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Stability of Peakons for a Nonlinear Generalization of the Camassa-Holm Equation*

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Abstract In this paper, by using the dynamic system method and the known conservation laws of the gCH equation, and underlying features of the peakons, we study the peakon solutions and the orbital stability of the peakons for a nonlinear generalization of the Camassa-Holm equation (gCH). The gCH equation is first transformed into a planar system. Then, by the first integral and algebraic curves of this system, we obtain one heteroclinic cycle, which corresponds to a peakon solution. Moreover, we give a proof of the orbital stability of the peakons for the gCH equation.

Keywords Camassa-Holm equation, Peakon, Stability, Heteroclinic cycle, Orbital stability.

MSC(2010) 34C25, 34C60, 37C27, 37C75.

1. Introduction

Due to its importance in breaking waves, the Camassa-Holm (CH) equation

$$u_t - u_{txx} - 3uu_x + 2u_x u_{xx} + uu_{xxx} = 0 \tag{1.1}$$

and related theories have been widely studied, see for instance [2–4, 6, 7, 11] and references therein. Recently, in [1], Anco and Recio have obtained single peakon and multi-peakon solutions to the following nonlinear generalization of the CH equation (gCH)

$$u_t - u_{xxt} = \frac{1}{2}(p+1)(p+2)u^p u_x - \frac{1}{2}p(p-1)u^{p-2}u_x^3 - 2pu^{p-1}u_x u_{xx} - u^p u_{xxx}, \tag{1.2}$$

where p is an arbitrary nonlinearity power. When $p = 1$, the gCH equation (1.2) becomes the CH equation (1.1) .

Similar to the CH equation, the gCH equation (1.2) also has the form of conservation law

$$m_t - \left(\frac{1}{2}pu^{p-1}\left(u^2 - u_x^2\right) + u^p m\right)_x = 0,\ m = u - u_{xx}. \tag{1.3}$$

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Thus, the integral

$$P=\int_{-\infty}^{+\infty} m dx \tag{1.4}$$

is conserved under the appropriately asymptotic decay condition on u.

The Hamiltonian

$$E_{(p)}=\int_{-\infty}^{+\infty} \frac{1}{2} u^p\left(u^2+u_x^2\right) dx, p \neq 0 \tag{1.5}$$

gives another conservation integral, which gives rise to the conservation law

$$D_t\left(\frac{1}{2} u^p\left(u^2+u_x^2\right)\right)+D_x X=0 \tag{1.6}$$

with

$$\begin{aligned} X= & -u^p u_t u_x+\frac{1}{2}\left(\left(1-D_x^2\right)^{-1}\left(\frac{1}{2} p u^{p-1}\left(u^2-u_x^2\right)+u^p\left(u-u_{xx}\right)\right)\right)^2 \\ & -\frac{1}{2}\left(D_x\left(\left(1-D_x^2\right)^{-1}\left(\frac{1}{2} p u^{p-1}\left(u^2-u_x^2\right)+u^p\left(u-u_{xx}\right)\right)\right)\right)^2 . \end{aligned} \tag{1.7}$$

The classification of conservation laws of the gCH equation is based on both [1] and [22]. When $p=3$, equation (1.2) becomes

$$u_t-u_{txx}-\left(\frac{3}{2} u^2\left(u^2-u_x^2\right)+u^3\left(u-u_{xx}\right)\right)_x=0. \tag{1.8}$$

In [1], Anco and Recio discovered that the equation (1.2) has peakons. Equation (1.8) has the peakon solution

