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Existence Results for Fractional Differential Equations with the Riesz-Caputo Derivative

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Abstract In this paper, we apply some fixed point theorems to attain the existence of solutions for fractional differential equations with the space-time Riesz-Caputo derivative. We study the boundary value problems that the nonlinearity term f is relevant to fractional integral and fractional derivative. In addition, the boundary conditions involve integral. Two examples are given to show the effectiveness of theoretical results.

Keywords Fractional differential equation, Riesz-Caputo derivative, Fixed point theorem.

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1. Introduction

With the progress of modern science and technology and the advancement of fractional order theory, fractional order differential equations have been widely used in signal and image processing, electromagnetism, mechanics, optics and other fields of science and engineering. It is of profound significance to solve practical problems. See [1–3, 10–13, 15, 16] and the references therein.

Specially, there are few papers that studied the fractional differential equations problems with the Riesz-Caputo derivative. For the Riesz fractional derivative is a two-sided operator which holds memory effects. In [5], Chen et al. investigated the following equations:

$$\begin{cases} {}_0^{RC}D_T^{\gamma} y(\tau) = g(\tau, y(\tau)), \quad 0 \le \tau \le T, 0 < \gamma \le 1, \\ y(0) = y_0, \quad y(T) = y_T, \end{cases}$$

where ${}^{RC}D^{\gamma}$ is a Riesz-Caputo derivative and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, y_0 and y_T are two constants. In [7], Gu et al. employed the Leray-Schauder and Krasnosel'skii fixed point theorems to show the existence of positive solutions for the above boundary value problems in [5], where $0 \le \tau \le 1$.

In [6], Chen et al. considered the anti-periodic boundary value problems:

$$\begin{cases} {}_0^{RC}D_T^{\gamma} y(\tau) = g(\tau, y(\tau)), \quad 0 \le \tau \le T, 1 \le \gamma \le 2, \\ y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0, \end{cases}$$

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where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. It reflected the future and the past nonlocal memory effects. In [2], Ahmad et al. investigated the existence of solutions for a nonlinear fractional integro-differential equation involving two Caputo fractional derivatives of different orders and a Riemann-Liouville integral, and equipped with dual anti-periodic boundary conditions. The authors introduced a new concept of dual anti-periodic boundary conditions.

In [14], the authors considered the following singular fractional boundary value problems of fractional differential equations:

$$\begin{cases} {}^C D^{\alpha}_{0+} u(t) = f(t, u(t), u'(t), {}^C D^{\beta}_{0+} u(t)), \\ u(0) + u(1) = 0, \quad u'(0) = 0, \end{cases}$$

where $f(t,x,y,z)$ is singular at the value 0 of its space variables x, y and z, $1 < \alpha < 2$, $0 < \beta < 1$, ${}^C D^{\theta}_{0+}$ is Caputo fractional derivative. By using the Vitali convergence theorems and fixed point theorem, the existence results of monotone solutions are attained.

In [4], the authors studied the existence of solutions for Caputo type sequential fractional integro-differential equations and inclusions:

$$\begin{aligned} &\left({}^C D^{\alpha} + \lambda {}^C D^{\alpha-1}\right) u(t) = f(t, u(t), {}^C D^{p} u(t), I^q u(t)), && t \in (0,1), \\ &\left({}^C D^{\alpha} + \lambda {}^C D^{\alpha-1}\right) u(t) \in F(t, u(t), I^q u(t)), && t \in (0,1), \end{aligned}$$

supplemented with the nonlocal boundary conditions

$$u(0) = h(u), u'(0) = u''(0) = 0, a I^{\beta} u(\xi) = \int_0^1 u(s) dH(s),$$

where ${}^C D^{\alpha}$ is the Caputo fractional derivative of order α, I^q is the Riemann-Liouville fractional integral of order q, $\alpha \in (3,4]$, $p \in (0,1)$, $\lambda > 0$, $\xi \in (0,1]$, $a \in \mathbb{R}$, $\beta > 0$, f is a nonlinear function, F is a nonlinear multivalued function. In [3], authors studied a new nonlocal boundary value problem of integro-differential equations involving mixed left and right Caputo and Riemann-Liouville fractional derivatives and Riemann-Liouville fractional integrals of different orders. The existence of solutions are obtained by using Leray-Schauder nonlinear alternative, Krasnosel'skii fixed point theorem and Banach contraction mapping principle.

Inspired by the works mentioned above, we study the existence and uniqueness of solutions of fractional differential equations with Riesz-Caputo derivative:

$$\begin{cases} {}^{RC}_{0} D^{\alpha}_{1} u(t) = f(t, u(t), {}_0 I^{\beta}_{t} u(t), {}^C_0 D^{\alpha-1}_{t} u(t)), \quad 0 \le t \le 1, \\ u(0) = 0, \quad u'(0) + u'(1) = 0, \quad u(1) = \int_0^1 u(t) dt, \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, $\beta > 0$, ${}^{RC}D$ is a Riesz-Caputo derivative, ${}_0 I^{\beta}_{t}$ is the left Riemann-Liouville fractional integral of order β, ${}^C_0 D^{\alpha-1}_{t}$ is the left Caputo derivative of order $\alpha - 1$, and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. By using the fixed point theorems, the existence results for fractional differential equations with the Riesz-Caputo derivative are obtained under some conditions.

