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On the Main Aspects of the Inverse Conductivity Problem

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Abstract We consider a nonlinear inverse problem for an elliptic partial differential equation known as the Calderón problem or the inverse conductivity problem. Based on several results, we briefly summarize them to motivate this research field. We give a general view of the problem by reviewing the available results for C^2 conductivities. After reducing the original problem to the inverse problem for a Schrödinger equation, we apply complex geometrical optics solutions to show its uniqueness. After extending the ideas of the uniqueness proof result, we establish a stable dependence between the conductivity and the boundary measurements. By using the Carleman estimate, we discuss the partial data problem, which deals with measurements that are taken only in a part of the boundary.

Keywords Calderón problem, Inverse conductivity problem, Dirichlet-to-Neumann map, Complex geometrical optics solutions, Carleman estimate.

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1. Introduction

You may ask what the inverse conductivity problem is. Well, to answer this question, as the name of the problem indicates, we should consider the direct conductivity problem first, given by

$$\begin{cases} \nabla \cdot \gamma \nabla w = 0 \text{ in } \Omega, \\ \qquad\quad w = f \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\gamma \in C^2(\bar{\Omega})$ is a positive real-valued function that represents the electrical conductivity of the domain Ω. Physically interpreted, the application of a voltage $f \in H^{1/2}(\partial\Omega)$ on the boundary induces an electrical potential w in the interior of Ω, where $w \in H^1(\Omega)$ is the unique weak solution of this elliptic boundary value problem.

We define the Dirichlet-to-Neumann map (DN map) Λ_γ by relating a boundary voltage f (Dirichlet data) to the flux at the boundary $\gamma \frac{\partial w}{\partial \nu}$ (Neumann data) as follows:

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

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$$f \mapsto \Lambda_\gamma(f) = \gamma \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega},$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative at $\partial\Omega$.
From the variational formulation of the precedent problem, it is clear that

$$\langle \Lambda_\gamma f, g\rangle = \left\langle \gamma \frac{\partial w}{\partial \nu}, g \right\rangle = \int_\Omega \gamma \nabla w \nabla z dx \quad \forall f, g \in H^{1/2}(\partial\Omega),$$

where $z \in H^1(\Omega), z\big|_{\partial\Omega} = g$. It follows from this definition that Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. In this context, Calderón in his pioneer paper [9] formulated the Calderón problem as being the problem studying the inversion of the map $\gamma \mapsto \Lambda_\gamma$, i.e., the posed question is whether we can determine γ from the knowledge of $\Lambda_\gamma f$ in each $f \in H^{1/2}(\partial\Omega)$. This inversion method is also called electrical impedance tomography (EIT). It is a medical imaging technology with several applications, including the detection of breast cancer and pulmonary imaging. For more detailed arguments on this technique, see the review papers [6, 18].

The determination of γ from the DN map has different aspects. In this paper, we answer the preceding question in the interior of the studied domain by giving results on the three aspects: uniqueness, stability and partial data. For the boundary determination, in the case that smooth conductivities Kohn and Vogelius [21] proved that Λ_γ determines γ and all its normal derivatives on the boundary. More general results were shown in [2, 31]. In particular, Brown [7] proved that we could recover the boundary values of a $W^{1,1}$ or a C^0 conductivity from the knowledge of Λ_γ.

While the current paper only deals with the inverse conductivity problem in three and higher dimensions, we mention that the approach for the two-dimensional problem is quite different, which is essentially based on complex analysis. We refer readers to the work of Astala and Päivärinta [5] on bounded measurable conductivities for a deeper understanding of the problem in the plane.

In the following, we only consider isotropic conductivities, which are not dependent on direction. If a conductivity depends on direction, it is called an anisotropic conductivity. In this case, we are in the presence of the anisotropic Calderón problem. In the plane, uniqueness was shown for L^∞ anisotropic conductivities in [4]. For $n \geq 3$, this problem is also called Calderón's inverse problem on Riemannian manifolds, and as was pointed out in [23], this is a geometrical problem that has up to now remained open. For more detailed arguments, please also see [12] .

There are several problems related to the main one. The fractional Calderón problem is a nonlocal version of the classical one [11]. It was first introduced in [15]. In the present work, it is a question to study a Schrödinger operator containing an electrical potential. However, if there is also a nonzero magnetic potential, we are in the presence of another variant of the standard problem, namely the Calderón problem for the magnetic Schrödinger operator [22]. By combining the two precedent problems, we can also define another closely related one, which is the inverse conductivity problem for the fractional magnetic operator, and it is the subject of [24, 25].

Under the broad research field of the Calderón problem, we focus on its main aspects. We propose a simplified review of Salo's lecture notes [28] and some chapters from [13]. The rest of this article is organized in the following way: the applied notation and background knowledge are summarized in Section 2. In Section 3, we review the known uniqueness and stability results for the full data problem for C^2 conductivities. Section 4 presents the partial data type problem. Section 5 contains some perspectives.

2. Preliminaries

Throughout this article,

- Ω denotes a bounded open set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.
- $n \geq 3$ denotes the space dimension.
- $\langle , \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.
- dS denotes the surface on $\partial\Omega$.
- $q : \Omega \to \mathbb{R}$ denotes an electrical potential.
- $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.
- $B_R(0)$ denotes the closed ball with center 0 and radius $R > 0$.

2.1. Fourier transform and function spaces

For $\xi \in \mathbb{R}^n$, the applied notation for the Fourier transform is

$$\hat{w}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi.x} w(x)dx.$$

For $s \in \mathbb{R}$, we define Sobolev spaces $H^s(\mathbb{R}^n)$ via Fourier transform as follows:

$$H^s(\mathbb{R}^n) = \{w \in \ \mathcal{S}'(\mathbb{R}^n) : \quad \langle\xi\rangle^s \widehat{w} \in L^2(\mathbb{R}^n)\},$$

where $\langle\xi\rangle = (|\xi|^2 + 1)^{1/2}$.
The associated norm is

$$\|w\|_{H^s(\mathbb{R}^n)} = \|\langle\xi\rangle^s \widehat{w}\|_{L^2(\mathbb{R}^n)}.$$

We give the following properties, which will be needed later in Section 3.

