\left(\tilde{x}, 0\right) = v_0 \geq 0. \end{cases} \tag{2.9}$$

Owing to (2.8)−(2.9), we obtain the following boundary value system

$$\begin{cases} \dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \sigma \tilde{u}\left(1 - \dfrac{\tilde{u}}{\beta}\right) - \dfrac{1}{\alpha}\left(\dfrac{1}{\tilde{u}} - 1\right) = 0, \qquad \tilde{x} \in (0, T), \\ \tilde{u}'_{\tilde{x}}|_{\tilde{x}=0} = u'_{\tilde{x}}|_{\tilde{x}=T} = 0. \end{cases} \tag{2.10}$$

Theorem 2.2. *(Existence of continuous nonnegative solutions.) Under the hypotheses* $(H_1) - (H_2)$*, there exists a set* $\Gamma \in R^+$ *of diffusion constants for which boundary value system (2.10) admits a positive solution.*

Proof. Combining with system (2.10), we consider the corresponding system of the first-order equations

$$z = \tilde{u}', \ z' = -h(\tilde{u}), \tag{2.11}$$

where

$$h(\tilde{u}) = \sigma \tilde{u}\left(1 - \frac{\tilde{u}}{\beta}\right) - \frac{1}{\alpha}\left(\frac{1}{\tilde{u}} - 1\right).$$

The first integral of system (2.11) satisfies the following equation

$$\frac{z^2}{2} + H(\tilde{u}) = E, \tag{2.12}$$

where

$$H(\tilde{u}) = \frac{\sigma}{2}\tilde{u}^2 - \frac{\sigma}{3\beta}\tilde{u}^3 - \frac{1}{\alpha}\ln\tilde{u} + \frac{1}{\alpha}\tilde{u}.$$

Denote $H'(\tilde{u}) = h(\tilde{u})$ and $h(\tilde{u}_*) = 0$, where $\tilde{u}_*$ is satisfied Lemma 2.1. Based on $H(\tilde{u})$, we can obtain

$$H''(\tilde{u}_*) = \sigma\left(1 - \frac{\tilde{u}_*}{\beta}\right) + \frac{1}{\alpha \tilde{u}_*^2} - \sigma \frac{\tilde{u}_*}{\beta}.$$

From the hypothesis (H_1), it can be reached that $\alpha\sigma\beta^2 < 1$. Then, we have

$$\frac{1}{\alpha \tilde{u}_*^2} - \frac{\sigma \tilde{u}_*}{\beta} > 0.$$

As a result, we get $H''(\tilde{u}_*) > 0$. Therefore, $H(\tilde{u})$ has a local minimum at $\tilde{u}_*$. It is notable that the trajectories of equation (2.12) satisfies $z(0) = z(T) = 0$ for a certain $T > 0$ and $H(u(0)) = H(u(T)) = E$. Furthermore, it follows from equation (2.12) that all trajectories are symmetric with respect to the z-axis. Consequently, there exists a periodic solution $\tilde{u} = \tilde{u}(\tilde{x}), z = z(\tilde{x})$ of the dynamic system $z = \tilde{\tilde{u}}', z' = -h(\tilde{u})$ subject to initial conditions $z(0) = \tilde{u}'(0) = z(T) = \tilde{u}'(T) = 0$. Moreover, for any natural n, we can derive $\tilde{u}'\left(\frac{nT}{2}\right) = z(0) = 0$, for even n and $\tilde{u}'\left(\frac{nT}{2}\right) = z(T) = 0$, for ode n. This completes the proof. □

Remark 2.2. Theorem 2.2 verifies the existence of nonconstant steady-state solutions of systems (2.4)−(2.5).

In the following, we consider the stability of non-constant stationary solution. To proceed, a significant lemma is presented here.