Few literature studied the fractional differential equations with the Riesz-Caputo derivative. Compared with the existing literature [5–7], the new feature lying in this paper is that we investigate BVP(1.1), in which the nonlinearity term f is relevant to fractional integral and fractional derivative. In addition, the boundary conditions involve integral. As far as we know, there is no literature that studies this problem. Our achievements fill this margin to some extension. Moreover, the basic space used in this paper is the space $X = \{u \in C[0,1], {}^C D_{0+}^{\alpha-1} u \in C[0,1]\}$ equipped with the norm $\|u\|_X = \|u\| + \|{}^C D_{0+}^{\alpha-1} u\|$. Here, $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$, this Banach space is different from [5–7]. Thus, our results are new and meaningful.

The rest of this paper is organized as follows: In Section 2, we present some definitions and lemmas, which will be used to prove our main results. In Section 3, the existence results of solutions are obtained by using the fixed poind theorem. In Section 4, two examples are given to show the effectiveness of theoretical results.

2. Basic definitions and preliminaries

For the convenience of readers, we present some definitions and lemmas, which will be used in the proofs of our results. Let $\alpha > 0$, and $n-1 < \alpha \leq n$, $n = [\alpha]$; $[\alpha]$ denotes the integer part of the real number α.

Definition 2.1 ([10]). The left and right Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : [0,1] \to \mathbb{R}$ are given respectively by

$$_0I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

$$_tI_1^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} u(s) ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$, $\alpha > 0$.

Definition 2.2. The Riesz integral of order $\alpha > 0$ of a function $u \in C[0,1]$ is defined by

$${}_0^R I_1^\alpha u(t) = \frac{1}{2\Gamma(\alpha)} \int_0^1 \mid t-s \mid^{\alpha-1} u(s) ds.$$

Remark 2.1. From Definition 2.1 and Definition 2.2, we conclude that

$${}_0^R I_1^\alpha u(t) = \frac{1}{2} \left[{}_0I_t^\alpha u(t) + {}_tI_1^\alpha u(t) \right].$$

Definition 2.3 ([5]). The classical Riesz-Caputo derivative of order $\alpha > 0$ is given by

$$\begin{aligned} {}_0^{RC} D_1^\alpha u(t) =& \frac{1}{\Gamma(n-\alpha)} \int_0^1 \mid t-s \mid^{n-\alpha-1} \left(\frac{d}{ds} \right)^n u(s) ds \\ =& \frac{1}{2} \left[{}_0^C D_t^\alpha u(t) + (-1)^n {}_t^C D_1^\alpha u(t) \right], \end{aligned}$$

where ${}_t^C D_1^\alpha$ is the right hand side Caputo derivative, ${}_0^C D_t^\alpha$ is the left hand side Caputo derivative, which are respectively given by

$${}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds} \right)^n u(s) ds,$$

$$ {}_t^C D_1^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^1 (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)ds. $$

Particularly, if $1 < \alpha \leq 2$ and $u(t) \in C[0,1]$, the Riesz-Caputo fractional derivative of order α is given by

$$ {}_0^{RC} D_1^\alpha u(t) = \frac{1}{2} \left({}_0^C D_t^\alpha u(t) + {}_t^C D_1^\alpha u(t) \right). $$

Lemma 2.1 (Lemma 2.1, [6]). *If $u(t) \in C^n[0,1]$, then*

$$ {}_0 I_t^\alpha {}_0^C D_t^\alpha u(t) = u(t) - \sum_{l=0}^{n-1} \frac{u^{(l)}(0)}{l!} (t-0)^l $$

and

$$ {}_t I_1^\alpha {}_t^C D_1^\alpha u(t) = (-1)^n \Big[u(t) - \sum_{l=0}^{n-1} \frac{(-1)^l u^{(l)}(1)}{l!} (1-t)^l \Big]. $$

Then, we have

$$ \begin{aligned} {}_0^R I_1^\alpha {}_0^{RC} D_1^\alpha u(t) =& \frac{1}{2} \big[{}_0 I_t^\alpha {}_0^C D_t^\alpha u(t) + {}_t I_1^\alpha {}_0^C D_t^\alpha u(t) \big] \\ &+ \frac{(-1)^n}{2} \big[{}_0 I_t^\alpha {}_t^C D_1^\alpha u(t) + {}_t I_1^\alpha {}_t^C D_1^\alpha u(t) \big] \\ =& \frac{1}{2} \big[({}_0 I_t^\alpha {}_0^C D_t^\alpha u(t) + (-1)^n \cdot {}_t I_1^\alpha {}_t^C D_1^\alpha u(t) \big]. \end{aligned} $$

