Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 4, Number 1, March 2022, 64–79 DOI:10.12150/jnma.2022.64

Bifurcation Difference Induced by Different Discrete Methods in a Discrete Predator-prey Model*

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Abstract In this paper, we revisit a discrete predator-prey model with Allee effect and Holling type-I functional response. The most important is for us to find the bifurcation difference: a flip bifurcation occurring at the fixed point E_3 in the known results cannot happen in our results. The reason leading to this kind of difference is the different discrete method. In order to demonstrate this, we first simplify corresponding continuous predator-prey model. Then, we apply a different discretization method to this new continuous model to derive a new discrete model. Next, we consider the dynamics of this new discrete model in details. By using a key lemma, the existence and local stability of nonnegative fixed points E_0, E_1, E_2 and E_3 are completely studied. By employing the Center Manifold Theorem and bifurcation theory, the conditions for the occurrences of Neimark-Sacker bifurcation and transcritical bifurcation are obtained. Our results complete the corresponding ones in a known literature. Numerical simulations are also given to verify the existence of Neimark-Sacker bifurcation.

Keywords Discrete predator-prey model with Holling type-I funcational response, Flip bifurcation, Neimark-Sacker bifurcation, Transcritical bifurcation, Allee effect.

MSC(2010) 39A10.

1. Introduction and preliminaries

In the past few years, the predator-prey models have been widely studied. For a generalized predator-prey system

$$\begin{cases} \dot{x} = xp(x) - g(x)y, \\ \dot{y} = y(rg(x) - q(y)), \end{cases} \tag{1.1}$$

where x and y indicate the density of prey and predator respectively, $p(x)$ represents the growth rate of prey with the absence of predator, $q(y)$ denotes the death rate of

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*The authors were supported by National Natural Science Foundation of China (No. 61473340), the Distinguished Professor Foundation of Qianjiang Scholar in Zhejiang Province and the National Natural Science Foundation of Zhejiang University of Science and Technology (No. F701108G14).

predator, r is predator's efficiency rate in predating and $g(x)$ describes the predator functional response. For different predator-prey systems, Holling [7] introduced three kinds of functional response, namely, $g(x) = x$, $\frac{x}{m+x}$, $\frac{x^2}{m+x^2}$, which are called Holling type I, II and III respectively.

In order to get a more realistic model, one often considers the Allee effect for a given model. Allee effect [3], introduced by ecologist W. C. Allee, is a fundamental phenomenon in the biological system, which describes a positive relationship between the population density and per capita growth rate of the population at low densities. The Allee effect can be divided into two kinds: strong Allee effect and weak Allee effect. A critical threshold is proposed for the strong Allee effect, under which the per capita growth rate of population is negative below the threshold and the growth rate becomes positive above the threshold, while the pre capita growth rate remain positive at a low population density for the weak Allee effect. Lots of researches about predator-prey models are done with the Allee effect [1, 19, 21, 23].

Generally speaking, it is impossible to solve a complicate system of ordinary differential equations. Therefore, one often solves its discrete version by using computer. Due to the realistic meanings of discrete models, more and more studies have applied the theory of discrete dynamical system [5, 8, 14, 15, 17].

Zhang et. al [22] first considered a Lotka-Volterra [11, 16] type predator-prey system with Holling type-I functional response as follows:

$$\begin{cases} \dot{x}(t) = r_0 x(1 - \frac{x}{k}) - axy, \\ \dot{y}(t) = bxy - dy, \end{cases} \tag{1.2}$$

where x is the prey population and y is the predator population, r_0 is the intrinsic growth rate of prey, k is the carrying capacity of the environment for prey, a is the prey capture rate by their predators, b is the conversion efficiency from prey to predator and d is the intrinsic death rate of predator. The initial values of system (1.2) satisfy $x(0) > 0$, $y(0) > 0$ and all the parameters are positive.