$$u(x, t)=c \varphi(x+c t)=c e^{-|x+c t|}, c \in \mathbb{R}. \tag{1.9}$$

Constantin and Strauss showed in [9] that peakons of a certain nonlinear dispersive equation are orbital stable. By using the method in [8], the orbital stability of peakons for other nonlinear wave equations was proved [5, 12, 17–19, 23]. More recently, Lu, Chen and Deng [20] have studied the peakon solutions of the gCH equation when $p=2$ and the orbital stability of the peakons. By using the dynamic system method, Lu, Lu and Chen [21] obtained some peakon and periodic peakon solutions to the modified Camassa-Holm equation, and Li [14] studied the dynamical behavior for the generalized Burger-Fisher equation and the Sharma-Tasso-Olver equation under different parametric conditions. In this paper, by the dynamic system method [10, 13, 15, 16] and the method in [9], we mainly study the case of $p=3$ in the gCH equation (1.2), which has higher degree of nonlinearity and integration than the former [20]. It is much more complex to construct the fifth degree polynomial to prove the stability of peakons, and we discover that the planar system has only one heteroclinic, which is different from the case of $p=2$ in [20].

Now, we state the main result of this paper.

Theorem 1.1. *For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $u \in C\left([0,T); H^1(\mathbb{R})\right)$ is a solution to* (1.8) *with*

$$\|u(\cdot,0) - \varphi\|_{H^1(\mathbb{R})} < \delta. \tag{1.10}$$

Then,

$$\|u(\cdot,t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{R})} < \varepsilon \tag{1.11}$$

for $t \in (0,T)$, where $\xi(t) \in \mathbb{R}$ is any point where the function $u(\cdot,t)$ attains its maximum.

The rest of the organization is as follows: In the second section, we analyze the peakon solution of the gCH equation with the dynamic system method [10,13,15,16]. In the third section, we prove the orbital stability of peakon by the method in [9]. In Section 4, we give a brief conclusion.

2. Peakon solutions

The first thing we need to do in this part is to convert the equation (1.8) into a dynamic system. By using $\tau = x + ct$ to substitute $u(x,t) = \varphi(\tau)$ into equation (1.8), we obtain

$$c\varphi' - c\varphi''' - \left(\frac{5}{2}\varphi^4 - \frac{3}{2}\varphi^2(\varphi')^2 - \varphi^3\varphi''\right)' = 0, \tag{2.1}$$

where φ' is the derivative with respect to τ. By integrating the above formula, we get

$$c\varphi - c\varphi'' - \left(\frac{5}{2}\varphi^4 - \frac{3}{2}\varphi^2(\varphi')^2 - \varphi^3\varphi''\right) = g, \tag{2.2}$$

where g is the integral constant. By taking transformation $y = \frac{d\varphi}{d\tau}$, we have the planar dynamical system

$$\begin{cases} \dfrac{d\varphi}{d\tau} = y \\ \dfrac{dy}{d\tau} = \dfrac{-3\varphi^2y^2 + 5\varphi^4 - 2c\varphi + 2g}{2(\varphi^3 - c)} \end{cases} \tag{2.3}$$

and the first integral

$$H(\varphi, y) = \left(\varphi^3 - c\right)\left[y^2 - \varphi^2 - \frac{2g\varphi}{\varphi^3 - c}\right] = h. \tag{2.4}$$

When $g = 0$, the following planar dynamical system

$$\begin{cases} \dfrac{d\varphi}{d\tau} = y \\ \dfrac{dy}{d\tau} = \dfrac{-3\varphi^2y^2 + 5\varphi^4 - 2c\varphi}{2(\varphi^3 - c)} \end{cases} \tag{2.5}$$

and the first integral

$$H(\varphi, y) = \left(\varphi^3 - c\right)\left(y^2 - \varphi^2\right) \tag{2.6}$$

are obtained. Since the system (2.5) is discontinuous on the singular line $\varphi^3 = c$, we do the conversion $d\tau = \left(\varphi^3 - c\right) d\xi$ to avoid this line for the time being. Through the above transformation, the system (2.5) becomes

$$\begin{cases} \dfrac{d\varphi}{d\xi} = \left(\varphi^3 - c\right) y, \\ \dfrac{dy}{d\xi} = \dfrac{-3\varphi^2 y^2 + 5\varphi^4 - 2c\varphi}{2}. \end{cases} \tag{2.7}$$

Since the system (2.7) has the same first integral (2.6) as the system (2.5), so the system (2.7) has the same topological phase portrait as the system (2.5), except for the singular line $\varphi^3 = c$. It is an invariant straight line solution for the system (2.7), and that is pretty obvious. When $c > 0$, system (2.7) has two equilibrium points $O(0,0)$ and $A\left(\sqrt[3]{\frac{2c}{5}}, 0\right)$. Beyond that, the straight line has other two equilibrium points $B(\sqrt[3]{c}, \sqrt[3]{c})$ and $C(\sqrt[3]{c}, -\sqrt[3]{c})$.