Particularly, if $1 < \alpha \leqslant 2$ and $u(t) \in C^1[0,1]$, then

$$ {}_0^R I_1^\alpha {}_0^{RC} D_1^\alpha u(t) = u(t) - \frac{1}{2} \big[u(0) + u(1) \big] - \frac{1}{2} \big[u'(0)t - u'(1)(1-t) \big]. $$

Lemma 2.2. *Assume that $p \in L^1[0,1]$, $1 < \alpha \leq 2$, $0 \leq t \leq 1$, then the problem*

$$ \begin{cases} {}_0^{RC} D_1^\alpha u(t) = p(t), \\ u(0) = 0, u'(0) + u'(1) = 0, u(1) = \displaystyle\int_0^1 u(t)dt, \end{cases} \tag{2.1} $$

has a unique solution

$$ \begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)} \int_0^1 \big[(1-s)^{\alpha-1} - (\alpha-1)(1-s)^{\alpha-2} \big] p(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds + \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} p(s)ds. \end{aligned} \tag{2.2} $$

Proof. Lemma 2.1 guarantees that

$$ \begin{aligned} u(t) =& \frac{u(0) + u(1)}{2} + \frac{u'(0)t - u'(1)(1-t)}{2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} p(s)ds. \end{aligned} $$

Then,

$$u'(t) = \frac{u'(0)+u'(1)}{2} + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}p(s)ds - \frac{1}{\Gamma(\alpha-1)}\int_t^1 (s-t)^{\alpha-2}p(s)ds.$$

From the boundary conditions $u(0) = 0$, $u'(0) + u'(1) = 0$, $u(1) = \int_0^1 u(t)dt$, we have

$$u'(1) = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}p(s)ds,$$
$$u'(0) = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}p(s)ds.$$

Then,

$$\begin{aligned} u(t) =& \frac{1}{2}\int_0^1 u(t)dt - \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}p(s)ds - \frac{1-t}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}p(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 (s-t)^{\alpha-1}p(s)ds \end{aligned} \tag{2.3}$$

and

$$\frac{1}{2}\int_0^1 u(t)dt = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}p(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}p(s)ds. \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain the unique solution of (2.1) in $[0, 1]$. □

From Lemma 2.2, we note (1.1) is equivalent to the following equation:

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)}\int_0^1 \left[(1-s)^{\alpha-1} - (\alpha-1)(1-s)^{\alpha-2}\right] f(t, u(t), I_{0^+}^{\beta}u(t), {}^C D_{0^+}^{\alpha-1}u(t))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(t, u(t), I_{0^+}^{\beta}u(t), {}^C D_{0^+}^{\alpha-1}u(t))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_t^1 (s-t)^{\alpha-1} f(t, u(t), I_{0^+}^{\beta}u(t), {}^C D_{0^+}^{\alpha-1}u(t))ds. \end{aligned} \tag{2.5}$$

□

For the sake of convenience, in the rest of this paper, we use $I_{0^+}^{\beta}$, ${}^C D_{0^+}^{\alpha-1}$ and ${}^{RC}D^{\alpha}$ to denote ${}_0I_t^{\beta}$, ${}_0^C D_t^{\alpha-1}$ and ${}_0^{RC}D_1^{\alpha}$ respectively.

Let the space $X = \{u \in C[0,1], {}^C D_{0^+}^{\alpha-1}u \in C[0,1]\}$ equipped with the norm $\|u\|_X = \|u\| + \|{}^C D_{0^+}^{\alpha-1}u\|$, where $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. Then, $(X, \|\cdot\|_X)$ is a Banach space.

Define an operator $A : X \to X$ by the formula

$$\begin{aligned} Au(t) =& \frac{1}{\Gamma(\alpha)}\int_0^1 \left[(1-s)^{\alpha-1} - (\alpha-1)(1-s)^{\alpha-2}\right] f_u(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_u(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 (s-t)^{\alpha-1} f_u(s)ds. \end{aligned} \tag{2.6}$$

for $t \in [0,1]$, $u \in X$, where $f_u(s) = f(s, u(s), I_{0^+}^{\beta}u(s), {}^C D_{0^+}^{\alpha-1}u(s))$, $s \in [0,1]$.

The function $u(t)$ is a solution of BVP(1.1), if and only if u is a fixed point of the operator A.

3. Main results

Let

$$\Lambda = \frac{2L_1L_2(\Gamma(3-\alpha)+1)}{\Gamma(\alpha)\Gamma(3-\alpha)}, \quad L_2 = 1 + \frac{1}{\Gamma(\beta+1)}.$$

We list the following assumptions adopted in this paper.

(H$_1$) $\alpha \in (1,2]$, $\beta > 0$, the function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. There is a constant $L_1 > 0$ such that

$$| f(t,x,y,z) - f(t,x_1,y_1,z_1) | \leq L_1(| x - x_1 | + | y - y_1 | + | z - z_1 |),$$

for all $t \in [0,1]$, $x, x_1, y, y_1, z, z_1 \in \mathbb{R}$.