Then, they introduced the strong Allee effect for the prey into system (1.2), and rewrite system (1.2) as

$$\begin{cases} \dot{x}(t) = r_0 x(1 - \frac{x}{k})(x - c) - axy, \\ \dot{y}(t) = bxy - dy. \end{cases} \tag{1.3}$$

Finally, the authors employed the forward Euler method to get the discrete form of system (1.3) in the following

$$\begin{cases} x_{n+1} = x_n + r_0 x_n(1 - \frac{x_n}{k})(x_n - c) - a x_n y_n, \\ y_{n+1} = y_n + b x_n y_n - d y_n. \end{cases} \tag{1.4}$$

Although the authors of [22] obtained some good results for system (1.4), some problems still exist. On one hand, when considering the dynamical properties of a given system of ordinary differential equations or differential equations, one hopes to study its equivalent simple form with as less parameters and variables as possible. System (1.3) has 6 parameters, and is not a simple form. In fact, by letting $\frac{x}{k} \to x$,

$\frac{a}{r_0 k} y \to y$, $\frac{c}{k} \to \alpha$, $r_0 b \to \beta$, $\frac{d}{r_0 k} \to \delta$, $r_0 kt \to t$, we obtain a simpler form of system (1.3) as follows:

$$\begin{cases} \frac{dx}{dt} = x(1-x)(x-\alpha) - xy, \\ \frac{dy}{dt} = \beta\, xy - \delta\, y, \end{cases} \tag{1.5}$$

because system (1.5) has only 3 parameters.

Generally, one thinks the threshold of prey population is less than the carrying capacity of the environment for the prey, namely, $c < k$. This leads to our assumption $\alpha \in (0,1)$ in this paper because of $\alpha = \frac{c}{k}$. We only consider the discretization of system (1.5) in the sequel.

On the other hand, from $\dot{x}(t) = \lim\limits_{x \to 0} \frac{x(t+h)-x(t)}{h}$, according to the forward Euler method, the discreteness of autonomous differential equation $\dot{x}(t) = f(x)$ takes this form

$$\frac{x(t_{n+1}) - x(t_n)}{h} = f(x(t_n)), \quad \text{or} \quad x_{n+1} = x_n + hf(x_n),$$

where $x_n = x(t_n)$, $t_n = t_0 + nh$, and h is a step length, requiring $0 < h \ll 1$. It is to let step length $h = 1$ that the authors in [22] derived system (1.4), which violates the requirement of $0 < h \ll 1$. Therefore, the forward Euler method used in [22] with $h = 1$ can not satisfy the requirement of accuracy. Hence, system (1.4) has mathematical means, but does not have the same biological meanings as system (1.3).

For further investigations into system (1.5), we apply the method of semidiscretization to it here. Suppose that $[t]$ denotes the greatest integer not exceeding t, we consider the following semidiscretization version of (1.5)

$$\begin{cases} \frac{1}{x(t)} \frac{dx(t)}{dt} = (1 - x([t]))(x([t]) - \alpha) - y([t]), \\ \frac{1}{y(t)} \frac{dy(t)}{dt} = \beta\, x([t]) - \delta. \end{cases} \tag{1.6}$$

It is easy to see that system (1.6) has piecewise constant arguments, and that a solution $(x(t), y(t))$ of system (1.6) for $t \in [0, +\infty)$ has the following features:

(i) $x(t)$ and $y(t)$ are continuous on $[0, +\infty)$;

(ii) $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist everywhere, when $t \in [0, +\infty)$ except for the points $t \in \{0, 1, 2, 3, ...\}$.

For any $n \in \{0, 1, 2, 3, ...\}$,$t \in [n, n+1)$, we integrate (1.6) on the interval $[n, t]$ and get the following system:

$$\begin{cases} x(t) = x(n)\, e^{(1-x(n))(x(n)-\alpha)-y(n)}\, (t-n), \\ y(t) = y(n)\, e^{\beta\, x(n) - \delta}\, (t-n). \end{cases} \tag{1.7}$$

Letting $t \to (n+1)^-$, (1.7) becomes

$$\begin{cases} x_{n+1} = x_n\, e^{(1-x_n)(x_n-\alpha)-y_n}, \\ y_{n+1} = y_n\, e^{\beta\, x_n - \delta}, \end{cases} \tag{1.8}$$

where $x_n = x(n)$ and $y_n = y(n)$.