The coefficient matrix of the linearized system (2.7) at the equilibrium point (φ_m, y_m) is denoted as $M(\varphi_m, y_m)$, and $J = \det M(\varphi_m, y_m)$ is defined. Then, by this definition, we can figure out $J_O = -c^2 < 0$, $J_A = \frac{9c^2}{5} > 0$ and $J_{B,C} = -9c^2 < 0$. According to planar dynamical system theory, O and B, C are three saddle points, and A is a center point. Figure 1 shows the phase portrait of the system (2.5) and the heteroclinic cycle, which is highlighted in blue in Figure 1, corresponding to one peakon constitutes the algebraic curves defined by $H(\varphi, y) = 0$. The closer $H(\varphi, y)$ is to 0, the closer the red solution curve corresponding to $H(\varphi, y) < 0$ and the green solution curve corresponding to $H(\varphi, y) > 0$ approach the heteroclinic cycle, but never intersect.

$$y^2 = \varphi^2 \tag{2.8}$$

is obtained by the algebraic curves defined by $H(\varphi, y) = 0$.

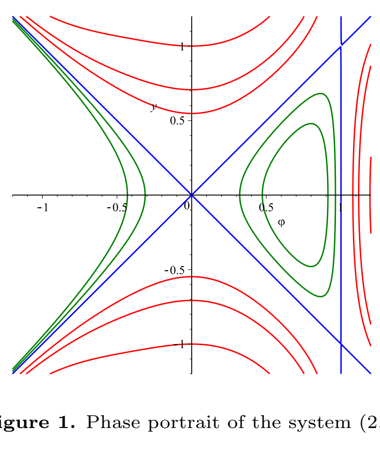

Figure 1. Phase portrait of the system (2.5)

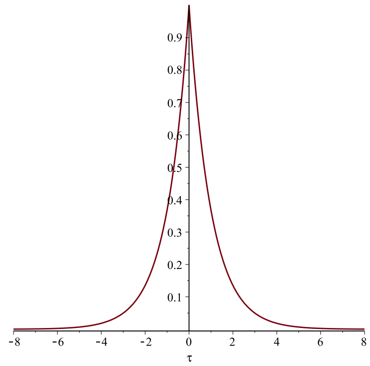

Figure 2. The profile of $\varphi(\tau) = e^{-|\tau|}$

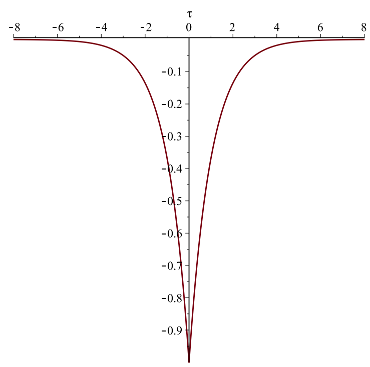

Figure 3. The profile of $\varphi(\tau) = -e^{-|\tau|}$

The peakon solution in the form of exponential function

$$\varphi(\tau) = \pm\sqrt[3]{c}e^{-|\tau|}, \tag{2.9}$$

whose profiles are shown in Figure 2 and Figure 3, is obtained by integrating with the first equation of the system (2.5).

We define $*$ as the convolution on $\mathbb{R}$, and its structure is $(f * g)(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$. Therefore, we have $u = \left(1-\partial_x^2\right)^{-1} m = \widetilde{p} * m$.