(H$_2$) There is a nondecreasing function $\varphi \in C([0,1], \mathbb{R}^+)$ such that

$$| f(t,x,y,z) | \leq \varphi(t), \quad \forall t \in [0,1], \quad (x,y,z) \in \mathbb{R}^3.$$

(H$_3$) There are functions $\varphi, \psi \in C([0,1], \mathbb{R}^+)$, where ψ is a nondecreasing function, satisfying

$$| f(t,x,y,z) | \leq \varphi(t)\psi(x+z), \quad \forall t \in [0,1], \quad (x,y,z) \in \mathbb{R}^3.$$

(H$_4$) There is $m > 0$ satisfying

$$\frac{m}{\|\varphi\|\psi(m)} > \frac{(3+2^{1-\alpha})\Gamma(3-\alpha)+2^{2-\alpha}}{\Gamma(\alpha+1)\Gamma(3-\alpha)}.$$

Theorem 3.1. *Assume that* (H_1) *holds. Then, BVP* (1.1) *has a unique solution on* $[0,1]$*, providing that* $\Lambda < 1$, $L_1L_2 \neq 1$.

Proof. Put $\sup_{t\in[0,1]} |f(t,0,0,0)| = M_1 < \infty$, and $r > \frac{M_1}{1-L_1L_2}$. Suppose $B_r \subset X$ is bounded, and the set

$$B_r = \{u \in X : \|u\|_X \leq r\}.$$

First and foremost, we prove that A maps B_r into itself, A is defined by (2.6). By the condition (H_1), for any $t \in [0,1]$, $u \in B_r$, we have

$$\begin{aligned}
|f_u(t)| &= |f(t, u(t), I_{0^+}^{\beta} u(t), {}^C D_{0^+}^{\alpha-1} u(t))| \\
&\leq |f(t, u(t), I_{0^+}^{\beta} u(t) + {}^C D_{0^+}^{\alpha-1} u(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\
&\leq L_1 \left(|u(t)| + |I_{0^+}^{\beta} u(t)| + |{}^C D_{0^+}^{\alpha-1} u(t)|\right) + M_1 \\
&\leq L_1 \left(\|u\| + \frac{\|u\|}{\Gamma(\beta+1)} + \|{}^C D_{0^+}^{\alpha-1} u\|\right) + M_1 \\
&\leq L_1 \|u\|_X \left(1 + \frac{1}{\Gamma(\beta+1)}\right) + M_1 \\
&\leq L_1 L_2 r + M_1.
\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
|(Au)(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f_u(s)| ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} |f_u(s)| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_u(s)| ds + \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} |f_u(s)| ds \\
\leq & \frac{L_1 L_2 r + M_1}{\Gamma(\alpha)} \left(\frac{1}{\alpha} + 1 + \frac{t^\alpha}{\alpha} + \frac{(1-t)^\alpha}{\alpha} \right) \\
\leq & \frac{5(L_1 L_2 r + M_1)}{\Gamma(\alpha+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
|(Au)'(t)| \leq & \left| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f_u(s) ds - \frac{1}{\Gamma(\alpha-1)} \int_t^1 (s-t)^{\alpha-1} f_u(s) ds \right| \\
\leq & \frac{L_1 L_2 r + M_1}{\Gamma(\alpha-1)} \left(\frac{t^{\alpha-1}}{\alpha-1} + \frac{(1-t)^{\alpha-1}}{\alpha-1} \right) \\
\leq & \frac{2(L_1 L_2 r + M_1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Hence, we conclude

$$
\|Au\| \leq \frac{5(L_1 L_2 r + M_1)}{\Gamma(\alpha+1)}, \quad \|(Au)'\| \leq \frac{2(L_1 L_2 r + M_1)}{\Gamma(\alpha)}.
$$

${}^C D_{0^+}^{\alpha-1}$ is the left Caputo derivative of order $\alpha - 1$ with $\alpha \in (1, 2]$. Then, we deduce

$$
\begin{aligned}
|{}^C D_{0^+}^{\alpha-1}(Au)(t)| \leq & \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} |(Au)'(s)| ds \\
\leq & \frac{2(L_1 L_2 r + M_1)}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} ds \\
\leq & \frac{2(L_1 L_2 r + M_1)}{\Gamma(\alpha)\Gamma(3-\alpha)}
\end{aligned}
$$

and

$$
\|{}^C D_{0^+}^{\alpha-1} Au\| \leq \frac{2(L_1 L_2 r + M_1)}{\Gamma(\alpha)\Gamma(3-\alpha)}.
$$

Therefore,

$$
\begin{aligned}
\|Au\|_X = & \|Au\| + \|{}^C D_{0^+}^{\alpha-1}\| \\
\leq & \frac{5(L_1 L_2 r + M_1)}{\Gamma(\alpha+1)} + \frac{2(L_1 L_2 r + M_1)}{\Gamma(\alpha)\Gamma(3-\alpha)} \\
\leq & r, \quad \forall t \in [0, 1].
\end{aligned}
$$