Lemma 2.2. *[2] Assume that $(\bar{u}(x), \bar{v}(x))$ is a non-constant regular solution of the stationary systems (2.6)−(2.7), then there exists $x_0 \in \Omega$ such that the vector $(\bar{u}(x_0), \bar{v}(x_0))$ is a constant solution of the systems (2.6)−(2.7).*

Theorem 2.3. *There exists no stable spatially non-constant stationary solution $(\bar{u}(x), \bar{v}(x))$ of the systems (2.4)−(2.5).*

Proof. Let $(\bar{u}(x), \bar{v}(x))$ be a non-constant regular solution of the stationary systems (2.4)−(2.5). From Lemma 2.2, we can obtain that there exists $x_0 \in (0,1)$ such that the vector $(\bar{u}(x_0), \bar{v}(x_0))$ is a constant solution of the systems (2.4)−(2.5). Since all the three constant solutions of the systems (2.4)−(2.5) are unstable under the hypotheses $(H_1) - (H_2)$, $(\bar{u}(x_0), \bar{v}(x_0))$ must be an unstable constant solution of the systems (2.4)−(2.5). Hence, $(\bar{u}(x), \bar{v}(x))$ must be unstable. This completes the proof. □

3. Diffusion-driven Turing instability when $d_{22} \neq 0$

In this section, we investigate the diffusion-driven Turing instability and Turing patterns, when both predator and prey have diffusions, and the diffusion coefficient of prey is greater than that of the predator. Let $d_{22} = \gamma d_{11}$ with $0 < \gamma < 1$, then system (2.1) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}\Delta u + \sigma u\left(1 - \dfrac{u}{\beta}\right) - (1+\alpha v)\, uv, \\ \dfrac{\partial v}{\partial t} = \gamma d_{11}\Delta v + v\,(1+\alpha v)\, u - v. \end{cases} \tag{3.1}$$

Song [17] pointed out in his paper that if $d_{22} > d_{11}$, there is no diffusion-driven Turing instability. However, we will consider the opposite situation that $0 < d_{22} < d_{11}$ with the homogeneous Neumann (zero flux) boundary conditions and nonnegative initial conditions. For the positive equilibrium $E_* = (u_*, v_*)$, the linearized system of (3.1) is

$$\begin{pmatrix} \dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} d_{11}\Delta u & 0 \\ 0 & \gamma d_{11}\Delta v \end{pmatrix} + \begin{pmatrix} -\dfrac{\sigma u_*}{\beta} & -(1+2\alpha v_*)\, u_* \\ v_* + \alpha v_*^2 & \alpha u_* v_* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.2}$$

It is obvious to know that the characteristic equation of the linearized system (3.2) is

$$\lambda^2 + T_k\lambda + D_k = 0, \tag{3.3}$$

where

$$T_k = d_{11}(1+\gamma)k^2\pi^2 + T_0,$$

$$D_k = \gamma d_{11}^2 k^4\pi^4 - \left(-\frac{\sigma u_*}{\beta}\gamma + \alpha u_* v_*\right) d_{11}k^2\pi^2 + D_0,$$

and

$$T_0 = u_*\left(\frac{\sigma}{\beta} - \alpha v_*\right), \quad D_0 = u_*^2 v_*\left(-\frac{\alpha\sigma}{\beta} - \frac{u_* - 2}{u_*^3}\right).$$

Assume that
(H_3) $0<\gamma<\gamma_1(d_{11})=\dfrac{\alpha u_* v_*}{d_{11}\pi^2 k^2+\dfrac{\sigma u_*}{\beta}}$.
(H_4) $0<\gamma<\gamma_2=\dfrac{\beta^2}{\sigma^2 u_*^2}\left(2D_0+\dfrac{\alpha\sigma u_*^2 v_*}{\beta}-2\sqrt{D_0^2+D_0\dfrac{\alpha\sigma u_*^2 v_*}{\beta}}\right)$.

In order to demonstrate that hypotheses $(H_3)-(H_4)$ can be easily understood, we give the following remark.

Remark 3.1.
(1) From hypothesis (H_4), it can be reached that $\min\limits_{k\in\mathbb{R}_+} D_k<0$ and $-\dfrac{\sigma u_*}{\beta}\gamma+\alpha u_* v_*>0$.
(2) Combining the condition $T_0=u_*\left(\dfrac{\sigma}{\beta}-\alpha v_*\right)>0$ with $-\dfrac{\sigma u_*}{\beta}\gamma+\alpha u_* v_*>0$, we have $0<\gamma<1$.