Proposition 2.1. *(Sobolev embedding) If* $w \in H^{s+k}(\mathbb{R}^n), s > n/2, k \in \mathbb{N},$ *then* $w \in C^k(\mathbb{R}^n)$ *and*

$$\|w\|_{C^k(\mathbb{R}^n)} \leq c\|w\|_{H^{s+k}(\mathbb{R}^n)}.$$

Proposition 2.2. *(Multiplication by functions) If* $w \in H^s(\mathbb{R}^n)$, $s \geq 0$, $f \in C^k(\mathbb{R}^n)$, $k \in \mathbb{N}$, $k \geq s$, *then* $fw \in H^s(\mathbb{R}^n)$ *and*

$$\|fw\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{C^k(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}.$$

Proposition 2.3. *(Logarithmic convexity) If* $0 \leq a \leq b, \ 0 \leq \tau \leq 1,$ *then*

$$\|w\|_{H^c(\mathbb{R}^n)} \leq \|w\|^{1-\tau}_{H^a(\mathbb{R}^n)}\|w\|^{\tau}_{H^b(\mathbb{R}^n)},$$

where $c = (1-\tau)a + \tau b$.

We record a Poincaré type inequality in the strip $S = \{x \in \mathbb{R}^n : c_1 < x \cdot \delta < c_2\}$, which will be used in Section 4.

Proposition 2.4. *(Poincaré inequality) For* δ *a unit vector in* $\mathbb{R}^n$, *there exists a constant* $C(S)$ *such that for all* $w \in C_c^\infty(S)$, *we have*

$$\|w\|_{L^2(S)} \leq C\|\delta \cdot Dw\|_{L^2(S)}.$$

3. The full data problem

In this section, we consider the inverse conductivity problem with measurements that are taken on the whole boundary $\partial\Omega$. We discuss the uniqueness and the stability issues of the problem in the next two subsections.

3.1. Uniqueness

Here, we consider the result of Sylvester and Uhlmann [30], which states the unique recovery of γ from Λ_γ.

Theorem 3.1. *For $j = 1, 2$, let $\gamma_j \in C^2(\bar{\Omega})$ be two positive functions. Then, we have*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2 \quad \text{in } \Omega.$$

This theorem can be reduced to the following one for a Schrödinger equation.

Theorem 3.2. *For $j = 1, 2$, let $q_j \in \mathcal{Q}$. Then, we have*

$$\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2 \quad \text{in } \Omega.$$

Next, we proceed to the reduction of Theorem 3.1 to Theorem 3.2.

3.1.1. Reduction of the conductivity equation to the Schrödinger equation

By providing a certain amount of smoothness (in our case γ has two derivatives), we can reduce the Calderón problem to the inverse boundary value problem for a Schrödinger equation. This reduction is based on the well-known Liouville transformation: if z is a weak solution of the conductivity equation $\nabla \cdot \gamma \nabla z = 0$, then $w = \gamma^{1/2} z$ is a solution to the Schrödinger equation $(-\Delta + q)w = 0$, where the potential $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.
For $q \in L^\infty(\Omega)$ and for all $f \in H^{1/2}(\partial\Omega)$, we give the following boundary value problem for the Schrödinger equation

$$\begin{cases} -\Delta w + qw = 0 \text{ in } \Omega, \\ \qquad\qquad w = f \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

From now on, we will consider the standard assumption that 0 is not a Dirichlet eigenvalue for the Schrödinger equation. Under this condition, the problem (3.1) is well-posed in the sense that it admits a unique solution. Moreover, we define $\mathcal{Q}$ as being the subset of all potentials $q \in L^\infty(\Omega)$ such that 0 is not a Dirichlet eigenvalue for $(-\Delta + q)w = 0$.
For all $q \in \mathcal{Q}$, we define the DN map Λ_q associated with (3.1) by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$
$$f \mapsto \Lambda_q(f) = \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega}.$$

From the variational formulation, it is clear that

$$\langle \Lambda_q f, g\rangle = \int_\Omega (qwz + \nabla w \cdot \nabla z)\, dx \quad \forall f, g \in H^{1/2}(\partial\Omega), \tag{3.2}$$

which implies that Λ_q is a self-adjoint bounded linear mapping from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. Since $q \in \mathcal{Q}$, we can give another useful identity, when $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. It is clear that the DN map Λ_q can be obtained from the DN map Λ_γ. The explicit expression related to those two maps is given by

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2}f) + \frac{1}{2}\gamma^{-1}\left.\frac{\partial\gamma}{\partial\nu}f\right|_{\partial\Omega}. \tag{3.3}$$

The next corollary shows that Theorem 3.2 implies Theorem 3.1.

Corollary 3.1. *If the conditions of Theorem 3.1 are satisfied and Theorem 3.2 holds, then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2 \quad \textit{in } \Omega.$$

Proof. Under the hypothesis of Theorem 3.1, suppose that for $j = 1, 2$, $q_j = \gamma_j^{-1/2}\Delta\gamma_j^{1/2}$, then $q_j \in \mathcal{Q}$. By applying Theorem 1.3 (uniqueness at the boundary) from [2], it follows that $\gamma_1 = \gamma_2$ and $\frac{\partial\gamma_1}{\partial\nu} = \frac{\partial\gamma_2}{\partial\nu}$ on $\partial\Omega$.
By using relation (3.3) with q_2 for all $f \in H^{1/2}(\partial\Omega)$ and replacing with the boundary values of γ and its normal derivative, it follows that $\Lambda_{q_1} f = \Lambda_{q_2} f\ \forall f \in H^{1/2}(\partial\Omega)$. Therefore, by Theorem 3.2, we deduce that $q_1 = q_2$ in the whole domain Ω.
Since $q \in \mathcal{Q}$, $(-\Delta + q)w = 0$ has a unique solution in Ω. In particular, we substitute with $q = \gamma_1^{-1/2}\Delta\gamma_1 = \gamma_2^{-1/2}\Delta\gamma_2$, which implies that both of $\gamma_1^{1/2}$ and $\gamma_2^{1/2}$ solve the precedent Schrödinger equation with the same boundary value (by using the precedent boundary identification). It follows from the uniqueness that $\gamma_1^{1/2} = \gamma_2^{1/2}$. □