Definition 2.1. Given initial data $u_0 \in W^{1,3}(\mathbb{R})$, the function $u \in L^{\infty}\left([0,T], W^{1,3}(\mathbb{R})\right)$ is said to be a weak solution to (1.8), if it satisfies the following identity

$$\int_0^T \int_{\mathbb{R}} \left[u\psi_t - \frac{1}{4}u^4\psi_x - \widetilde{p} * \left(\frac{9}{4}u^4 + \frac{3}{2}u^2u_x^2\right)\psi\right] dxdt + \int_{\mathbb{R}} u(x,0)\,\psi(x,0)\,dx = 0 \tag{2.10}$$

for any smooth test function $\psi(t,x) \in C_c^{\infty}([0,T] \times \mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, it is called a global weak solution.

Theorem 2.1 ([1])**.** *The peaked function of the form*

$$u(t,x) = \pm\sqrt[3]{c}e^{-|x+ct|} \tag{2.11}$$

is a global weak solution to (1.8) *in the sense of Definition 2.1.*

3. Proof of stability

Notice that a small change in the shape of a peakon can produce another peakon at a different speed. Thus, the proper concept of stability is orbital stability: a wave whose initial profile is close to peakon will remain close to some translation of it all the time thereafter. In another words, the shape of the wave remains roughly the same at all times.

Equation (1.8) has the following three conservation laws

$$H_0[u]=\int_{\mathbb{R}} udx,\ H_1[u]=\int_{\mathbb{R}}\left(u^2+u_x^2\right)dx,\ H_2[u]=\frac{5}{4}\int_{\mathbb{R}} u^3\left(u^2+u_x^2\right)dx. \quad (3.1)$$

Therefore, just for the sake of calculation, let us take $c=1$ and then $\varphi(x)=e^{-|x|}$. We get

$$H_1[\varphi]=2,\quad H_2[\varphi]=1 \quad (3.2)$$

by applying (3.1). We complete the proof of Theorem 1.1 through the following four lemmas in several steps. The first consideration is the expansion of the conservation law H_1 around the peakon φ in the $H^1(\mathbb{R})$-norm.

Lemma 3.1. *For any $u\in H^1(\mathbb{R})$ and $\xi\in\mathbb{R}$,*

$$H_1[u]-H_1[\varphi]=\|u-\varphi(\cdot-\xi)\|^2_{H^1(\mathbb{R})}+4(u(\xi)-1). \quad (3.3)$$

Proof. We calculate

$$\begin{aligned}\|u-\varphi(\cdot-\xi)\|^2_{H^1(\mathbb{R})}&=\int_{\mathbb{R}}[u(x)-\varphi(x-\xi)]^2+[u_x(x)-\varphi_x(x-\xi)]^2dx\\&=H_1[u]+H_1[\varphi]-2\int_{\mathbb{R}}u(x)\varphi(x-\xi)dx \quad (3.4)\\&\quad-2\int_{\mathbb{R}}u_x(x)\varphi_x(x-\xi)dx.\end{aligned}$$

Since

$$\varphi(x-\xi)=\begin{cases}e^{-(x-\xi)}, & x>\xi\\ e^{x-\xi}, & x<\xi\end{cases} \quad (3.5)$$

and

$$\varphi_x(x-\xi)=\begin{cases}-e^{-(x-\xi)}, & x>\xi,\\ e^{x-\xi}, & x<\xi,\end{cases} \quad (3.6)$$

we obtain

$$\varphi_x(x-\xi)=\begin{cases}-\varphi(x-\xi), & x>\xi,\\ \varphi(x-\xi), & x<\xi.\end{cases} \quad (3.7)$$

Using (3.7) and integration by parts, we find

$$\begin{aligned}&\int_{\mathbb{R}}u_x(x)\varphi_x(x-\xi)\,dx\\&=\int_{-\infty}^{\xi}u_x(x)\varphi(x-\xi)dx-\int_{\xi}^{+\infty}u_x(x)\varphi(x-\xi)dx \quad (3.8)\\&=2u(\xi)-\int_{\mathbb{R}}u(x)\varphi(x-\xi)\,dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|u-\varphi(\cdot-\xi)\|^2_{H^1(\mathbb{R})} &= H_1[u]+H_1[\varphi]-4u(\xi)\\ &= H_1[u]-H_1[\varphi]-4(u(\xi)-1).\end{aligned} \tag{3.9}$$

