Journal of Nonlinear Modeling and Analysis
Volume 4, Number 1, March 2022, 103–113
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.103

A Note on the Stefan-Boltzmann Problem for Heat Transfer in a Fin

Boris P. Belinskiy[1], John R. Graef[1,†] and Lingju Kong[1]

Abstract A fin is traditionally thought of as an extension of a surface to facilitate the transfer of heat away from a larger body to which it is attached. In this paper, the authors study some mathematical properties of a nonlinear heat transfer model for a fin and its relation to an associated linear model. Specifically, they prove that the solution exists and is unique, and they determine bounds for the temperature. Further, they prove the monotonicity of the temperature distribution, and they obtain an estimate for the maximal difference between the temperatures as determined by the nonlinear and linear models.

Keywords Heat transfer, Fin, Stefan-Boltzmann law, Existence and uniqueness, Dependence.

MSC(2010) 34B15.

1. Introduction

Extended surfaces, often called fins, are used in heat exchange devices to facilitate the transfer of the heat away from the main body. The usual physical assumptions in the heat transfer analysis of a fin are the following (see Lienhard IV and Lienhard V [15]):

(i) Heat transfer is 1-D.

(ii) Heat transfer is steady-state.

(iii) The conduction coefficient k, the convective heat transfer coefficient h, and the emmisivity ϵ are constant.

(iv) The temperature T_b at the base of the fin is constant.

(v) The temperature T_∞ of the fluid surrounding the fin is constant.

(vi) The body of the fin is a solid of revolution.

Then, it is the temperature variation along the fin that needs to be determined as a function of the distance from the base.

[†]the corresponding author.
Email address: Boris-Belinskiy@utc.edu (B. Belinskiy), John-Graef@utc.edu (J. R. Graef), Lingju-Kong@utc.edu (L. Kong)
[1]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, Tennessee 37403, USA

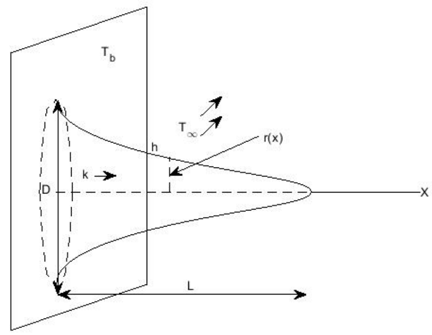

Figure 1. Profile of a fin

The geometry of the fin is as follows: First, it is located on the interval $[0, L]$ and is attached to a heated surface at $x = 0$. The radius of the cross section is given by $r(x) : [0, L] \to \mathbb{R}_+$, and the cross-sectional area, denoted by $A(x)$, is $A(x) = \pi r^2(x)$. The differential of surface area becomes $dA_s := P(x)dx = 2\pi r(x)\sqrt{1 + [r'(x)]^2}dx$.

We will denote the temperature distribution in the fin by $T(x) : [0, L] \to \mathbb{R}_+$. Then, the (steady-state) heat equation can be obtained from the energy balance for the region between x and $x + dx$; it has the form (see [6, 15, 17, 19])

$$k\frac{d}{dx}\left(A\frac{dT}{dx}\right) - h\,\frac{dA_s}{dx}\,(T - T_\infty) - \epsilon\sigma\frac{dA_s}{dx}(T^4 - T_\infty^4) = 0, \quad x \in (0, L). \tag{1.1}$$

The third term on the left hand side of this equation represents the amount of heat transferred from the fin per unit area due to radiation. Here, ϵ is the emmisivity of the fin face, and σ is the Stefan-Boltzmann constant. It is the presence of the radiation effect that makes the equation nonlinear. Since the constant $\sigma \approx 5.67 \times 10^{-8} Wm^{-2}K^{-4}$ is small, at low temperatures the third term on the left hand side of (1.1) may be neglected so that the equation reduces to the linear one

$$k\frac{d}{dx}\left(A\frac{dT}{dx}\right) - h\,\frac{dA_s}{dx}\,(T - T_\infty) = 0, \quad x \in (0, L). \tag{1.2}$$

In addition to equations (1.1) and (1.2), we need to formulate the boundary conditions (BCs) that are to be satisfied.