Remark 3.1. For later use in the next subsection, we follow another direction to prove that $\gamma_1^{-1/2}\Delta\gamma_1 = \gamma_2^{-1/2}\Delta\gamma_2 \Rightarrow \gamma_1 = \gamma_2$.
By using the identification $\triangle(\log\gamma_j^{1/2}) = \gamma_j^{-1/2}\Delta\gamma_j - |\nabla(\log\gamma_j^{1/2})|^2$, the equation $q_1 = q_2$ may be written as $\triangle z + \nabla\theta\cdot\nabla z = 0$, which is a linear equation for $z = \log\gamma_1^{1/2}\gamma_2^{-1/2} \in C^2(\Omega)$ with $\theta = \log(\gamma_1\gamma_2)^{1/2}$.
By the identity $\nabla\cdot(e^\theta\nabla z) = e^\theta(\Delta z + \nabla\theta\cdot\nabla z)$ and the boundary identification $\gamma_1\big|_{\partial\Omega} = \gamma_2\big|_{\partial\Omega}$, we see that z is a solution to the well-posed Dirichlet problem

$$\begin{cases} \nabla\cdot((\gamma_1\gamma_2)^{1/2}\nabla z) = 0 \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad z = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, z is identically zero in the whole domain Ω, and we get the aimed conclusion $\gamma_1 = \gamma_2$ in Ω.

From now on, we will be concentrating on Theorem 3.2.

3.1.2. Construction of special solutions

Next, we use Fourier analysis to construct special solutions for the Schrödinger equation.

Let $A = \{\zeta \in \mathbb{C}^n : \ \zeta\cdot\zeta = 0\}$. If $w = e^{i\zeta.x}, \zeta \in A$, then w is a harmonic function, which solves $(-\Delta + q)w = 0$ with $q = 0$. However, if $q \neq 0$, $w = e^{i\zeta.x}$ cannot be a solution to the precedent equation anymore. However, we can still find solutions that look like the previous ones. This idea originates from Sylvester and Uhlmann [30].

Here, we look for special solutions $w(x,\zeta), \zeta \in A$ to the equation $(-\Delta + q)w = 0$, which are asymptotically exponential, i.e., $w \sim e^{i\zeta . x}$ for $|\zeta| \to \infty$. This last asymptotic property asserts that the corrector function $r(x) = \ell(x) - 1$ with $\ell = e^{-i\zeta . x} w$ decays to zero, when $|\zeta| \to \infty$. Therefore, we write

$$w(x) = e^{i\zeta . x}(1 + r), \tag{3.4}$$

with $r \in H^1(\Omega)$ is just a correction term that is needed to transit from an approximate solution to the exact one by taking $|\zeta| \to \infty$.
We substitute with (3.4) in $(-\Delta + q)w = 0$. By using the fact that $-\Delta = D^{2*}$, we obtain

$$(D^2 + 2\zeta \cdot D + q)r = -q \text{ in } \Omega, \tag{3.5}$$

which shows that (3.4) is a solution to $(-\Delta + q)w = 0$, if and only if (3.5) holds. The functions $w(x) = e^{i\zeta . x}(1 + r)$ are called complex geometrical optics solutions (CGOs). To approximate them to the exact solutions, we should establish certain asymptotic bounds on the corrector term r, when $|\zeta| \to \infty$.
First, we give the following basic estimate for $q = 0$.

Proposition 3.1. *There exists a constant $M(n, \Omega)$ such that for any $\zeta \in A, |\zeta| \geq 1$ and $f \in L^2(\Omega)$, the function $r \in H^1(\Omega)$ solves the equation*

$$(D^2 + 2\zeta \cdot D)r = f \text{ in } \Omega, \tag{3.6}$$

with the following estimates

$$\|r\|_{L^2(\Omega)} \leq \frac{M}{|\zeta|} \|f\|_{L^2(\Omega)},$$

$$\|\nabla r\|_{L^2(\Omega)} \leq M \|f\|_{L^2(\Omega)}.$$

Proof. The idea of the proof is to apply Fourier transform to (3.6), since it is a linear equation with constant coefficients. If we do that directly, we get

$$(\xi^2 + 2\zeta \cdot \xi)\hat{r}(\xi) = \hat{f}(\xi),$$

which is an expression of r with a vanishing denominator. For instance, for $\xi = 0$. To simplify, we extend f to be zero in the cube $Q = [-\pi, \pi]^n$ outside the domain Ω. Let $\zeta = h(a_1 + a_2)$, where $h = \frac{|\zeta|}{2^{1/2}}$ and $a_1, a_2 \in \mathbb{R}^n$ are two orthogonal unit vectors that we identify with the vectors of the canonic base e_1 and e_2. Clearly, (3.6) becomes

$$(D^2 + 2h(D_1 + iD_2))r = f \text{ in } Q.$$

We write $\omega_k = e^{i(k+1/2e_2).x}, k \in \mathbb{Z}^n$, where $\{\omega_k\}$ is an orthonormal complete set of $L^2(Q)$. By applying the theory of Hilbert spaces, we are using the smart technique of expressing the second member and the corrector term of the last equation as Fourier series on Q in slightly shifted lattices from $\mathbb{Z}^n$ to $\mathbb{Z}^n + 1/2e_2$. After some calculus, we get

$$\|r\|_{L^2(Q} \leq \frac{1}{h} \|f\|_{L^2(Q)} \text{ and } \|\nabla r\|_{L^2(Q)} \leq 4\|f\|_{L^2(Q)}.$$

*$Dw = (\frac{1}{i}\partial_1 w, ..., \frac{1}{i}\partial_n w)$.