□

Lemma 3.2. *For any $u\in H^1(\mathbb{R})$, let $M=\max_{x\in\mathbb{R}}\{u(x)\}$. Then,*

$$H_2[u]\le \frac{5}{4}M^3H_1[u]-\frac{3}{2}M^5. \tag{3.10}$$

Proof. Let us take $M=u(\xi)$ where $\xi\in\mathbb{R}$ and define

$$g(x):=\begin{cases}u(x)-u_x(x), & x<\xi,\\ u(x)+u_x(x), & x>\xi.\end{cases} \tag{3.11}$$

We calculate

$$\begin{aligned}\int_{\mathbb{R}} g^2(x)dx &= \int_{-\infty}^{\xi}[u(x)-u_x(x)]^2dx+\int_{\xi}^{+\infty}[u(x)+u_x(x)]^2dx\\ &= H_1[u]-u^2(x)\Big|_{-\infty}^{\xi}+u^2(x)\Big|_{\xi}^{+\infty}\\ &= H_1[u]-2M^2.\end{aligned} \tag{3.12}$$

Similarly,

$$\begin{aligned}&\int_{\mathbb{R}} u^3g^2(x)\,dx\\ &=\int_{-\infty}^{\xi}u^3(u-u_x)^2dx+\int_{\xi}^{+\infty}u^3(u+u_x)^2dx\\ &=\int_{-\infty}^{\xi}u^3\left(u^2+u_x^2-2uu_x\right)dx+\int_{\xi}^{+\infty}u^3\left(u^2+u_x^2+2uu_x\right)dx\\ &=\frac{4}{5}H_2[u]-\frac{2}{5}u^5(x)\Big|_{-\infty}^{\xi}+\frac{2}{5}u^5(x)\Big|_{\xi}^{+\infty}\\ &=\frac{4}{5}H_2[u]-\frac{4}{5}M^5.\end{aligned} \tag{3.13}$$

Since

$$\int_{\mathbb{R}} u^3(x)\,g^2(x)\,dx\le M^3\int_{\mathbb{R}} g^2(x)\,dx, \tag{3.14}$$

we have

$$\frac{4}{5}H_2[u]-\frac{4}{5}M^5\le M^3\left(H_1[u]-2M^2\right). \tag{3.15}$$

Simplify (3.15) to get

$$H_2[u]\le \frac{5}{4}M^3H_1[u]-\frac{3}{2}M^5.$$

□

Lemma 3.3. *For any* $u \in H^1(\mathbb{R})$, *if* $\|u-\varphi\|_{H^1} < \delta$, *then*

$$|H_1[u] - H_1[\varphi]| \le \delta\left(\delta + 2\sqrt{2}\right) \tag{3.16}$$

and

$$|H_2[u] - H_2[\varphi]| \le \frac{5}{4}\delta\left(\frac{\sqrt{2}}{4}\delta^4 + \frac{5}{2}\delta^3 + 5\sqrt{2}\delta^2 + 10\delta + 5\sqrt{2}\right). \tag{3.17}$$

Proof. Identity (3.12) shows that for all $v \in H^1(\mathbb{R})$,

$$\sup_{x\in\mathbb{R}} |v(x)| \le \sqrt{\frac{1}{2}H_1[v]} = \frac{\sqrt{2}}{2}\|v\|_{H^1}. \tag{3.18}$$

Equality holds if and only if v is proportional to a translate of φ. Note that

$$\begin{aligned} |H_1[u] - H_1[\varphi]| &= \left|\|u\|_{H^1}^2 - \|\varphi\|_{H^1}^2\right| \\ &= |(\|u\|_{H^1} + \|\varphi\|_{H^1})(\|u\|_{H^1} - \|\varphi\|_{H^1})| \\ &\le (\|u-\varphi\|_{H^1} + 2\|\varphi\|_{H^1})\|u-\varphi\|_{H^1} \\ &\le \delta\left(\delta + 2\sqrt{2}\right). \end{aligned} \tag{3.19}$$