The temperature at $x = 0$ is assumed to be the same as that of the base, which is namely T_b. At the right hand endpoint $x = L$, we assume that we have an adiabatic condition. Thus, our boundary conditions become

$$T(0) = T_b \quad \text{and} \quad \left(A\frac{d}{dx}T(x)\right)\Big|_{x=L} = 0. \tag{1.3}$$

Boundary conditions other than (1.3) have been considered in the literature. For example, see [6, 15, 19].

Remark 1.1. In [17], the (steady-state) heat equation for a circular fin is given in the form (as adjusted to our notation)

$$\frac{d}{dr}\left(kA\frac{dT}{dr}\right) - h\,P\,(T - T_\infty) - \epsilon\sigma\,P\,(T^4 - T_\infty^4) = 0, \quad r \in (r_b, r_t), \tag{1.4}$$

where P is the wetted perimeter. We wish to point out the similarity to our equation (1.1).

We will apply methods used in the qualitative theory of differential equations (e.g., see [1,9]) to study existence and uniqueness of the solutions, determine bounds for the temperature, prove its monotonic distribution and obtain an estimate for the maximal difference between the temperatures as determined by the nonlinear and the linear models. For these mathematical studies, the class of the coefficients of the differential equation (1.1) is of central importance and to this we refer to standard engineering practice where fins that have certain optimal properties are used (see a brief review on optimality studies below). Analysis of these optimal fins shows that $A(x) > 0$ on $[0, L)$, but $A(L) = 0$. It is well-known that if the leading coefficient of a differential equation vanishes at the end point (so-called singular point), the aforementioned mathematical problems may be quite non-trivial. Our goal is to find a broad class of functions $A(x)$ such that the existence, uniqueness, and other aforementioned properties are present.

In the sequel, we assume that the following condition holds:

(H) $T_b > T_\infty$, $A(x) > 0$ on $[0, L)$, and

$$\int_0^L \frac{1}{A(\tau)} \int_\tau^L r(t)\sqrt{1+[r'(t)]^2}\, dt d\tau < \infty. \tag{1.5}$$

The physical meaning of the first requirement is that the temperature at the base of fin exceeds the temperature of the surrounding medium, so that the physical application of a fin makes sense. The second requirement means that the only sharp point of a fin is its tip. Condition (1.5) describes the class of fins for which we are able to prove our mathematical results. This means that the fin is not too sharp at the tip. Later, we show that the fins found in [2,10,11] satisfy this condition.

In a recent paper, the present authors [3] made a qualitative study of boundary value problem (BVP) (1.1), (1.3) and its relation to the classical linear problem (1.2), (1.3) under the condition

$$\int_0^L \frac{1}{A(\tau)} d\tau < \infty. \tag{1.6}$$

We will show later that this condition is not always physically realistic. In particular, it is not satisfied for the known fins in [2,10,11].

Remark 1.2. Observing that for any given fin, its surface area is finite, so there is a constant $M > 0$ such that

$$\int_0^L r(t)\sqrt{1+[r'(t)]^2}\, dt < M.$$

Thus,

$$\begin{aligned}\int_0^L \frac{1}{A(\tau)} \int_\tau^L r(t)\sqrt{1+[r'(t)]^2}\, dt d\tau &\le \int_0^L \frac{1}{A(\tau)} \int_0^L r(t)\sqrt{1+[r'(t)]^2}\, dt d\tau \\ &< M \int_0^L \frac{1}{A(\tau)} d\tau.\end{aligned}$$

Hence, we see that (1.6) implies (1.5). In Subsection 2.3 below, we examine conditions (1.5) and (1.6), and show by means of two examples that in fact (1.5) is much less restrictive than (1.6).