Through $(H_3)-(H_4)$, we have $T_0>0$, $D_0>0$ and $T_k>0$. Let $k_{\min}^2$ be the minimal point of function D_k on $k^2\in\mathbb{R}_+$, then

$$k_{\min}^2=\frac{-\dfrac{\sigma u_*}{\beta}\gamma+\alpha u_* v_*}{2\gamma d_{11}\pi^2}.$$

Meanwhile, according to hypothesis (H_3), it yields that $k_{\min}^2>1/2$. On account of the fact that $\gamma=\gamma_1(d_{11})$ decreases monotonically in d_{11} and intersects with $\gamma=\gamma_2$ at $d_{11}=d_0$, where

$$d_0\triangleq\sqrt{\frac{\alpha^2 u_*^2 v_*^2}{\gamma_2^2}-\frac{\sigma u_*}{\beta}}.$$

We denote the relationship between $\gamma_1(d_{11})$ and γ_2 in the following

$$\gamma_*\left(d_{11}\right)=\begin{cases}\gamma_2, & 0<d_{11}\le d_0,\\ \gamma_1\left(d_{11}\right), & d_{11}\ge d_0.\end{cases}$$

In order to derive the condition of the Turing instability, a lemma is presented at first.

Lemma 3.1. *The hypotheses* $(H_3)-(H_4)$ *hold, if and only if* $0<\gamma<\gamma_*\left(d_{11}\right)$, $d_{11}>0$.

Denote

$$\gamma_{\mathrm{T}}\left(k,d_{11}\right)=\frac{d_{11}k^2\pi^2\alpha u_* v_*-D_0}{d_{11}k^2\pi^2\left(d_{11}k^2\pi^2+\dfrac{\sigma u_*}{\beta}\right)},\quad\text{for}\quad d_{11}>d_{11}(k),$$

where $d_{11}(k)=\dfrac{D_0}{k^2\pi^2\alpha u_* v_*}$. Then, $D_k=0$, when $\gamma=\gamma_{\mathrm{T}}\left(k,d_{11}\right)$.

Taking the above calculations into account, we have $d_{11}^M(k)$ be the point at which the monotonicity of the function $\gamma_{\mathrm{T}}(k, d_{11})$ changes, where

$$d_{11}^M(k) = \frac{D_0 + \sqrt{D_0^2 + k^4\pi^4\alpha u_* v_* \frac{\sigma u_*}{\beta}}}{k^2\pi^2\alpha u_* v_*}.$$

Hence, $\gamma_{\mathrm{T}}(k, d_{11})$ can take the maximum value γ_2 at $d_{11}^M(k)$.

As for $k \in \mathbb{N}$, there is only one root $d_{k,k+1} \in \left(d_{11}^M(k+1), d_{11}^M(k)\right)$ that meets $\gamma_{\mathrm{T}}(k, d_{11}) = \gamma_{\mathrm{T}}(k+1, d_{11})$ for $d_{11} > 0$, and $\gamma_{\mathrm{T}}(k, d_{11}) > \gamma_{\mathrm{T}}(k+1, d_{11}) > \gamma_{\mathrm{T}}(k+2, d_{11}) > \cdots$, for $d_{11} > d_{k,k+1}$. Moreover, $\gamma_{\mathrm{T}}(d_{11}) \triangleq \gamma_{\mathrm{T}}(k, d_{11}), d_{11} \in (d_{k,k+1}, d_{k-1,k}), k \in \mathbb{N}$, and mark $d_{0,1} = +\infty$. Then, $\gamma_{\mathrm{T}}(d_{11}) \leq \gamma_*(d_{11}), 0 < d_{11} < +\infty$. Furthermore, $\gamma_{\mathrm{T}}(d_{11}) = \gamma_*(d_{11})$, if and only if $d_{11} = d_{11}^M(k)$, $k \in \mathbb{N}$.

Theorem 3.1. *Assume* $(H_1) - (H_4)$ *hold.*
For any given $k_1 \in \mathbb{N}$, *when* $\gamma = \gamma_{\mathrm{T}}(k_1, d_{11})$, *the system (3.1) occurs* $k_1 -$ *mode Turing bifurcation at* (u_*, v_*). $\gamma = \gamma_{\mathrm{T}}(d_{11})$ *is the critical curve of Turing instability.*
(i) If $\gamma > \gamma_{\mathrm{T}}(d_{11})$, *the system (3.1) is asymptotically stable at positive equilibrium* (u_*, v_*).
(ii) If $0 < \gamma < \gamma_{\mathrm{T}}(d_{11})$, *Turing instability occurs in the system (3.1) at positive equilibrium* (u_*, v_*).