This means that under the condition $|\zeta| \geq 1$, both r and ∇r are in L^2. Thus, $r \in H^1(\Omega)$ and satisfies the above two estimations. $\square$

If $q \neq 0$, we are no longer in the presence of a linear equation with constant coefficients. Then, the previous method is inapplicable. Now, under the condition $|\zeta| \geq \max(1, M\|q\|_{L^\infty})$ and by analogy to Proposition 3.1, we can show that the function $r \in H^1$ solving the equation

$$(D^2 + 2\zeta \cdot D + q)r = f \text{ in } \Omega, \tag{3.7}$$

satisfies the same precedent L^2 norm estimates for the corrector term r and its gradient ∇r.

To prove this, we define the solution operator of equation (3.5) G_ζ from $L^2(\Omega)$ to $H^1(\Omega)$ by $G_\zeta(f) = r$.

We know that since $q \neq 0$, the solution of (3.7) may be given by $G_\zeta \tilde{f} = r$ for some $\tilde{f} \in L^2(\Omega)$, which implies that $(I + qG_\zeta)\tilde{f} = f$.

Since $|\zeta| \geq \max(1, M\|q\|_{L^\infty})$, it follows that

$$\|qG_\zeta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{M\|q\|_{L^\infty}}{|\zeta|} \leq \frac{1}{2}.$$

To conclude, we can get the desired estimates by inverting the operator $I + qG_\zeta$.

Next, we consider the existence result of CGOs from [30].

Theorem 3.3. *Let $q \in L^\infty(\Omega)$. There exists a constant $M(n, \Omega)$ such that for any $\zeta \in A, |\zeta| \geq \max(1, M\|q\|_{L^\infty})$, $\lambda \in H^2(\Omega)$, $\zeta \cdot \nabla\lambda = 0$ in Ω. The equation $(-\Delta + q)w = 0$ has a solution $w(x) = e^{i\zeta \cdot x}(\lambda + r)$ with $r \in H^1(\Omega)$ satisfying*

$$\|r\|_{L^2(\Omega)} \leq \frac{M}{|\zeta|}\|(-\Delta + q)\lambda\|_{L^2(\Omega)},$$

$$\|\nabla r\|_{L^2(\Omega)} \leq M\|(-\Delta + q)\lambda\|_{L^2(\Omega)}.$$

This theorem guarantees the existence of CGOs for the Schrödinger equation, but what about the uniqueness question for the Calderón problem? Since it is the subject of this uniqueness subsection, the answer will be given in the rest of the present subsection.

3.1.3. Uniqueness proof

From Corollary 3.1, we know that Theorem 3.2 implies Theorem 3.1. Then, it is sufficient to prove the uniqueness result for Theorem 3.2. To do that, we still need an integral identity that relates boundary measurements with interior potentials. From (3.2), it follows that

$$\left\langle (\Lambda_{q_1} - \Lambda_{q_2}) w_1\big|_{\partial\Omega}, w_2\big|_{\partial\Omega} \right\rangle = \int_\Omega (q_1 - q_2) w_1 w_2 \, dx, \tag{3.8}$$

for $q_j \in \mathcal{Q}$ and $w_j \in H^1$ uniquely solve $-\Delta w_j + q_j w_j = 0$, for $j = 1, 2$.

Now, we can give the outline of the proof of Theorem 3.2.

Proof. Since $\Lambda_{q_1} = \Lambda_{q_2}$, the integral identity (3.8) can be simplified into

$$\int_\Omega (q_1 - q_2) w_1 w_2 \, dx = 0. \tag{3.9}$$

The idea of the proof is to look for an approximation of $e^{i\zeta.x}$ by the products $w_1 w_2$. That is possible by following this reasoning: Fix $\xi \in \mathbb{R}^n$, since $n \geq 3$, we introduce two unit vectors $\omega_1, \omega_2 \in \mathbb{R}^n$ such that the set $\{\omega_1, \omega_2, \xi\}$ is orthogonal.
Let $\zeta = h(\omega_1 + i\omega_2)$, then $\zeta \in A$. If h is sufficiently large, the application of Theorem 3.3 guarantees the existence of two CGOs for $(-\triangle + q_j)w_j = 0, j = 1, 2$:

$$w_1 = e^{i\zeta.x}(e^{ix.\xi} + r_1) \text{ and } \ w_2 = e^{-i\zeta.x}(1 + r_2).$$

Moreover, $\|r_j\|_{L^2(\Omega)} < C/h$. By taking the limits as $h \to \infty$ in (3.9), the correction terms in w_1 and w_2 will vanish. Consequently, as mentioned before, the CGOs will look like the complex exponential $e^{i\xi.x}$. Thus, the precedent integral identity (3.9) becomes

$$\int_\Omega (q_1 - q_2) e^{i\xi.x} \, dx = 0.$$

Since this identity holds for every $\xi \in \mathbb{R}^n$, we extend by zero the function $(q_1 - q_2)(\xi)$ on $\mathbb{R}^n \backslash \Omega$. Then, by the uniqueness theorem of the Fourier transform [29], we deduce that $q_1 = q_2$ in Ω. This means that the map $q \mapsto \Lambda_q$ is one-to-one. Thus, the inverse problem for the Schrödinger equation has a unique solution. Therefore, the uniqueness of the inverse conductivity problem holds (Theorem 3.2). □

3.2. Stability

By definition: the stability of a problem is that the behavior of the solution changes continuously with the change of initial conditions. This means that a small change in data leads to a small change in the solution. Thus, this section aims to establish the estimation

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \varpi \left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)} \right).$$

We observe that the difference of the conductivities is taken in the L^∞ norm. Thanks to an example given by Alessandrini [1], if we take $\gamma_j \in L^\infty(\Omega), j = 1, 2$, the precedent estimate is invalid. Then, logical reasoning is to impose some a priori constraints on the conductivities γ_j, as it is announced in the following result of Alessandrini [1].