A great deal of work in the literature has been devoted to the study of heat transfer problems for fins. For example, basic concepts in the theory of heat transfer can be found in [15,19]. Some versions of the linear problem were studied in [6,19]. Two important characteristics of a fin are its efficiency and effectiveness [6,19].

Here, it is that the problem of *optimization* of a fin appears. To the best of our knowledge, the first result in this direction belongs to E. Schmidt [18] who formulated the optimality hypothesis that the temperature distribution along a minimum volume one-dimensional fin is linear. In [16], the problem of finding a fin of minimum weight was re-formulated based on the Pontryagin Maximum Principle, and the resulting nonlinear two-point BVP was solved numerically. Since then, extensive research has been done on finding the optimal form of a fin of given volume that would maximize efficiency, of given efficiency that would minimize the volume and other similar types of problems. Both linear (1.2) and nonlinear (1.1) models of fins were studied. In [17], the optimum design of a circular fin is studied. The nonlinear model (see (1.4)) is considered but the length-of-arc idealization is used among other assumptions. The Runge–Kutta method is used to solve the optimization problem. For some recent examples, we refer the reader to [2,4,5,7,8,10–14,20–22] and the references contained therein. We mention only a few particular results. In [10,11], the analytic solution for an optimal fin was found without the "length-of-arc" assumption. For example, a generalized methodology for the optimum design of fins of three basic geometries is developed in [12]. Papers devoted to the study of fins under more complex physical conditions such as porous fins and wet fins can be found in [14] where the minimum shapes of porous fins are studied. The heat transfer coefficient is a function of temperature and calculus of variations techniques are often used in the analysis. The dependence of the optimal volume on porosity has also been examined. Below, we discuss [2] and only note here that the authors proceeded without the "length-of-arc" assumption.

An outline of our study is as follows: The existence and uniqueness of solutions to the BVP (1.1), (1.3) is proved in Theorem 2.1. The estimates (2.3) ($T_\infty \leq T(x) \leq T_b$) are proved in Lemma 2.2. The monotonicity of the temperature along the fin is given in Theorem 2.2, and the dependence of the temperature on the Stefan–Boltzmann constant is found in Theorem 2.3. Examples of two fins are discussed in Section 2.3 where it is shown in particular that the fins found in [2, 10, 11] satisfy the condition (1.5). Hence, the aforementioned results (existence, uniqueness, monotonicity, estimates (2.3)) hold for these fins.

2. Main results

2.1. Existence and uniqueness of solutions of BVP (1.1), (1.3)

In what follows, we let X be the Banach space of all continuous functions $T(x)$ on $[0, L]$ equipped with the norm

$$\|T\| = \max_{x\in[0,L]} |T(x)|, \quad T \in X.$$

For any $T \in X$, define a function $\widetilde{T} : [0, L] \to \mathbb{R}$ by

$$\widetilde{T}(x) = \max\{T_\infty,\ \min\{T(x), T_b\}\}, \quad x \in [0, L].$$

We consider the continuous functional $F : [0, L] \times X \to \mathbb{R}$ given by

$$F(x, T(x)) = h\frac{dA_s}{dx}\left(\widetilde{T} - T_\infty\right) + \epsilon\sigma\frac{dA_s}{dx}\left(\widetilde{T}^4 - T_\infty^4\right) + \frac{T(x) - T_\infty}{1 + T^2(x)}\chi_{\{T(x)<T_\infty\}}(x) + \frac{T(x) - T_b}{1 + T^2(x)}\chi_{\{T(x)>T_b\}}(x), \tag{2.1}$$

where the characteristic function χ_I on any set $I \subset \mathbb{R}$ is defined by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

We observe that Lemmas 1 and 3 in [3] are still true under the new condition (1.5) since their proofs did not need condition (1.6). These are stated below as Lemmas 2.1 and 2.2 respectively.