Thus, we conclude that the operator A maps B_r into itself. Moreover, for all

$s \in [0,1]$, $u, v \in B_r$, we have

$$\begin{aligned}|f_u(s) - f_v(s)| &\le L_1 \left(|u(s) - v(s)| + |I_{0^+}^{\beta} u(s) - I_{0^+}^{\beta} v(s)| + |{}^C D_{0^+}^{\alpha-1} u(s) - {}^C D_{0^+}^{\alpha-1} v(s)| \right) \\ &\le L_1 \left(\|u - v\| + \|{}^C D_{0^+}^{\alpha-1} u - {}^C D_{0^+}^{\alpha-1} v\| + \frac{\|u - v\|}{\Gamma(\beta+1)} \right) \\ &\le L_1 \|u - v\|_X \left(1 + \frac{1}{\Gamma(\beta+1)} \right) \\ &\le L_1 L_2 \|u - v\|_X.\end{aligned}$$

Then,

$$\begin{aligned}|(Au)(t) - (Av)(t)| \le & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f_u(s) - f_v(s)| ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} |f_u(s) - f_v(s)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_u(s) - f_v(s)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} |f_u(s) - f_v(s)| ds \\ \le & \frac{L_1 L_2 \|u - v\|_X}{\Gamma(\alpha)} \left(\frac{1}{\alpha} + 1 + \frac{t^{\alpha}}{\alpha} + \frac{(1-t)^{\alpha}}{\alpha} \right) \\ \le & \frac{L_1 L_2 \|u - v\|_X}{\Gamma(\alpha+1)}\end{aligned}$$

and

$$\begin{aligned}|(Au)'(t) - (Av)'(t)| \le & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f_u(s) - f_v(s)| ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_t^1 (s-t)^{\alpha-1} |f_u(s) - f_v(s)| ds \\ \le & \frac{2 L_1 L_2 \|u - v\|_X}{\Gamma(\alpha)},\end{aligned}$$

which imply that

$$\|Au - Av\| \le \frac{L_1 L_2 \|u - v\|_X}{\Gamma(\alpha+1)}, \quad \|(Au)' - (Av)'\| \le \frac{2 L_1 L_2 \|u - v\|_X}{\Gamma(\alpha)}.$$

Thus, we obtain

$$\begin{aligned}|{}^C D_{0^+}^{\alpha-1} (Au)(t) - {}^C D_{0^+}^{\alpha-1} (Au)(t)| \le & \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} |(Au)'(s) - (Av)'(s)| ds \\ \le & \frac{2 L_1 L_2 \|u - v\|_X}{\Gamma(\alpha)\Gamma(3-\alpha)}\end{aligned}$$

and

$$\|{}^C D_{0^+}^{\alpha-1} Au - {}^C D_{0^+}^{\alpha-1} Av\| \le \frac{2 L_1 L_2 \|u - v\|_X}{\Gamma(\alpha)\Gamma(3-\alpha)}.$$

From the above inequalities, we obtain

$$\begin{aligned}\|Au-Av\|_X =&\|Au-Av\|+\|^C D_{0^+}^{\alpha-1}Au-^C D_{0^+}^{\alpha-1}Av\|\\ \le&2L_1L_2\|u-v\|_X\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)\Gamma(3-\alpha)}\right)\\ =&\frac{2L_1L_2(\Gamma(3-\alpha)+1)}{\Gamma(\alpha)\Gamma(3-\alpha)}\|u-v\|_X\\ =&\Lambda\|u-v\|_X.\end{aligned}$$

In view of the condition (H_1), $\Lambda<1$, the operator A is a contraction. By Banach's fixed point theorem, BVP(1.1) has a unique solution on $[0,1]$. □

Lemma 3.3 (Krasnosel'skii fixed point theorem, [8])**.** *Let Q be a closed, convex, bounded and nonempty subset in Banach space E. Suppose that A_1, A_2 be operators such that*

(i) $A_1u_1+A_2u_2\in Q$, $\forall u_1,u_2\in Q$;

(ii) A_1 *is a contraction mapping;*

(iii) A_2 *is compact and continuous.*

Then, the equation

$$A_1z+A_2z=z$$

has solution in E.

Theorem 3.2. *Assume that (H_1) and (H_2) hold. Then, the BVP (1.1) has at least one solution on $[0,1]$, providing that $\frac{3L_1}{\Gamma(\alpha+1)}<1$.*

Proof. Define a set $B_{r_1}=\{u\in X:\|u\|_X\le r_1\}$. Let the operators $A_1, A_2: B_{r_1}\to X$ be defined by

$$\begin{aligned}A_1u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^1\left[(t-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}\right]f_u(s)ds,\\ A_2u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_u(s)ds+\frac{1}{\Gamma(\alpha)}\int_t^1(s-t)^{\alpha-1}f_u(s)ds,\end{aligned}.$$

for any $t\in[0,1]$, $u\in B_{r_1}$, where $f_u(t)=f(t,u(t),I_{0^+}^{\beta}u(t),^C D_{0^+}^{\alpha-1}u(t))$, $s\in[0,1]$. Choose a number $r_1\ge\|\varphi\|\left(\frac{5}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(\alpha)\Gamma(3-\alpha)}\right)$, where φ is the function given by the condition (H_2). Obviously, B_{r_1} is a closed, convex and bounded set.