System (1.8) is derived without requiring the step length $h = 1$. Hence, in this sequel, we study the properties of system (1.8). Although both discrete systems (1.4) and (1.8) are derived from the same system (1.3), the discrete methods are different. To our surprise, at the same equilibrium, system (1.4) has a flip bifurcation, but system (1.8) does not have. This will be shown in the sequel.

The rest of this paper is organized as follows: In Section 2, the existence and local stability of nonnegative fixed points of system (1.8) are studied. In Section 3, the conditions are formulated for the occurrences of Neimark-Sacker bifurcation and transcritical bifurcation of system (1.8). In Section 4, our derived analytical results for the Neimark-Sacker bifurcation of system (1.8) are numerically simulated. Finally, this paper is ended with some discussions and conclusions in Section 5.

2. The existence and stability of fixed points

In this section, we will not only study the existence of fixed points of system (1.8), but also determine the local stability of these fixed points. It is noted that the fixed points of system (1.8) satisfy

$$\begin{cases} x = x\, e^{(1-x)(x-\alpha)-y}, \\ y = y\, e^{\beta\, x-\delta}. \end{cases} \tag{2.1}$$

Considering the biological meanings of system (1.8), only nonnegative fixed points are studied. By solving (2.1), we obtain three nonnegtive fixed points $E_0(0,0)$, $E_1(\alpha,0)$, $E_2(1,0)$, and if $\alpha\beta < \delta < \beta$, system (1.8) has a positive fixed point $E_3(\frac{\delta}{\beta}, \frac{(\beta-\delta)(\delta-\alpha\beta)}{\beta^2})$.

The Jacobian matrix of system (1.8) at a fixed point $E(x,y)$ is

$$J(E) = \begin{pmatrix} (1+x(1+\alpha-2x))e^{(1-x)(x-\alpha)-y} & -xe^{(1-x)(x-\alpha)-y} \\ \beta\, ye^{\beta\, x-\delta} & e^{\beta\, x-\delta} \end{pmatrix}, \tag{2.2}$$

The characteristic equation of $J(E)$ is

$$\lambda^2 - Tr(J)\lambda + Det(J) = 0, \tag{2.3}$$

where

$Tr(J) = (1+x(1+\alpha-2x))e^{(1-x)(x-\alpha)-y} + e^{\beta\, x-\delta}$,

$Det(J) = (1+x(1+\alpha-2x)+\beta xy)e^{(1-x)(x-\alpha)-y+\beta\, x-\delta}$.

In order to study the local stability and bifurcation of system (1.8), the following definition and lemma [17,18] are needed.

Definition 2.1. Let $E(x,y)$ be a fixed point of system (1.8) with multipliers λ_1 and λ_2.

(i) A fixed point $E(x,y)$ is called sink, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so a sink is locally asymptotically stable.

(ii) A fixed point $E(x,y)$ is called source, if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so a source is locally asymptotically unstable.

(iii) A fixed point $E(x, y)$ is called saddle, if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

(iv) A fixed point $E(x, y)$ is called to be non-hyperbolic, if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 2.1. *Let $F(\lambda) = \lambda^2 + P\lambda + Q$, where P and Q are two real contants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then, the following statements hold.*

(i) If $F(1) > 0$, then

(i.1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$, if and only if $F(-1) > 0$ and $Q < 1$;

(i.2) $\lambda_1 = -1$ and $\lambda_2 \neq -1$, if and only if $F(-1) = 0$ and $P \neq 2$;

(i.3) $|\lambda_1| < 1$ and $|\lambda_2| > 1$, if and only if $F(-1) < 0$;

(i.4) $|\lambda_1| > 1$ and $|\lambda_2| > 1$, if and only if $F(-1) > 0$ and $Q > 1$;

(i.5) λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1| = |\lambda_2| = 1$, if and only if $-2 < P < 2$ and $Q = 1$;

(i.6) $\lambda_1 = \lambda_2 = -1$, if and only if $F(-1) = 0$ and $P = 2$.