Lemma 2.1. *BVP* (1.1), (1.3) *has at most one solution.*

Lemma 2.2. *Assume that $T(x)$ is a solution of the BVP consisting of the equation*

$$k\frac{d}{dx}\left(A\frac{dT}{dx}\right) - F(x, T(x)) = 0, \quad x \in (0, L), \tag{2.2}$$

and the BCs (1.3)*. Then,*

$$T_\infty \le T(x) \le T_b \quad \text{for all } x \in [0, L]. \tag{2.3}$$

Consequently, $T(x)$ is a solution of BVP (1.1), (1.3)*.*

In [3], inequality (1.6) above was a key assumption in the construction of a Green's function for a linear problem (see [3, equation (2.5)]) so that an equivalent integral operator equation could be obtained for an associated BVP. With the new condition (1.5), the Green function in [3] is not well-defined. However, in this paper, we will be able to circumvent this problem and achieve the same goal by using the following lemma.

Lemma 2.3. *For any $l \in C[0, L]$ with $\int_0^L \frac{1}{A(\tau)} \int_\tau^L l(t)dtd\tau < \infty$, a function $T(x)$ is a solution of the BVP consisting of the equation*

$$k\frac{d}{dx}\left(A\frac{dT}{dx}\right) - l(x) = 0, \quad x \in (0, L) \tag{2.4}$$

and the BCs (1.3)*, if and only if*

$$T(x) = T_b - \frac{1}{k}\int_0^x \frac{1}{A(\tau)}\int_\tau^L l(t)dtd\tau. \tag{2.5}$$

Proof. Assume first that $T(x)$ is a solution of BVP (2.4), (1.3). Integrating (2.4) from x to L and using the second condition in (1.3), we obtain

$$\frac{d}{dx}T(x) = -\frac{1}{kA(x)}\int_x^L l(t)dt, \quad x \in (0, L).$$

Then, (2.5) follows from integrating the above equality from 0 to x and using the first condition in (1.3). On the other hand, if $T(x)$ satisfies (2.5), it is easy to check that it is a solution of BVP (2.4), (1.3). This completes the proof of the lemma. □

Now, we show that our nonlinear problem has a unique solution.

Theorem 2.1. *BVP* (1.1), (1.3) *has a unique solution* $T(x)$. *Moreover, this solution satisfies* (2.3).

Proof. First, we will prove that BVP (2.2), (1.3) has a solution in X. Define an operator $K : X \to X$ by

$$(KT)(x) = T_b - \frac{1}{k}\int_0^x \frac{1}{A(\tau)}\int_\tau^L F(t, T(t))dtd\tau, \quad T \in X.$$

From (1.5) and (2.1), we have

$$\int_0^x \frac{1}{A(\tau)}\int_\tau^L F(t, T(t))dtd\tau < \infty.$$

Then, by Lemma 2.3, $T(x)$ is a solution of BVP (2.2), (1.3), if and only if T is a fixed point of K. A standard argument can be used to verify that K is completely continuous. Then, Schauder's fixed point theorem implies that there exists a fixed point T of K in X, Thus, $T(x)$ is a solution of BVP (2.2), (1.3). In view of Lemmas 2.1 and 2.2, this completes the proof of the theorem. □

Remark 2.1. As in Theorem 2.1, BVP (1.2), (1.3) also has a unique solution $T(x)$ that satisfies (2.3).

The following theorem is easy to verify.

Theorem 2.2. *Let* $T(x)$ *be the unique solution of BVP* (1.1), (1.3) *or BVP* (1.2), (1.3). *Then,* $T'(x) \leq 0$ *on* $[0, L]$, *and so* $T(x)$ *is nonincreasing in* $[0, L]$.

We note that monotonicity of the temperature was first conjectured by Schmidt [18] for optimal fins.

2.2. Dependence of the temperature on the parameter σ

Let $T_\sigma(x)$ be the unique solution of BVP (1.1), (1.3) and $T_0(x)$ be the unique solution of BVP (1.2), (1.3). An explicit estimate for the norm $\|T_\sigma - T_0\|$ of the temperature difference between the nonlinear and linear problems is given in the next theorem.