Theorem 3.4. *For $j = 1, 2$, let $\gamma_j \in H^{s+2}(\Omega), s \geq n/2$ be two positive functions satisfying $1/N \leq \gamma_j \leq N$ and $\|\gamma_j\|_{H^{s+2}} \leq N$. There exist constants $t(s, n) \in (0, 1)$ and $C(n, \Omega, N, s) > 0$ such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \varpi \left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)} \right),$$

where ϖ is a modulus of continuity satisfying $\varpi(\tau) \leq C \mid \log \tau \mid^{-t}, \ 0 < \tau < 1/e$.

Remark 3.2. Since $\gamma_j \in H^{s+2}(\Omega)$ for $s \geq n/2$, then $\gamma_j \in C^2(\overline{\Omega})$ by Proposition 2.1.

Boundary stability: Under the same assumptions of Theorem 3.4, we have the following result for stability at the boundary.

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right). \tag{3.10}$$

We also give the stability result for the Schrödinger equation.

Theorem 3.5. *For $j = 1, 2$, let $q_j \in \mathcal{Q}$, with $\|q_j\|_{L^\infty(\Omega)} \leq N$. There exists a constant $C(n, \Omega, N, s) > 0$ such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq \varpi\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right),$$

where ϖ satisfies $\varpi(\tau) \leq C \mid \log\tau \mid^{-\frac{2}{n+2}},\ 0 < \tau < 1/e$.

To prove this, we consider $\xi \in \mathbb{R}^n$ and define

$$B = \{\zeta_j \in \mathbb{C}^n :\ \zeta_j \cdot \zeta_j = 0,\ |\zeta_1| = |\zeta_2| = h,\ \zeta_1 + \zeta_2 = \xi, j = 1, 2\}.$$

The application of Theorem 3.3 under the condition $h \geq \max(1, MN)$ ensures the existence of CGOs:

$$w_1 = e^{i\zeta_1 . x}(1 + r_1) \text{ and } w_2 = e^{i\zeta_2 . x}(1 + r_2),$$

for $(-\Delta + q_j)w_j = 0$ with $\|r_j\|_{L^2(\Omega)} \leq \frac{M}{h}\|q_j\|_{L^\infty(\Omega)}$, and M depending only on n and Ω.
We note by $\widetilde{q_j}$ the extension of q_j by zero to $\mathbb{R}^n$, by using the previous integral identity (3.8) with some calculus (considering $\Omega \subseteq B_R(0)$ and $h \leq e^{Rh}$), we get

$$|\widehat{(\widetilde{q_1} - \widetilde{q_2})}(\xi)| \leq C\left(e^{4Rh}\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + 1/h\right).$$

This last estimation means that the Fourier transform of the extension of $q_1 - q_2$ to $\mathbb{R}^n$ is a L^1 function.
For a constant $\rho > 0$, from the definition of the norm in $H^{-1}(\mathbb{R}^n)$, it is clear that

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq \|q_1 - q_2\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{|\xi|\leq\rho} \left|\langle\xi\rangle^{-1}\widehat{(q_1 - q_2)}(\xi)\right|^2 d\xi + \int_{|\xi|>\rho} \left|\langle\xi\rangle^{-1}\widehat{(q_1 - q_2)}(\xi)\right|^2 d\xi.$$

Then, with an appropriate choice of h and ρ, we get the desired conclusion.

Now, to reduce the stability result for the conductivity equation to the one for the Schrödinger equation, we need more facts about Sobolev spaces. Then, we recall properties 2.1, 2.2 and 2.3 from Section 2. Since we are working on a bounded domain with a smooth boundary, we need to use the corresponding of the previous properties on Ω and $\partial\Omega$. This is possible by introducing the extension operator and via $H^s(\mathbb{R}^{n-1})$ respectively. Notice that the continuous functions are defined on $\bar{\Omega}$, and the condition on s becomes $s \geq \frac{n-1}{2}$ on $\partial\Omega$ since that $\partial\Omega \subset \mathbb{R}^{n-1}$.
Moreover, we need the following inequality that relates the difference of the DN maps for conductivities to the one for potentials.

Lemma 3.1. *Under the same conditions of Theorem 3.4, there exists a constant $C(n,\Omega,N,s)$ such that*

$$\begin{aligned}&\|\Lambda_{q_1}-\Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\\ &\quad\leq C\left(\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}+\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|^{\frac{2}{2s+3}}_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right).\end{aligned} \tag{3.11}$$

Proof. For $f\in H^{1/2}(\partial\Omega)$, we use equation (3.3) to calculate $(\Lambda_{q_1}-\Lambda_{q_2})f$. We estimate the $H^{-1/2}$ norm of the resulting expression by the triangle inequality. Then, we use the a priori conditions and Proposition 2.1 to get

$$\begin{aligned}&\|\Lambda_{q_1}-\Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\\ &\qquad\leq C\left(\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}+\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)}\right).\end{aligned} \tag{3.12}$$

By Proposition 2.1 again, Proposition 2.3, the trace Theorem and the estimates on γ_j, we can change the C^1 norm in (3.12) to the L^∞ norm.
Then, the estimate (3.11) follows from the stability at the boundary result (3.10). □

To conclude this section, we give the idea of the proof of Theorem 3.4. We recall from Remark 3.1, the function $z=\log\gamma_1^{1/2}\gamma_2^{-1/2}\in C^2(\bar{\Omega})$ satisfying

$$\begin{cases}\nabla\cdot(\gamma_1\gamma_2)^{1/2}\nabla z=(\gamma_1\gamma_2)^{1/2}(q_1-q_2)\text{ in }\Omega,\\ z=\frac{1}{2}\log\gamma_1-\frac{1}{2}\log\gamma_2\text{ on }\partial\Omega.\end{cases}$$

By applying Theorem 3.5 and Lemma 3.1, we can obtain a bound for z in $H^1(\Omega)$ in terms of z in $H^{1/2}(\partial\Omega)$. Similar to the precedent proof, we change those norms to the L^∞ norm. Then, by the a priori constraints, Proposition 2.1,2.3 and estimate (3.10), we can deduce the continuous dependence of the initial data with the solution. Thus, the stability for the Calderón problem has been proved.