(ii) If $F(1) = 0$, namely, 1 is one root of $F(\lambda) = 0$. Then, the other root λ satisfies $|\lambda| = (<, >)1$, if and only if $|Q| = (<, >)1$.

(iii) If $F(1) < 0$, then $F(\lambda) = 0$ has one root lying in $(1, \infty)$. Moreover,

(iii.1) the other root λ satisfies $\lambda < (=) - 1$, if and only if $F(-1) < (=)0$;

(iii.2) the other root λ satisfies $-1 < \lambda < 1$, if and only if $F(-1) > 0$.

In view of the above Definition 2.1 and Lemma 2.1, one can easily derive the following results.

Theorem 2.1. *The following statements about fixed points $E_0(0, 0)$, $E_1(\alpha, 0)$, $E_2(1, 0)$ of system (1.8) are true:*

(i) The fixed point $E_0(0, 0)$ of system (1.8) is a sink.

(ii) For the fixed point $E_1(\alpha, 0)$,

(ii.1) if $\delta < \alpha\beta$, then E_1 is a source;

(ii.2) if $\delta = \alpha\beta$, then E_1 is non-hyperbolic;

(ii.3) if $\delta > \alpha\beta$, then E_1 is a saddle.

(iii) For the fixed point $E_2(1, 0)$,

(iii.1) if $\delta < \beta$, then E_2 is a source;

(iii.2) if $\delta = \beta$, then E_2 is non-hyperbolic;

(iii.3) if $\delta > \beta$, then E_2 is a sink.

Proof. The Jacobian matrix of system (1.8) at fixed point $E_0(0, 0)$ is

$$J(E_0) = \begin{pmatrix} e^{-\alpha} & 0 \\ 0 & e^{-\delta} \end{pmatrix}. \tag{2.4}$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = e^{-\alpha}$ and $\lambda_2 = e^{-\delta}$ with $|\lambda_1| < 1$ and $|\lambda_2| < 1$. By Definition 2.1(i), $E_0(0, 0)$ is a sink.

The Jacobian matrix of system (1.8) at fixed point $E_1(\alpha, 0)$ reads

$$J(E_1) = \begin{pmatrix} 1+\alpha(1-\alpha) & -\alpha \\ 0 & e^{\alpha\beta-\delta} \end{pmatrix}, \tag{2.5}$$

with the eigenvalues $\lambda_1 = 1+\alpha(1-\alpha) > 1$ and $\lambda_2 = e^{\alpha\beta-\delta}$.

Therefore, $|\lambda_2| > (=, <)1$ is equivalent to $\delta < (=, >)\alpha\beta$. Accordingly, E_1 is a source (non-hyperbolic, saddle).

The Jacobian matrix of system (1.8) at fixed point $E_2(1, 0)$ is

$$J(E_2) = \begin{pmatrix} \alpha & -1 \\ 0 & e^{\beta-\delta} \end{pmatrix}. \tag{2.6}$$

The eigenvalues of $J(E_2)$ are $\lambda_1 = \alpha < 1$ and $\lambda_2 = e^{\beta-\delta}$.

It is easy to see that $\delta < (=, >)\beta$ implies $|\lambda_2| > (=, <)1$. Correspondingly, E_2 is a saddle (non-hyperbolic, sink).

Theorem 2.2. *When $\alpha\beta < \delta < \beta$, $E_3(\frac{\delta}{\beta}, \frac{(\beta-\delta)(\delta-\alpha\beta)}{\beta^2})$ is a positive fixed point of system (1.8). Let $\beta - \delta - (\delta - \alpha\beta)(1 - \beta + \delta) \triangleq q_0$, then the following statements about E_3 are true:*

(i) If $q_0 < 0$, E_3 is a sink.