Theorem 2.3. *Assume that* $0 < \theta < 1$, *where*

$$\theta = \frac{h}{k}\int_0^L \frac{1}{A(\tau)}\int_0^L g(t)dtd\tau$$

with $g(x) = \frac{dA_s}{dx} = 2\pi r(x)\sqrt{1+[r'(x)]^2}$*. Then, we have*

$$\|T_\sigma - T_0\| \le \frac{\epsilon\sigma\left(T_b^4 - T_\infty^4\right)\theta}{(1-\theta)h}. \tag{2.6}$$

Proof. First recall that ϵ, h, and σ are physical constants appearing in equations (1.1) and (1.2). In view of (1.5), we may apply Lemma 2.1 to obtain

$$T_\sigma(x) = T_b - \frac{1}{k}\int_0^x \frac{1}{A(\tau)}\int_\tau^L \left[hg(t)(T_\sigma(t) - T_\infty) + \epsilon\sigma g(t)\left(T_\sigma^4(t) - T_\infty^4\right)\right] dtd\tau.$$

and

$$T_0(x) = T_b - \frac{1}{k}\int_0^x \frac{1}{A(\tau)}\int_\tau^L hg(t)(T_0(t) - T_\infty)dtd\tau.$$

Then,

$$T_\sigma(x) - T_0(x) = -\frac{1}{k}\int_0^x \frac{1}{A(\tau)}\int_\tau^L \left[hg(t)(T_\sigma(t) - T_0(t)) + \epsilon\sigma g(t)\left(T_\sigma^4(t) - T_\infty^4\right)\right] dtd\tau.$$

Since $T_\infty \le T_\sigma(x) \le T_b$ for all $x \in [0, L]$, we have

$$\begin{aligned}|T_\sigma(x) - T_0(x)| &\le \left[h\|T_\sigma - T_0\| + \epsilon\sigma\left(T_b^4 - T_\infty^4\right)\right]\frac{1}{k}\int_0^x \frac{1}{A(\tau)}\int_\tau^L g(t)dtd\tau \\ &\le \left[h\|T_\sigma - T_0\| + \epsilon\sigma\left(T_b^4 - T_\infty^4\right)\right]\frac{1}{k}\int_0^L \frac{1}{A(\tau)}\int_\tau^L g(t)dtd\tau.\end{aligned}$$

Thus,

$$\|T_\sigma - T_0\| \le \theta\|T_\sigma - T_0\| + \epsilon\sigma\left(T_b^4 - T_\infty^4\right)\frac{\theta}{h},$$

from which (2.6) follows. This completes the proof of the theorem. □

Remark 2.2. From (2.6), we see that

$$\lim_{\sigma\to 0^+} T_\sigma(x) = T_0(x) \quad \text{uniformly on } [0, L].$$

2.3. Examples

In this subsection, we provide two examples to show that the condition (1.5) in this paper is much less restrictive than the corresponding condition (1.6) used in [3].

Example 2.1. Let the radius of the cross section of the fin be given by $r(x) = C(L-x)^n$, where $C > 0$ and $n > 0$ are constants. We will verify that (1.6) only holds only for $n \in (0, 1/2)$ and (1.5) holds for all $n \in (0, 2)$.