4. The partial data problem

The partial data type problem aims to reduce as much as possible the part of the boundary, where measurements are taken and excitations on the studied body are imposed, because from a realistic view, it is not practical to consider measurements on the whole boundary of some domain.
From now on, $\beta,\delta\in\mathbb{R}^n$ denote orthogonal unit vectors. The function φ is defined by $\varphi(x)=\delta\cdot x$, and we also define the following subsets of $\partial\Omega$:

- $\partial\Omega_{+,\epsilon}=\{x\in\partial\Omega:\ \delta\cdot\nu(x)>\epsilon\}$.
- $\partial\Omega_{-,\epsilon}=\{x\in\partial\Omega:\ \delta\cdot\nu(x)<\epsilon\}$.
- $\partial\Omega_{\pm}=\{x\in\partial\Omega:\ \pm\delta\cdot\nu(x)>0\}$.

The first result in this direction is due to Bukhgeim and Uhlmann [8], and we will present it in the next theorem.

Theorem 4.1. *For $j = 1, 2$, let $\gamma_j \in C^2(\bar{\Omega})$ be two positive functions. If $\gamma_1\big|_{\partial\Omega} = \gamma_2\big|_{\partial\Omega}$, and for $\epsilon > 0$, we have*

$$\Lambda_{\gamma_1} f\big|_{\partial\Omega_{-,\epsilon}} = \Lambda_{\gamma_2} f\big|_{\partial\Omega_{-,\epsilon}} \quad \forall f \in H^{1/2}(\partial\Omega),$$

and then $\gamma_1 = \gamma_2$ in Ω.

Unlike the previous section, this time to construct more general CGOs, we do not use Fourier analysis. Since we still need some kind of control on parts of the boundary, we use the weighted norm estimate of Carleman, which will be presented in the next subsection.

4.1. Carleman estimates

We recall $\zeta \in A = \{\zeta \in \mathbb{C}^n : \ \zeta \cdot \zeta = 0\}$. From Subsection 3.1, if (3.4) solves Schrödinger equation, then $e^{-i\zeta.x}(-\Delta + q)e^{i\zeta.x} r = f$. We write $\zeta = \frac{1}{d}(\beta + i\delta)$, where d is a positive small parameter. The application of Theorem 3.3 gives

$$\|r\|_{L^2(\Omega)} \leq Md\|e^{\delta.x/d}(-\Delta + q)e^{-\delta.x/d} r\|_{L^2(\Omega).}$$

This inequality introduces Carleman estimate that we will give for two types of functions. First, for smooth functions compactly supported. Secondly, for functions vanishing on the boundary. Contrary to the first case, the second case will involve boundary terms of the normal derivative.

Proposition 4.1. *(Carleman estimate 1) Let $q \in L^\infty(\Omega)$. There exist constants $d_0, C > 0$ such that for $0 < d \leq d_0$, we have*

$$\|w\|_{L^2(\Omega)} \leq Cd\|e^{\varphi/d}(-\Delta + q)e^{-\varphi/d} w\|_{L^2(\Omega)} \quad w \in C_c^\infty(\Omega). \tag{4.1}$$

Remark 4.1. Since the estimate (4.1) can be written

$$\|we^{\varphi/d}\ \|_{L^2(\Omega)} \leq Cd\|e^{\varphi/d}(-\Delta + q)w\|_{L^2(\Omega)},$$

we can see it as a uniqueness result. If $w \in C_c^\infty(\Omega)$ is a solution to Schrödinger equation, then w must be identically zero in the whole domain.

Let us introduce the following useful operators:

- $P_0 = (dD)^2$.
- $P = P_0 + d^2 q$.
- $P_{0,\varphi} = e^{\varphi/d} P_0 e^{-\varphi/d}$.
- $P_\varphi = P_{0,\varphi} + d^2 q$.

Proof. To prove (4.1), we need to establish that

$$d\|w\| \leq C\|P_\varphi w\| \quad w \in C_c^\infty(\Omega). \tag{4.2}$$

1. If $q = 0$, the last estimation (4.2) can be simplified to

$$d\|w\| \leq C\|P_{0,\varphi} w\|,$$

where $P_{0,\varphi} = 2i\delta \cdot dD + (dD)^2 - 1$. It is a question to find a positive lower bound for the norm of $P_{0,\varphi}w$. For that, we decompose the operator $P_{0,\varphi}$ and we write

$$P_{0,\varphi} = I + iJ,$$

where

$$I = (dD)^2 - 1 \text{ and } \; J = 2\delta \cdot dD$$

are two self-adjoint constant-coefficients differential operators.
A direct calculus gives

$$\|P_{0,\varphi}w\|^2 = \|Jw\|^2 + \|Iw\|^2 + (i[I,J]w|w).$$

Since $[I,J] = 0^{\dagger}$, we deduce that

$$\|P_{0,\varphi}w\|^2 = \|Iw\|^2 + \|Jw\|^2 \geq 0.$$

Poincaré inequality from Proposition 2.4 guarantees that $\|Jw\| \geq cd\|w\|$ with $c = c(\Omega)$. Thus, we get the desired estimate (4.2) with zero potential.

2. If $q \neq 0$, the last lower bound for the norm of $P_{0,\varphi}w$ (4.2) and the choice of $d_0 = \frac{1}{2C\|q\|_{L^\infty(\Omega)}}$ with $0 < d \leq d_0$ gives the conclusion.

□

As mentioned earlier, Carleman estimates provide a new method to construct CGOs. Before giving those solutions, we first consider the following existence result.