For any $u,v\in B_{r_1}$, using the condition (H_2), we have

$$\begin{aligned}\|A_1u+A_2v\|&\le\|A_1u\|+\|A_2v\|\\ &\le\frac{\|\varphi\|}{\Gamma(\alpha+1)}+\frac{\|\varphi\|}{\Gamma(\alpha)}+\frac{\|\varphi\|}{\Gamma(\alpha+1)}+\frac{\|\varphi\|}{\Gamma(\alpha+1)}\\ &\le\frac{5\|\varphi\|}{\Gamma(\alpha+1)},\end{aligned}$$

$$\|(A_1u)'+(A_2v)'\|\le\frac{2\|\varphi\|}{\Gamma(\alpha)}$$

and

$$\|{}^C D_{0^+}^{\alpha-1}(A_1u + A_2v)\| \leq \frac{2\|\varphi\|}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} ds \leq \frac{2\|\varphi\|}{\Gamma(\alpha)\Gamma(3-\alpha)}.$$

From the definition of r_1, for all $u, v \in B_{r_1}$, we can obtain

$$\begin{aligned}\|A_1u + A_2v\|_X &= \|A_1u + A_2v\| + \|{}^C D_{0^+}^{\alpha-1}(A_1u + A_2v)\| \\ &\leq \frac{5\|\varphi\|}{\Gamma(\alpha+1)} + \frac{2\|\varphi\|}{\Gamma(\alpha)\Gamma(3-\alpha)} \\ &\leq r_1.\end{aligned}$$

Thus, $A_1u + A_2v \in B_{r_1}$, for all $u, v \in B_{r_1}$.

Next, we show that A_1 is a contraction mapping. By the condition (H_1), for all $u, v \in B_{r_1}$, $s \in [0,1]$, we obtain

$$\begin{aligned}| A_1u - A_1v | \leq & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f_u(s) - f_v(s)| ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} |f_u(s) - f_v(s)| ds \\ \leq & \frac{L_1}{\Gamma(\alpha)} (\|u-v\| + \|I_{0^+}^{\beta}u - I_{0^+}^{\beta}v\| + \|{}^C D_{0^+}^{\alpha-1}u - {}^C D_{0^+}^{\alpha-1}v\|) \left(\frac{1}{\alpha} + 1\right) \\ \leq & \frac{3L_1}{\Gamma(\alpha+1)} \left(\|u-v\| + \frac{\|u-v\|}{\Gamma(\beta+1)} + \|{}^C D_{0^+}^{\alpha-1}u - {}^C D_{0^+}^{\alpha-1}v\|\right) \\ \leq & \frac{3L_1\|u-v\|_X}{\Gamma(\alpha+1)}\end{aligned}$$

and

$$\|(Au)' - (Av)'\| = 0, \quad \|{}^C D_{0^+}^{\alpha-1}A_1u - {}^C D_{0^+}^{\alpha-1}A_1v\| = 0,$$

which yield

$$\|A_1u - A_1v\|_X \leq \frac{3L_1\|u-v\|_X}{\Gamma(\alpha+1)},$$

for all $u, v \in B_{r_1}$ with $\frac{3L_1}{\Gamma(\alpha+1)} < 1$. Then, operator A_1 is a contraction mapping.

Next, we prove that the operator A_2 is compact and continuous. Obviously, A_2 is continuous stem from the continuity of f. By the condition (H_2), we can prove that A_2 is uniformly bounded on B_{r_1}, as

$$\|A_2u\| \leq \frac{2\|\varphi\|}{\Gamma(\alpha+1)}, \quad \|(A_2u)'\| \leq \frac{2\|\varphi\|}{\Gamma(\alpha)}, \quad \|{}^C D_{0^+}^{\alpha-1}(A_2u)\| \leq \frac{2\|\varphi\|}{\Gamma(\alpha)\Gamma(3-\alpha)}.$$

Thus, for all $u \in B_{r_1}$, we get

$$\|A_2u\|_X = \|A_2u\| + \|{}^C D_{0^+}^{\alpha-1}(A_2u)\| \leq 2\|\varphi\| \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(3-\alpha)}\right).$$