In fact, it is trivial to notice that (1.6) holds for $n \in (0, 1/2)$. As for condition (1.5), we first assume that $n \in (0, 1)$. Then, since

$$[r'(t)]^2 = C^2n^2(L-t)^{2(n-1)} \ge C^2n^2L^{2(n-1)} > 0 \quad \text{for all } t \in [0, L],$$

there exists a constant $C_1 > 0$ such that $1 \le C_1[r'(t)]^2$ on $[0, L]$. Thus,

$$1 + [r'(t)]^2 \le C_1[r'(t)]^2 + [r'(t)]^2 = (C_1+1)[r'(t)]^2 = C_2^2(L-t)^{2(n-1)}$$

for all $t \in [0, L)$, where $C_2 = Cn\sqrt{C_1 + 1}$. Hence, we have

$$\begin{aligned}\int_0^L \frac{1}{A(\tau)} \int_\tau^L r(t)\sqrt{1 + [r'(t)]^2}\, dtd\tau \\ &\le C_3 \int_0^L \frac{1}{(L-\tau)^{2n}} \int_\tau^L (L-t)^n \sqrt{(L-t)^{2(n-1)}}\, dtd\tau \\ &\le C_3 \int_0^L \frac{1}{(L-\tau)^{2n}} \int_\tau^L (L-t)^{2n-1}\, dtd\tau = \frac{C_3 L}{2n} < \infty, \end{aligned} \tag{2.7}$$

where $C_3 = \frac{C_2}{\pi C}$ is a positive constant.

Next, we assume that $n \in [1, 2)$. Then,

$$1 + [r'(t)]^2 = 1 + C^2 n^2 (L-t)^{2(n-1)} \le C_4^2 \quad \text{for all } t \in [0, L],$$

where $C_4 = \sqrt{1 + C^2 n^2 L^{2(n-1)}}$. Thus,

$$\begin{aligned}\int_0^L \frac{1}{A(\tau)} \int_\tau^L r(t)\sqrt{1 + [r'(t)]^2}\, dtd\tau \le C_5 \int_0^L \frac{1}{(L-\tau)^{2n}} \int_\tau^L (L-t)^n\, dtd\tau \\ = \frac{1}{n+1} C_5 \int_0^L \frac{1}{(L-\tau)^{2n}} (L-\tau)^{n+1} d\tau = \frac{1}{(n+1)(2-n)} C_5 L^{2-n} < \infty, \end{aligned} \tag{2.8}$$

where $C_5 = \frac{C_4}{\pi C}$ is a positive constant.

Now, from (2.7) and (2.8), we see that (1.5) holds for all $n \in (0, 2)$.

Example 2.2. Hanin and Campo [11] showed that the optimal fin profile in the form of a body of revolution has the radius of the cross-section $r(x)$ given by (see [11, equation (32)])

$$r(x) = \frac{1}{\gamma}\left(\rho - \sqrt{1 - (1-\gamma x)^2}\right), \quad 0 \le x \le L := \frac{1}{\gamma}\left(1 - \sqrt{1-\rho^2}\right), \tag{2.9}$$

where $\gamma = \frac{h}{k}$ and $\rho = \frac{q_0}{k\theta_0} < 1$ with q_0 and θ_0 being the heat transfer rate at the fin semi-base and the temperature excess at the fin base respectively.

We claim that for $r(x)$ given in (2.9), condition (1.5) holds, but (1.6) does not. To show that (1.5) holds, note that from (2.9), we see that

$$r'(x) = \frac{\gamma x - 1}{\sqrt{1 - (1-\gamma x)^2}}, \quad 0 \le x \le L.$$

Thus,

$$r'(0) = -\infty \quad \text{and} \quad -\infty < r'(x) \le r'(L) = -\frac{\sqrt{1-\rho^2}}{\rho} < 0 \quad \text{for all } 0 < x \le L.$$

Hence, there exists a constant $C_6 > 0$ such that

$$\sqrt{1 + [r'(x)]^2} \le -C_6 r'(x) \quad \text{for all } 0 \le x \le L.$$

This implies

$$\begin{aligned}\int_0^L \frac{1}{A(\tau)} \int_\tau^L & r(t)\sqrt{1+[r'(t)]^2}\ dtd\tau \\ &\le \frac{1}{\pi} C_6 \int_0^L \frac{1}{[r(\tau)]^2} \int_\tau^L r(t)(-r'(t))\ dtd\tau \\ &\qquad \le \frac{1}{\pi} C_6 \int_0^L \frac{1}{[r(\tau)]^2} \frac{1}{2} [r(\tau)]^2\ d\tau = \frac{1}{2\pi} C_6 L < \infty.\end{aligned}$$

Thus, (1.5) holds.