Theorem 4.2. *Let $q \in L^\infty(\Omega)$. There exist constants $d_0, C > 0$ such that for $0 < d \leq d_0$, for all $f \in L^2(\Omega)$, and the function $r \in L^2(\Omega)$ solves the inhomogeneous equation*

$$e^{\varphi/d}(-\Delta + q)e^{-\varphi/d}r = f \quad \textit{in } \Omega$$

with the estimate $\|r\|_{L^2(\Omega)} \leq Cd\|f\|_{L^2(\Omega)}$.

Proof.

We define the operator P_φ^* by:

$$P_\varphi^* : L^2(\Omega) \to L^2(\Omega),$$

$$P_\varphi^* = P_{0,-\varphi} + d^2\bar{q}.$$

If d_0 is as in Proposition 4.1, we have

$$\|w\| \leq C/d\|P_\varphi^* w\|. \tag{4.3}$$

We denote by $E \subset L^2(\Omega)$ the image of $C_c^\infty(\Omega)$ by P_φ^*. The linear functional

$$K : E \to \mathbb{C},$$

$$K(P_\varphi^*\vartheta) = (\vartheta \mid f) \; \forall \vartheta \in C_c^\infty(\Omega)$$

is well-defined since that Carleman estimate (4.3) for the operator P_φ^* ensures that P_φ is surjective.
By estimate (4.3) again, we have

$$\mid K(P_\varphi^*\vartheta) \mid \leq C/d\|P_\varphi^*\vartheta\|\|f\|.$$

†$[I,J]=IJ-JI$ is the commutator of the operators I and J. As both I and J are constant-coefficients differential operators, then we have always $[I,J]=0$.

This means that K is a bounded linear functional. Therefore, Hahn-Banach Theorem guarantees the existence of an extension of K to $L^2(\Omega)$, noted $\hat{K}$ with $\|\hat{K}\| \leq \|K\|$.
Since $\hat{K}$ is a bounded linear functional on $L^2(\Omega)$, it can be written as an inner product by the Riesz representation theorem. this means that there exists $\acute{r} \in L^2(\Omega)$ such that $\hat{K}(y) = (y|\acute{r}),\ y \in L^2(\Omega)$ and $\|\hat{K}\| = \|\acute{r}\|$. Thus, $P_\varphi \acute{r} = f$ weakly. The choice of $r = d^2\acute{r}$ gives the desired conclusion.

□

By possessing the valuable tool of Carleman estimate, we can give a slightly more general reconstruction for CGOs than the one given in Section 3. The form of solutions is

$$w = e^{\frac{-1}{d}(\varphi + i\psi)}(\lambda + r), \tag{4.4}$$

where ψ has values in $\mathbb{R}$, and λ is a complex amplitude. It is clear that (4.4) solves Schrödinger equation, if and only if

$$e^{\varphi/d}(-\Delta + q)e^{-\varphi/d}(e^{-i\psi/d}r) = f \quad \text{in } \Omega.$$

The application of Theorem 4.2 guarantees the existence of CGOs for this inhomogeneous equation provided that $\| f \| \leq C$, which is possible with an appropriate choice of ψ and λ (we can choose $\psi = \beta \cdot x$ such that $(\delta + i\beta) \cdot \nabla\lambda = 0$).

Next, we consider Carleman estimate for not compactly supported functions, which vanish at the boundary.

Proposition 4.2. *(Carleman estimate 2) Let $q \in L^\infty(\Omega)$. There exist constants $d_0, C > 0$ such that for $0 < d \leq d_0$, $\forall w \in C^\infty(\bar{\Omega})$ with $w|_{\partial\Omega} = 0$, we have the estimation*

$$\begin{aligned} &- d\left((\delta \cdot \nu)\partial_\nu w \mid \partial_\nu w\right)_{\partial\Omega_-} + \|w\|_{L^2(\Omega)} \\ &\qquad \leq Cd\|e^{\varphi/d}(-\Delta + q)e^{-\varphi/d}w\|_{L^2(\Omega)} + Cd((\delta \cdot \nu)\partial_\nu w \mid \partial_\nu w)_{\partial\Omega_+}. \end{aligned} \tag{4.5}$$

The idea of the proof is the same as the one for Proposition 4.1. As was mentioned before, this time integration by parts gives the above boundary terms.

4.2. Partial data uniqueness proof

The partial data uniqueness result for the Schrödinger equation is given in the next Theorem.

Theorem 4.3. *For $j = 1, 2$, let $q_j \in \mathcal{Q}$. If*

$$\Lambda_{q_1} f\big|_{\partial\Omega_{-,\epsilon}} = \Lambda_{q_2} f\big|_{\partial\Omega_{-,\epsilon}} \ \forall f \in H^{1/2}(\partial\Omega),$$

then $q_1 = q_2$ in Ω.

Remark 4.2. Analogous reasoning to the one in the proof of Corollary 3.1 allows us to conclude that Theorem 4.3 implies Theorem 4.1.

Proof. Recalling the integral identity (3.8)

$$\left\langle (\Lambda_{q_1} - \Lambda_{q_2}) w_1\big|_{\partial\Omega}, w_2\big|_{\partial\Omega} \right\rangle = \int_\Omega (q_1 - q_2) w_1 w_2 \ dx,$$

for $w_j \in H^1$ solve $-\triangle w_j + q_j w_j = 0$, $j = 1, 2$. By the condition on DN maps, the last boundary integral identity is restricted to $\partial\Omega_{+,\epsilon}$.
If $w_1 \in H^2(\Omega)$, then $\Lambda_{q_1}(w_1|_{\partial\Omega}) = \partial_\nu w_1\big|_{\partial\Omega}$ and $\Lambda_{q_2}(w_1|_{\partial\Omega}) = \partial_\nu \phi\big|_{\partial\Omega}$, with $\phi \in H^2(\Omega)$ solves the problem

$$\begin{cases} -\Delta\phi + q_2\phi = 0 \text{ in } \Omega, \\ \qquad\quad \phi = w_1 \text{ on } \partial\Omega. \end{cases}$$