For all $0 \le t_1 < t_2 \le 0$, $u \in B_{r_1}$, we obtain

$$
\begin{aligned}
&|(A_2u)(t_2) - (A_2u)(t_1)| \\
\le & \frac{\|\varphi\|}{\Gamma(\alpha)} \bigg| \int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
& + \int_{t_2}^{1} (s - t_2)^{\alpha-1} ds - \int_{t_1}^{1} (s - t_1)^{\alpha-1} ds \bigg| \\
\le & \frac{\|\varphi\|}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha + (1-t_2)^\alpha - (1-t_1)^\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
&|{}^C D_{0+}^{\alpha-1}(A_2u)(t_2) -{}^C D_{0+}^{\alpha-1}(A_2u)(t_1)| \\
\le & \frac{2\|\varphi\|}{\Gamma(\alpha)\Gamma(2-\alpha)} \left| \int_0^{t_1} (t_2 - s)^{1-\alpha} - (t_1 - s)^{1-\alpha} ds + \int_{t_1}^{t_2} (t_2 - s)^{1-\alpha} ds \right| \\
\le & \frac{2\|\varphi\|}{\Gamma(\alpha)\Gamma(3-\alpha)} (t_2^{2-\alpha} - t_1^{2-\alpha}).
\end{aligned}
$$

$|(A_2u)(t_2) - (A_2u)(t_1)| \to 0$, and $|{}^C D_{0+}^{\alpha-1}(A_2u)(t_2) -{}^C D_{0+}^{\alpha-1}(A_2u)(t_1)| \to 0$ as $t_2 \to t_1$. This prove that A_2 is equicontinuous on B_{r_1}. By the Arzelà-Ascoli theorem, the operator A_2 is compact. As all the assumptions of Krasnosel'skii fixed point theorem are satisfied, then there exists a $u \in X$ such that the equation $A_1u + A_2u = u$ has a solution. Then, the BVP (1.1) has at least one solution in $[0, 1]$. □

Lemma 3.4 (Leray-Schauder alternative theorem, [9]). *Let Q be a closed, convex subset in Banach space E, U is an open subset of Q and $\theta \in U$. Suppose that $A : \overline{U} \to Q$ be a continuous compact map (that is $A(\overline{U})$ is a relatively compact subset of Q), then either:*

(i) A has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial\overline{U}$ and $\lambda \in (0, 1)$ with $u = \lambda Au$.

Theorem 3.3. *Assume that (H_3) and (H_4) hold. Then, the BVP (1.1) has at least one solution.*

Proof. Define a set $B_{r_2} = \{u \in X : \|u\|_X \le r_2\}$, $B_{r_2} \subset X$. The proof of that the operator $A : B_{r_2} \to X$ is completely continuous, which is similar to that of Theorem 3.1. Therefore, we omit it here. Assume that u be a solution of BVP (1.1). If there exists $u \in B_{r_2}$ and $\lambda \in (0, 1)$ such that $u = \lambda Au$. Then, for all $t \in [0, 1]$, by (H_3), we have

$$
\begin{aligned}
|u(t)| =&|\lambda(Au)(t)| \\
\leq&\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_u(s)|ds+\frac{\lambda}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}|f_u(s)|ds \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f_u(s)|ds+\frac{\lambda}{\Gamma(\alpha)}\int_t^1(s-t)^{\alpha-1}|f_u(s)|ds \\
\leq&\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)\psi(u(s)+^C D_{0^+}^{\alpha-1}u(s))ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\varphi(s)\psi(u(s)+^C D_{0^+}^{\alpha-1}u(s))ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)\psi(u(s)+^C D_{0^+}^{\alpha-1}u(s))ds \\
&+\frac{1}{\Gamma(\alpha)}\int_t^1(s-t)^{\alpha-1}\varphi(s)\psi(u(s)+^C D_{0^+}^{\alpha-1}u(s))ds \\
\leq&\frac{\|\varphi\|\psi(\|u\|_X)}{\Gamma(\alpha)}\left(\frac{1}{\alpha}+1+\frac{t^\alpha}{\alpha}+\frac{(1-t)^\alpha}{\alpha}\right) \\
\leq&\|\varphi\|\psi(\|u\|)\frac{3+2^{1-\alpha}}{\Gamma(\alpha+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
|(Au)'(t)| \leq&\left|\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f_u(s)ds-\frac{1}{\Gamma(\alpha-1)}\int_t^1(s-t)^{\alpha-1}f_u(s)ds\right| \\
\leq&\frac{\|\varphi\|\psi(\|u\|_X)}{\Gamma(\alpha-1)}\left(\frac{t^{\alpha-1}}{\alpha-1}+\frac{(1-t)^{\alpha-1}}{\alpha-1}\right) \\
\leq&\|\varphi\|\psi(\|u\|_X)\frac{2^{2-\alpha}}{\Gamma(\alpha)}.
\end{aligned}
$$