(ii) If $q_0 = 0$, E_3 is non-hyperbolic.

(iii) If $q_0 > 0$, E_3 is a source.

Proof. The Jacobian matrix of system (1.8) at fixed point E_3 is

$$J(E_3) = \begin{pmatrix} 1 + \frac{\delta(\beta+\alpha\beta-2\delta)}{\beta^2} & -\frac{\delta}{\beta} \\ \frac{(\beta-\delta)(\delta-\alpha\beta)}{\beta} & 1 \end{pmatrix}. \tag{2.7}$$

The characteristic equation of (2.7) is

$$F(\lambda) = \lambda^2 + P\lambda + Q = 0,$$

where $P = -2 - \frac{\delta(\beta+\alpha\beta-2\delta)}{\beta^2}$, $Q = 1 + \frac{\delta[\beta+\alpha\beta-2\delta+(\beta-\delta)(\delta-\alpha\beta)]}{\beta^2} = 1 + \frac{\delta q_0}{\beta^2}$.

By calculation, we find

$$F(1) = \frac{\delta(\beta-\delta)(\delta-\alpha\beta)}{\beta^2} > 0,$$

and

$$\begin{aligned} F(-1) &= 4 + \frac{\delta[2\beta + 2\alpha\beta - 4\delta + (\beta-\delta)(\delta-\alpha\beta)]}{\beta^2} \\ &= \frac{4(\beta-\delta)(\beta+\delta)}{\beta^2} + \frac{2\delta(1+\alpha)}{\beta} + \frac{\delta(\beta-\delta)(\delta-\alpha\beta)}{\beta^2} > 0. \end{aligned}$$

We can see that $Q < (=, >)1$ is equivalent to $q_0 < (=, >)0$. Therefore, the following results are obtained.

If $q_0 < 0$, then $Q < 1$. By Lemma 2.1(i.1), $J(E_3)$ has two eigenvalues λ_1 and λ_2 with $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so E_3 is a sink.

If $q_0 > 0$, then $Q > 1$. By Lemma 2.1(i.4), $J(E_3)$ has two eigenvalues λ_1 and λ_2 with $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_3 is a source.

If $q_0 = 0$, then $Q = 1$. At this time, $P = -2 - \frac{\delta(\beta+\alpha\beta-2\delta)}{\beta^2} > -2$. Again, $F(-1) = 1 - P + Q = 2 - P > 0$. So, $P < 2$. Hence, $-2 < P < 2$. Lemma 2.1(i.2) tells us that $J(E_3)$ has a pair of conjugate complex roots λ_1 and λ_2 with $|\lambda_1| = |\lambda_2| = 1$, so E_3 is non-hyperbolic.

3. Bifurcation analysis

In this section, we use the Center Manifold Theorem and bifurcation theory [2,6,9,13,20] to analyze the flip bifurcation and Neimark-sacker bifurcation of system (1.8) happening at the fixed point E_3 and the transcritical bifurcations at the fixed points E_1 and E_2 respectively.

3.1. Bifurcation of system (1.8) at fixed point E_3

3.1.1. Flip bifurcation

Theorem 3.1. *System* (1.8) *cannot undergo a flip bifurcation at the fixed point* E_3*.*

Proof. The necessary condition for a flip bifurcation to occur at the fixed point E_3 of system (1.8) is $F(-1) = 0$. From the proof process of Theorem 2.2, we know that $F(-1) > 0$ always stays true under given parameter conditions. Therefore, system (1.8) can not undergo a flip bifurcation at the fixed point E_3.

Remark. According to [22, Theorem 1], system (1.4) undergoes a flip bifurcation at fixed point E_3. However, we prove that the flip bifurcation can not occur in its equivalent system we presented. Therefore, our results clearly show that different discrete methods can lead to completely different dynamic behaviors.