Proof. When $\gamma = \gamma_{\mathrm{T}}(k_1, d_{11})$, we can get $D_k = 0$. Then, the characteristic equation (3.3) becomes

$$\lambda^2 + T_{k_1}\lambda = 0. \tag{3.4}$$

Equation (3.4) has a zero root, and the other root $-T_{k_1}$ has negative real part. That is, the system (3.1) occurs Turing bifurcation at (u_*, v_*) .

If $\gamma > \gamma_{\mathrm{T}}(d_{11})$, then $D_{k_1} > 0$. Therefore, all roots of (3.3) have negative real parts. On the other hand, if $0 < \gamma < \gamma_{\mathrm{T}}(d_{11})$, $D_{k_1} < 0$, then the system (3.1) is Turing instability. This completes the proof. □

We show the relation of $\gamma = \gamma_2$, $\gamma = \gamma_1(d_{11})$ and $\gamma = \gamma_{\mathrm{T}}(k, d_{11})$ in Figure 1, where $d_{11} > 0$, $k = 1, 2, 3 \cdots$. Moreover, we present the critical curve of Turing instability $\gamma = \gamma_{\mathrm{T}}(d_{11})$ in Figure 2.

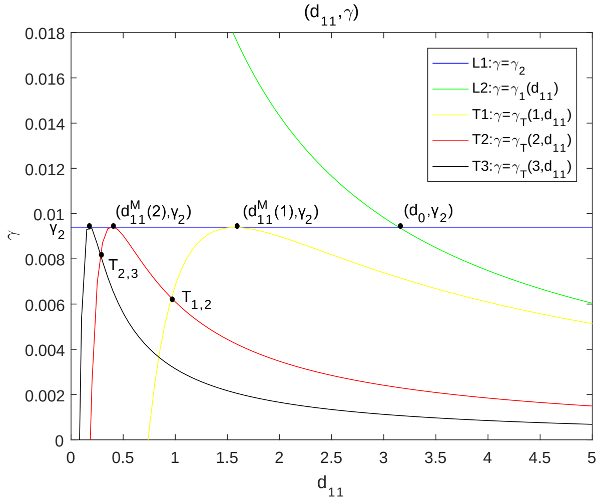

Figure 1. The figure of functions $\gamma = \gamma_2$, $\gamma = \gamma_1(d_{11})$ and $\gamma = \gamma_{\mathrm{T}}(k, d_{11})$, $d_{11} > 0$, $k = 1, 2, 3\cdots$ in (d_{11}, γ) plane

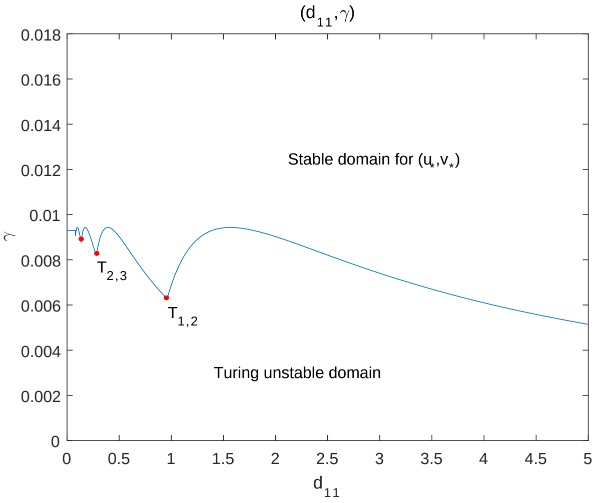

Figure 2. The Turing bifurcation line $T : \gamma = \gamma_{\mathrm{T}}(d_{11})$, $d_{11} > 0$

4. Numerical simulations of diffusion-driven Turing instability when $d_{22} = 0$ or $d_{22} \neq 0$

In this section, we will simulate the diffusion-driven Turing instability, when $d_{22} = 0$ or $d_{22} \neq 0$ respectively. For fixed $\alpha = 0.3$, $\beta = 1.5$, $\sigma = 1.4$, we obtain $(u_*, v_*) = (0.690938, 1.49071)$, which satisfies the hypotheses $(H_1) - (H_2)$.

Case I $d_{11} = 0.1, d_{22} = 0$

In this case, we present a numerical study showing emergence of periodic and spike pattern because of diffusion-driven instability.