Hence, we conclude

$$
\begin{aligned}
|^C D_{0^+}^{\alpha-1}u(t)| =&|\lambda^C D_{0^+}^{\alpha-1}(Au)(t)| \leq \frac{1}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}|(Au)'(s)|ds \\
\leq&\|\varphi\|\psi(\|u\|_X)\frac{2^{2-\alpha}}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}ds \\
\leq&\|\varphi\|\psi(\|u\|_X)\frac{2^{2-\alpha}}{\Gamma(\alpha)\Gamma(3-\alpha)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|u\|_X =&\|u\|+\|^C D_{0^+}^{\alpha-1}u)\| = \lambda\|Au\|_X = \lambda\|Au\|+\lambda\|^C D_{0^+}^{\alpha-1}(Au)\| \\
\leq&\|\varphi\|\psi(\|u\|_X)\frac{(3+2^{1-\alpha})\Gamma(3-\alpha)+2^{2-\alpha}}{\Gamma(\alpha+1)\Gamma(3-\alpha)},
\end{aligned}
$$

which yields

$$
\frac{\|u\|_X}{\|\varphi\|\psi(\|u\|_X)} \leq \frac{(3+2^{1-\alpha})\Gamma(3-\alpha)+2^{2-\alpha}}{\Gamma(\alpha+1)\Gamma(3-\alpha)}. \tag{3.1}
$$

Taking (H_3) and (3.1) into account, it implies $\|u\|_X \neq r_2$. Therefore, there is no $u \in B_{r_2}$ such that $u = \lambda Au$ for $\lambda \in (0,1)$. According to Leray-Schauder fixed point theorem, A has a fixed point in B_{r_2}. We deduce that BVP (1.1) has solution. $\square$

4. Examples

Two examples are given to test our results established in the previous section.

Example 4.1. Consider the boundary value problem

$$\begin{cases} {}_0^{RC}D_1^{\frac{3}{2}}u(t) = \dfrac{1}{\sqrt{t^2+7}} + \dfrac{u(t)}{129} + \dfrac{2}{83}I_{0+}^{\frac{5}{2}}u(t) + \dfrac{3}{95}({}^C D_{0+}^{\frac{1}{2}}u(t)), \\ u(0) = 0, \quad u'(0) + u'(1) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{5}{2}$, the nonlinear term is

$$f(t,x,y,z) = \frac{1}{\sqrt{t^2+7}} + \frac{x}{129} + \frac{2y}{83} + \frac{3z}{95},$$

for $t \in [0,1]$, $x, y, z \in \mathbb{R}$. There is a constant $1/12$ such that

$$|f(t,x,y,z) - f(t,x_1,y_1,z_1)| \leq \frac{1}{12}(|x-x_1| + |y-y_1| + |z-z_1|).$$

After simple calculations, we get

$$\begin{aligned} &L_1 = \frac{1}{12}, \quad M_1 = \frac{1}{\sqrt{7}}, \quad L_2 = 1 + \frac{1}{\Gamma(\beta+1)} \approx 1.3009, \\ &\Lambda = \frac{2L_1L_2(\Gamma(3-\alpha)+1)}{\Gamma(\alpha)\Gamma(3-\alpha)} \approx 0.5207 < 1, \quad L_1L_2 \approx 0.1084 \neq 1. \end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied. As a result, Theorem 3.1 guarantees that the problem (4.1) has a unique solution.

Example 4.2. We consider the following problem

$$\begin{cases} {}_0^{RC}D_1^{\frac{3}{2}}u(t) = \dfrac{1}{\sqrt{t^2+3}}\sin t\left(\dfrac{1}{8}u(t) + \dfrac{1}{5}I_{0+}^{\frac{1}{2}}u(t) + \dfrac{1}{3}({}^C D_{0+}^{\frac{1}{2}}u(t))\right), \quad 0 \leq t \leq 1, \\ u(0) = 0, \quad u'(0) + u'(1) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{4.2}$$

where $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, and the function f defined by

$$f(t,x,y,z) = \frac{1}{\sqrt{t^2+3}}\sin t\left(\frac{1}{8}x + \frac{1}{5}y + \frac{1}{3}z\right),$$

for $t \in [0,1]$, $x, y, z \in \mathbb{R}$.

In addition, we get

$$|f(t,u(t),I_{0^+}^{\frac{1}{2}},{}^{C}D_{0^+}^{\frac{1}{2}}u(t))| \leq \frac{1}{3\sqrt{t^2+3}}\left(\|u\|+\|{}^{C}D_{0^+}^{\frac{1}{2}}u\|\right).$$

For the above functions, we attain

$$\mid f(t,x,y,z) \mid \leq \varphi(t)\psi(x+z),$$

with $\varphi(t)=\frac{1}{\sqrt{t^2+3}}$, $\|\varphi\|=\frac{1}{2}$, and $\psi(x+z)=\frac{1}{3}(x+z)$.

Let $L_1=\frac{1}{12}$, $\widetilde{r}_2=3$, we find that

$$\frac{\widetilde{r}_2}{\|\varphi\|\psi(\widetilde{r}_2)}=6>\frac{(3+2^{1-\alpha})\Gamma(3-\alpha)+2^{2-\alpha}}{\Gamma(\alpha+1)\Gamma(3-\alpha)}\approx 1.9946.$$

Now, all conditions of Theorem 3.2 are satisfied. Thus, by Theorem 3.2, we conclude that problem (4.2) has at least one solutions.

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