3.1.2. Neimark-Sacker bifurcation

When $q_0 = 0$, equivalently, $\alpha \triangleq \alpha_0 = \frac{1+\delta}{\beta} - \frac{1}{1-\beta+\delta}$. Theorem 2.2(iii) shows that the fixed point $E_3(x_*, y_*)$ is non-hyperbolic. Then, we point out the occurrence of Neimark-Sacker bifurcation which is stated in the following steps.

The first step. Let $u = x - x_*$ and $v = y - y_*$, then the fixed point $E_3(x_*, y_*)$ is transformed into the origin $O(0,0)$, and system (1.8) into

$$\begin{cases} u \to (u + x_*)\, e^{(1-u-x_*)(u+x_*-\alpha)-(v+y_*)} - x_*, \\ v \to (v + y_*)\, e^{\beta\,(u+x_*)-\delta} - y_*. \end{cases} \tag{3.1}$$

The second step. Choose the parameter α as a bifurcation parameter. Given a small perturbation α_* of the parameter α around α_0, i.e., $\alpha_* = \alpha - \alpha_0$, with $0 < |\alpha_*| \ll 1$, a perturbation of system (3.1) is

$$\begin{cases} u \to (u + x_*)\, e^{(1-u-x_*)(u+x_*-\alpha_0-\alpha_*)-(v+y_*)} - x_*, \\ v \to (v + y_*)\, e^{\beta\,(u+x_*)-\delta} - y_*. \end{cases} \tag{3.2}$$

The corresponding characteristic equation of the linearized equation of system (3.2) at $(u,v)=(0,0)$ is

$$\lambda^2+p(\alpha_*)\lambda+q(\alpha_*)=0, \tag{3.3}$$

where

$$p(\alpha_*)=-2-\frac{\delta(\beta+\alpha_0\beta+\alpha_*\beta-2\delta)}{\beta^2},$$

$$q(\alpha_*)=1+\frac{\delta[\beta+\alpha_0\beta+\alpha_*\beta-2\delta+(\beta-\delta)(\delta-\alpha_0\beta-\alpha_*\beta)]}{\beta^2}.$$

When $(\beta-\delta)(\beta^2+\delta^2)<\beta^2$ holds, $p^2(0)-4q(0)<0$. Therefore, the two roots of equation (3.3) are

$$\lambda_{1,2}(\alpha_*)=\omega\pm\mu\, i,$$

where $\omega=-\frac{1}{2}p(\alpha_*)$, $\mu=\frac{1}{2}\sqrt{4q(\alpha_*)-p^2(\alpha_*)}$, then

$$|\lambda_{1,2}(\alpha_*)|=\sqrt{q(\alpha_*)}=\sqrt{1+\frac{\delta\alpha_*}{\beta}(1-\beta+\delta)}.$$

The occurence of Neimark-Sacker bifurcation needs the following two conditions:

$$(i)\qquad \left.\frac{d|\lambda_{1,2}(\alpha_*)|}{d\alpha_*}\right|_{\alpha_*=0}\neq 0,$$

$$(ii)\qquad \lambda_{1,2}^i\neq 1, i=1,2,3,4.$$

By calculation, one finds

$$\left.\frac{d|\lambda_{1,2}(\alpha_*)|}{d\alpha_*}\right|_{\alpha_*=0}=\frac{\delta(1-\beta+\delta)}{2\beta}\neq 0,$$

and obviously $\lambda_{1,2}^i\neq 1, i=1,2,3,4$. Therefore, both of the conditions (i) and (ii) are satisfied.

The third step. Look for the normal form of system (3.2), when $\alpha_*=0$. Expanding system (3.2) with $\alpha_*=0$ as a Taylor series at $(u,v)=(0,0)$ to the third order as follows:

$$\begin{cases} u\to a_{10}u+a_{01}v+a_{20}u^2+a_{11}uv+a_{02}v^2+a_{30}u^3\\ \qquad +a_{21}u^2v+a_{12}uv^2+a_{03}v^3+O(\rho^4),\\ v\to b_{10}u+b_{01}v+b_{20}u^2+b_{11}uv+b_{02}v^2+b_{30}u^3\\ \qquad +b_{21}u^2v+b_{12}uv^2+b_{03}v^3+O(\rho^4), \end{cases} \tag{3.4}$$

where $\rho=\sqrt{u_n^2+v_n^2}$,

$a_{10}=1+\frac{\delta(\beta+\alpha_0\beta-2\delta)}{\beta^2}$, $\quad a_{01}=-\frac{\delta}{\beta}$, $\quad a_{20}=\frac{\beta+\alpha_0\beta-3\delta}{\beta}+\frac{\delta(\beta+\alpha_0\beta-2\delta)^2}{2\beta^3}$,