To see that (1.6) does not hold, we first observe that

$$\int_0^L \frac{1}{A(\tau)} d\tau = \frac{\gamma^2}{\pi} \int_0^L \frac{1}{\left(\rho - \sqrt{1-(1-\gamma\tau)^2}\right)^2} d\tau.$$

Then, making the substitution $1 - \gamma\tau = \sin\vartheta$ yields

$$\int_0^L \frac{1}{A(\tau)} d\tau = \frac{\gamma}{\pi} \int_{\vartheta_1}^{\frac{\pi}{2}} \frac{\cos\vartheta}{(\rho - \cos\vartheta)^2} d\vartheta,$$

where $\vartheta_1 = \arcsin\left(\sqrt{1-\rho^2}\right) \in (0, \pi/2)$. Now, making another substitution $u = \cos\vartheta$, we see that

$$\int_0^L \frac{1}{A(\tau)} d\tau = \frac{\gamma}{\pi} \int_0^\rho \frac{u}{(\rho - u)^2\sqrt{1-u^2}} du = \infty.$$

Therefore, condition (1.6) does not hold.

Barman et al. [2] found the optimal form of the longitudinal wet fin. Similarly, with the Hanin–Campo fin, we can prove that the optimal form found in [2] satisfies (1.5) but not (1.6).

Examples 2.1 and 2.2 above confirm the fact that the class of functions $A(x)$ is much broader under the condition (1.5) than under (1.6).

3. Conclusions

We apply methods from the qualitative theory of differential equations to mathematically analyze the nonlinear heat transfer model for a fin and its relation to an associated linear model. Our consideration is made in the Banach space of continuous functions $T(x)$ on $[0, L]$. First, we prove that a solution of the BVP exists and is unique. To accomplish this, we reduce the solution of the BVP to the a fixed point for an equivalent integral operator equation and use Schauder's fixed point theorem. The uniqueness is proved by a careful mathematical analysis of the problem. We also prove that if the natural physical assumption that the temperature of the base T_b is higher than the temperature of the surrounding medium T_∞. Then, at any point x of the fin, $T_\infty \le T(x) \le T_b$. Moreover, it is proved mathematically (see Theorem 2.2 above) that the temperature dissipates, i.e. $T'(x) \le 0$ through the fin as is expected physically, and the temperature in the fin decreases to the temperature of the surrounding medium (see Remark 2.1). Since the temperatures T_b and T_∞ are given, the last inequality suggests the possibility to control a solution found by numerical or asymptotic methods.

Monotonicity of the temperature along the fin is summarized in Theorem 2.2. The integral equations for the temperatures, along with the inequality $T(x) \leq T_b$ result in the continuous dependence of the temperature on the Stefan–Boltzmann constant. This result allows us to find an estimate for the maximal difference between the temperatures as determined by the nonlinear model and the linear model. The examples of two (optimal) fins are discussed. It is shown that the fins found in [2, 10, 11] satisfy the condition (1.5). Hence, the aforementioned results (existence, uniqueness, monotonicity, estimates) hold for these fins.

Acknowledgements

The authors are grateful to the anonymous reviewer for the valuable remarks that helped improve the quality of our presentation and for suggesting references of which the authors were unaware.

References

[1] R. P. Agarwal, M. Meehan and D. O' Regan, *Fixed point theory and applications*, Cambridge Tracts in Mathematics, 141, Cambridge University Press, Cambridge, 2001.

[2] D. Barman, D. Goswami and B. Kundu, *An optimum longitudinal wet fin without the "length-of-arc" assumption*, HVAC&R Research, 2013, 19(4), 444–456.