Therefore, we get

$$\int_\Omega (q_1 - q_2) w_1 w_2 \ dx = \int_{\partial\Omega_{+,\epsilon}} \partial_\nu (w_1 - \phi) w_2 \ dS. \tag{4.6}$$

Let $\xi \in \mathbb{R}^n$, with $\xi \perp \delta$ and $\{\beta, \delta, \xi\}$ is an orthogonal triplet. We write $\psi(x) = \beta.x$, then the application of Theorem 3.3 gives the existence of CGOs:

$$w_1 = e^{1/d(\varphi + i\psi)} e^{ix.\xi}(1 + r_1) \ \text{ and } \ w_2 = e^{-1/d(\varphi+i\psi)}(1 + r_2),$$

with the usual estimates $\|r_j\| \leq C/d$, and $\|\nabla r_j\| \leq C$.
We write $w = w_1 - \phi \in H^2(\Omega) \cap H^1_0(\Omega)$. For $d \to 0$, the equation (4.6) becomes

$$\int_\Omega e^{ix.\xi}(q_1 - q_2) \ dx = \int_{\partial\Omega_{+,\epsilon}} (\partial_\nu w) w_2 \ dS. \tag{4.7}$$

We aim to show that

$$\lim_{d\to+\infty} \int_{\partial\Omega_{+,\epsilon}} (\partial_\nu w) w_2 \ dS = 0.$$

From the application of Cauchy-Schwarz to (4.7), it follows that

$$\left| \int_{\partial\Omega_{+,\epsilon}} (\partial_\nu w) w_2 \ dS \right|^2 \leq \left(\int_{\partial\Omega_{+,\epsilon}} |e^{-\varphi/d} \partial_\nu w|^2 \ dS \right) \left(\int_{\partial\Omega_{+,\epsilon}} |e^{\varphi/d} w_2|^2 \ dS \right).$$

The application of estimate (4.5) with the weight $-\varphi$ and $z = e^{\varphi/d} w$ to the function w with potential q_2 makes the first term on the right-hand side of the last inequality less than or equal to Cd. Furthermore, the trace theorem shows that the second term can be bounded by a constant C. Thus, for small d we have

$$\left| \int_{\partial\Omega_{+,\epsilon}} (\partial_\nu w) w_2 \ dS \right| \leq C d^{1/2}.$$

Letting $d \to 0$ in (4.7), we conclude that

$$\int_\Omega e^{ix.\xi}(q_1 - q_2) \ dx = 0 \ \ \forall \xi \in \mathbb{R}^n, \xi \perp \delta.$$

Since the condition on DN maps is true for a fixed $\epsilon > 0$, it is also true on $\partial\Omega_{-,\tau}(\eta)$ with a sufficiently close η to δ on $S^n(0,1)$. Then, the last equation holds also on an open cone of $\mathbb{R}^n$. The analyticity of Fourier transform ensures that $q_1 = q_2$ on the whole domain Ω. $\square$

5. Some perspectives

In recent years, great progress has been made by several authors in the research field of Calderón's problem, which was the motivation behind writing this paper to draw increasing attention to this problem to improve the known results. Thanks for the rapid development in this topic, we note that the results of the previous sections can be considered as an introduction to this domain. Therefore, a lot might lie beyond this paper. We propose some challenges and research perspectives, which can be subject to new results in several directions.

1. The problem of finding the lowest regularity condition on the conductivity under which uniqueness holds inspired many authors. Recently, Caro and Rogers [10] have applied Bourgain's spaces to prove uniqueness for Lipschitz conductivities in three and higher dimensions. Haberman [16] involves L^p harmonic analysis to show this result for $W^{1,n}$ conductivities in dimensions $n = 3, 4$, and for $W^{1+(1-\theta)(\frac{1}{2}-\frac{2}{n}),\frac{n}{1-\theta}}$, $\theta \in [0, 1)$ for $n = 5, 6$. By using the valuable tool of bilinear estimates, more improved results were given. For $\gamma \in W^{41/40+,5}$ and $\gamma \in W^{11/10+,6}$ for $n = 5$ and $n = 6$, respectively in [17]. Also, for $W^{1+\frac{n-5}{2p}+,p}$, $p \in [n, \infty)$ conductivities in five and higher dimensions [27]. The observation of those results makes us wonder how much it would be interesting to check whether it is possible to prove Brown's conjecture [6], which affirms that in three and higher dimensions $\gamma \in W^{1,n}$ is the minimum possible regularity for which uniqueness holds.
2. The construction technique of the CGOs beyond this paper was sufficient to prove uniqueness and stability results. However, this method cannot provide us with a procedure for reconstructing the conductivity from DN map. In [26], Nachman provided a constructive procedure to compute $\gamma \in C^{1,1}$ from Λ_γ. This process was followed by García and Zhang in [14] to reconstruct C^1 or Lipschitz conductivities with $|\nabla \log \gamma|$ sufficiently small. Based on the uniqueness result of [10], we wonder if we still can generalize Nachman's result to Lipschitz conductivities by taking off the smallness condition on $|\nabla \log \gamma|$ to improve the results in [14]. This problem seems more complicated and may require new ideas beyond the known techniques. We intend to study those reconstruction issues for the inverse conductivity problem soon.
3. The results presented in Section 4 can be considered as an introduction to the partial data type problems. The general strategy viewed there was followed by Kenig, Sjöstrand and Uhlmann in [20] to construct a wider class of CGOs by improving the previously discussed Carleman estimates to more general ones. Another research direction that seems worth pursuing is the stability of the partial data problem. For the problem of full data, it is known that the logarithmic modulus of continuity type stability is optimal [3]. In the light of the partial data uniqueness result in [20], one can ask for the optimal stability for the Carleman estimate approach. We refer the reader to [19, 32] on the recent progress in this partial data type problem.

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