Consider the system (2.1), when $d_{11} = 0.1$, $d_{22} = 0$ with the homogeneous Neumann (zero flux) boundary conditions and nonnegative initial conditions $(u_0, v_0) = (u_*, v_* + 0.07\cos(2\pi x^2)) = (0.690938, 1.49071 + 0.07\cos(2\pi x^2))$, which is chosen as a perturbation of the stationary solution (u_*, v_*). Then, periodic and spike pattern appears. Figure 3 shows that in the system of a single reaction-diffusion equation coupled with ordinary differential equation, and diffusion does not only destabilizes the constant steady state, but also leads to an unbounded growth of model solution.

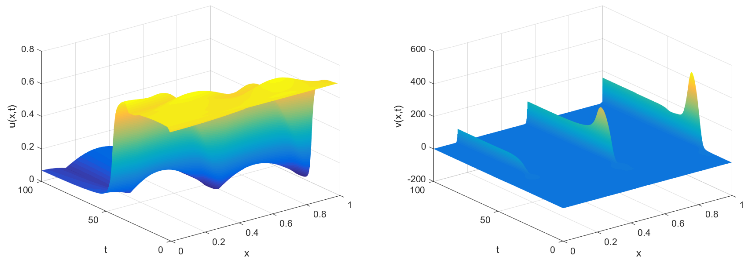

Figure 3. Periodic and spike pattern around (u_*, v_*) with initial condition $(u_0, v_0) = (u_*, v_* + 0.07\cos(2\pi x^2))$

Case II $d_{11} = 0.1, d_{22} = 0.02$

Through the hypothesis (H_4), we can get $\gamma_2 = 0.0094$. Then, we select $\gamma = 0.2$ which satisfies $\gamma > \gamma_2$. Consider the system (3.1), when $d_{11} = 0.1, d_{22} = 0.02$ with the homogeneous Neumann (zero flux) boundary conditions and nonnegative initial condition $(u_0, v_0) = (u_* + 0.0028\cos(3x) - 0.0034\cos(29x), v_* + 0.0057\cos(3x) - 0.0014\cos(29x))$. The Turing pattern in the neighborhood of a constant occurs. Meanwhile, pattern arising through a bifurcation is spatially periodic.

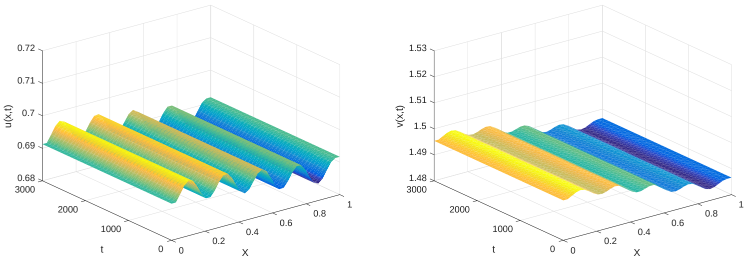

Figure 4. Stable spatially periodic pattern around (u_*, v_*) with nonnegative initial condition $(u_0, v_0) = (u_* + 0.0028\cos(3x) - 0.0034\cos(29x), v_* + 0.0057\cos(3x) - 0.0014\cos(29x))$

5. Conclusion

Predator-prey systems of carnivorous plants and insects occur widely in nature, and these kinds of systems are quite different from those of general predator-prey. In this paper, we find that in the absence of predator diffusion, diffusion-driven instability can occur if hunting cooperation existence. By employing the basic tools of the elliptic equation, we determine the existence of the nonconstant period steady-state solutions, and prove that they are all unstable. Under the condition in which the diffusion coefficient of prey is greater than that of the predator, we analyze the existence of Turing instability and Turing pattern at positive equilibrium, when both predator and prey with diffusion. The results obtained in this paper show the relationship in diffusions, hunting cooperation and Turing patterns (see Table 1).

Table 1. The distribution of diffusion-driven instability and Turing pattern

	with hunting cooperation ($\alpha > 0$)	without hunting cooperation ($\alpha < 0$)
$d_{22} = 0$	Turing instable no stable pattern	no Turing instability
$d_{22} < d_{11}$	Turing instable stable pattern	no Turing instability
$d_{22} > d_{11}$	no Turing instability	no Turing instability

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