Similarly,

$$\begin{aligned} &|H_2[u] - H_2[\varphi]| \\ &= \frac{5}{4}\left|\int_{\mathbb{R}} u^3\left(u^2+u_x^2\right)dx - \int_{\mathbb{R}} \varphi^3\left(\varphi^2+\varphi_x^2\right)dx\right| \\ &\le \frac{5}{4}\left|\int_{\mathbb{R}}\left[(u-\varphi)^2(u+2\varphi) + 3\varphi^2(u-\varphi)\right]\left(u^2+u_x^2\right)dx\right| \\ &\quad + \frac{5}{4}\left|\int_{\mathbb{R}}\varphi^3\left[(u-\varphi)^2 + (u_x-\varphi_x)^2\right]dx\right| \\ &\quad + \frac{5}{4}\left|\int_{\mathbb{R}}\varphi^3\left[2\varphi(u-\varphi) + 2\varphi_x(u_x-\varphi_x)\right]dx\right| \\ &\le \frac{5}{4}\left[\left\|(u-\varphi)^2\right\|_{L^\infty}\|u+2\varphi\|_{L^\infty} + 3\left\|\varphi^2\right\|_{L^\infty}\|u-\varphi\|_{L^\infty}\right]H_1[u] \\ &\quad + \frac{5}{4}\left\|\varphi^3\right\|_{L^\infty}\|u-\varphi\|_{H^1}^2 + \frac{5}{2}\left\|\varphi^3\right\|_{L^\infty}\|\varphi\|_{H^1}\|u-\varphi\|_{H^1} \\ &\le \frac{5}{4}\left[\frac{1}{2}\|u-\varphi\|_{H^1}^2 \cdot \frac{\sqrt{2}}{2}(\|u-\varphi\|_{H^1} + 3\|\varphi\|_{H^1}) + \frac{3\sqrt{2}}{2}\|u-\varphi\|_{H^1}\right]H_1[u] \\ &\quad + \frac{5}{4}\|u-\varphi\|_{H^1}^2 + \frac{5\sqrt{2}}{2}\|u-\varphi\|_{H^1} \\ &\le \frac{5}{4}\left[\frac{\sqrt{2}}{4}\delta^2\left(\delta+3\sqrt{2}\right) + \frac{3\sqrt{2}}{2}\delta\right]\left(\delta^2 + 2\sqrt{2}\delta + 2\right) + \frac{5}{4}\delta^2 + \frac{5\sqrt{2}}{2}\delta \\ &= \frac{5}{4}\delta\left(\frac{\sqrt{2}}{4}\delta^4 + \frac{5}{2}\delta^3 + 5\sqrt{2}\delta^2 + 10\delta + 5\sqrt{2}\right). \end{aligned} \tag{3.20}$$

□

Lemma 3.4. *For any* $u \in H^1(\mathbb{R})$, *let* $M = \max_{x\in\mathbb{R}}\{u(x)\}$. *If*

$$|H_1[u] - H_1[\varphi]| \le \delta\left(\delta + 2\sqrt{2}\right)$$

and

$$|H_2[u] - H_2[\varphi]| \le \frac{5}{4}\delta\left(\frac{\sqrt{2}}{4}\delta^4 + \frac{5}{2}\delta^3 + 5\sqrt{2}\delta^2 + 10\delta + 5\sqrt{2}\right)$$

for some δ, *then*

$$|M-1| \le \frac{\sqrt{5}}{2}\delta^{\frac{1}{2}}\sqrt{Q(\delta)}, \tag{3.21}$$

where

$$Q(\delta) = \left(1 + \sqrt{2}\delta + \frac{1}{2}\delta^2\right)^{3/2}\left(\delta + 2\sqrt{2}\right) + \frac{\sqrt{2}}{4}\delta^4 + \frac{5}{2}\delta^3 + 5\sqrt{2}\delta^2 + 10\delta + 5\sqrt{2}. \tag{3.22}$$