[3] B. P. Belinskiy, J. R. Graef and L. Kong, *Stefan-Boltzmann problem for heat transfer in a fin*, Mathematical Methods in the Applied Sciences, 2021, 44(6), 4745–4755.

[4] B. P. Belinskiy, J. W. Hiestand and J. V. Matthews, *Piecewise uniform optimal design of a bar with an attached mass*, Electronic Journal Differential Equations, 2015, 206, 1–17.

[5] B. P. Belinskiy, J. W. Hiestand and L. Weerasena, *Optimal design of a fin in steady-state*, Applied Mathematical Modelling, 2020, 77(2), 1188–1200.

[6] T. L. Bergman, A. S. Lavine, F. P. Incropera and D. P. Dewitt, *Fundamentals of Heat and Mass Transfer*, Seventh Edition, John Wiley & Sons, New York, 2011.

[7] R. Das and B. Kundu, *Direct and inverse approaches for analysis and optimization of fins under sensible and latent heat load*, International Journal of Heat and Mass Transfer, 2018, 124, 331–343.

[8] R. Das, K. Singh, B. Akay and T. K. Gogoi, *Application of artificial bee colony algorithm for maximizing heat transfer in a periodic fin*, Proceedings of the Institution of Mechanical Engineers E, Journal of Process Mechanical Engineering, 2016, 232(1), 38–48.

[9] J. R. Graef, J. Henderson, L. Kong and X. Liu, *Ordinary Differential Equations and Boundary Value Problems, Volume II: Boundary Value Problems*, Trends in Abstract and Applied Analysis, Volume 8, World Scientific, New Jersey, 2018.

[10] L. G. Hanin, *A New Optimum Pin Fin Beyond the "Length-of-arc" Assumption*, Heat Transfer Engineering, 2008, 29(7), 608–614.

[11] L. G. Hanin and A. Campo, *A new minimum volume straight cooling fin taking into account the length of arc*, International Journal of Heat and Mass Transfer, 2003, 46(26), 5145–5152.

[12] B. Kundu and P. K. Das, *Optimum Profile of Thin Fins with Volumetric Heat Generation: A Unified Approach*, Journal of Heat Transfer, 2005, 127(8), 945–948.

[13] B. Kundu and K. S. Lee, *A novel analysis for calculating the smallest envelope shape of wet fins with a nonlinear mode of surface transport*, Energy, 2012, 44(1), 527–543.

[14] B. Kundu and K. S. Lee, *Exact analysis for minimum shape of porous fins under convection and radiation heat exchange with surrounding*, International Journal of Heat and Mass Transfer, 2015, 81, 439–448.

[15] J. H. Lienhard IV and J. H. Lienhard V, *A Heat Transfer Textbook*, Fourth Edition, Phlogiston Press, Cambridge, 2018.

[16] C. J. Maday, *The minimum weight one-dimensional straight cooling fin*, Journal of Engineering Industry, 1974, 96(1), 161–165.

[17] P. Razelos and R. K. Krikiss, *The optimum design of radiating and convective-radiating circular fins*, Heat Transfer Engineering, 2003, 24(3), 17–41.

[18] E. Schmidt, *Die Wärmebertragung durch Rippen*, Zeitschrift des Vereines Deutscher Ingenieure, 1926, 70, 947–951.

[19] P. J. Schneider, *Conduction Heat Transfer*, First Edition, Addison-Wesley, Reading, 1955.

[20] R. K. Singla and R. Das, *A differential evolution algorithm for maximizing heat dissipation in stepped fin*, Neural Computing & Applications, 2018, 30, 3081–3093.

[21] J. Wang and X. Wang, *The heat transfer optimization of conical fin by shape modification*, Chinese Journal of Chemical Engineering, 2016, 24(8), 972–978.

[22] P. A. Wankhade, B. Kundu and R. Das, *Establishment of non-Fourier heat conduction model for an accurate transient thermal response in wet fin*, International Journal of Heat and Mass Transfer, 2018, 126(A), 911–923.