Proof. Because of the inequality (3.10) in Lemma 3.2,

$$H_2[u] - \frac{5}{4}M^3 H_1[u] + \frac{3}{2}M^5 \le 0. \tag{3.23}$$

Define the quintic polynomial

$$P(y) = H_2[u] - \frac{5}{4}y^3 H_1[u] + \frac{3}{2}y^5. \tag{3.24}$$

When $H_1[u] = H_1[\varphi] = 2$ and $H_2[u] = H_2[\varphi] = 1$, it takes the following form

$$P_0(y) = H_2[\varphi] - \frac{5}{4}y^3 H_1[\varphi] + \frac{3}{2}y^5 = \frac{1}{2}(y-1)^2\left[3y(y+1)^2 + y + 2\right]. \tag{3.25}$$

According to (3.24) and (3.25), we calculate that

$$P_0(M) = P(M) + \frac{5}{4}M^3\left(H_1[u] - H_1[\varphi]\right) - \left(H_2[u] - H_2[\varphi]\right). \tag{3.26}$$

Since $H_1[u]$ is near 2 and $H_2[u]$ is near 1, $\sqrt[3]{\frac{1}{2}} < \sqrt[3]{\frac{H_2[u]}{H_1[u]}} < M$. By (3.23), (3.25) and (3.26), we obtain

$$(M-1)^2 \le \frac{5}{4}M^3\left(H_1[u] - H_1[\varphi]\right) - \left(H_2[u] - H_2[\varphi]\right). \tag{3.27}$$

By using (3.27) and the relation

$$0 \le M^2 \le \frac{H_1[u]}{2} \le 1 + \sqrt{2}\delta + \frac{1}{2}\delta^2, \tag{3.28}$$

we find

$$|M-1| \le \sqrt{\frac{5}{4}M^3\left|H_1[u] - H_1[\varphi]\right| + \left|H_2[u] - H_2[\varphi]\right|} \le \frac{\sqrt{5}}{2}\delta^{\frac{1}{2}}\sqrt{Q(\delta)}.$$

□

Next, we prove Theorem 1.1.

Proof. Since $H_1[u]$ and $H_2[u]$ are both conserved by the equation (1.8), we have $H_1[u(\cdot,t)] = H_1[u_0], H_2[u(\cdot,t)] = H_2[u_0], t \in (0,T)$.

We apply Lemma 3.3 to u_0 and to δ, and by the hypotheses of Lemma 3.4 are satisfied for $u(\cdot,t)$. Hence,

$$|u(\xi(t),t) - 1| \le \frac{\sqrt{5}}{2}\delta^{\frac{1}{2}}\sqrt{Q(\delta)}. \tag{3.29}$$

Combining (3.3) with Lemma 3.1, we find

$$\begin{aligned} &\|u(\cdot,t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{R})} \\ &= H_1[u] - H_1[\varphi] - 4(u(\xi,t) - 1) \\ &\le |H_1[u] - H_1[\varphi]| + 4|u(\xi,t) - 1| \\ &\le \delta\left(\delta + 2\sqrt{2}\right) + 2\sqrt{5}\delta^{\frac{1}{2}}\sqrt{Q(\delta)}. \end{aligned} \tag{3.30}$$

Based on the (3.30), for any $\varepsilon > 0$, we can take a $\delta(\varepsilon)$ such that

$$\|u(\cdot,t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{R})} < \varepsilon.$$

□

Remark 3.1. Compared with the CH case, the main interest in the proof is to see the effect of the higher nonlinearity. Specifically, in the gCH case, it has a more complicated conservation law, and its integrand is to the fifth power which leads to the need for more computation. It also needs to construct a quintic polynomial to complete the proof.

4. Conclusion

In this paper, the orbital stability of the peakons for the generalized Camassa-Holm equation (gCH) with a quartic nonlinearity is studied. By the dynamic system method in [10,13,15,16], it shows that the plane dynamical system derived from the gCH equation has only one heteroclinic cycle corresponding to a peakon solution, and the orbital stability of the peakons for gCH equation is also proved by the method in [9].

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