$a_{11}=-\frac{\beta^2+\delta(\beta+\alpha_0\beta-2\delta)}{\beta^2}$, $\quad a_{02}=\frac{\delta}{2\beta}$,

$a_{30}=-1+\frac{(\beta+\alpha_0\beta-2\delta)^2-2\delta(\beta+\alpha_0\beta-2\delta)}{3\beta^2}+\frac{(\beta+\alpha_0\beta-2\delta)^2}{6\beta^2}-\frac{\delta(\beta+\alpha_0\beta-2\delta)^2}{3\beta^3}$
$\qquad +\frac{\delta(\beta+\alpha_0\beta-2\delta)^2}{6\beta^4}$,

$a_{21} = -\frac{\beta+\alpha_0\beta-3\delta}{\beta} - \frac{r^2\delta(\beta+\alpha_0\beta-2\delta)^2}{2\beta^3}, \qquad a_{12} = \frac{1}{2} + \frac{\delta(\beta+\alpha_0-2\delta)}{2\beta^2},$
$a_{03} = \frac{\delta}{6\beta}, \qquad b_{10} = \frac{(\beta-\delta)(\delta-\alpha_0\beta)}{\beta}, \qquad b_{01} = 1,$
$b_{20} = \frac{1}{2}(\beta-\delta)(\delta-\alpha_0\beta), \qquad b_{11} = b, \qquad b_{02} = 0,$
$b_{30} = \frac{1}{6}\beta(\beta-\delta)(\delta-\alpha_0\beta), \qquad b_{21} = \frac{1}{2}\beta^2, \qquad b_{12} = b_{03} = 0.$

The fourth step. Take matrix

$$T = \begin{pmatrix} 0 & a_{01} \\ \mu & 1-\omega \end{pmatrix}, \quad \text{then} \quad T^{-1} = \begin{pmatrix} \frac{\omega-1}{\mu\, a_{01}} & \frac{1}{\mu} \\ \frac{1}{a_{01}} & 0 \end{pmatrix}.$$

Using transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix},$$

system (3.4) is transformed into the following form:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \omega & -\mu \\ \mu & \omega \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} F(X,Y) + O(\rho^4) \\ G(X,Y) + O(\rho^4) \end{pmatrix}, \tag{3.5}$$

where $\rho = \sqrt{X^2+Y^2}$,

$$F(X,Y) = c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + c_{21}u^2v + c_{12}uv^2 + c_{03}v^3,$$
$$G(X,Y) = d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + d_{21}u^2v + d_{12}uv^2 + d_{03}v^3,$$

$u = a_{01}Y, \quad v = \mu X + (1-\omega)Y,$
$c_{20} = \frac{a_{20}(\omega-1)}{\mu\, a_{01}} + \frac{b_{20}}{\mu}, \qquad c_{11} = \frac{a_{11}(\omega-1)}{\mu\, a_{01}} + \frac{b_{11}}{\mu}, \qquad c_{02} = \frac{a_{02}(\omega-1)}{\mu\, a_{01}} + \frac{b_{02}}{\mu},$
$c_{30} = \frac{a_{30}(\omega-1)}{\mu\, a_{01}} + \frac{b_{30}}{\mu}, \qquad c_{21} = \frac{a_{21}(\omega-1)}{\mu\, a_{01}} + \frac{b_{21}}{\mu}, \qquad c_{12} = \frac{a_{12}(\omega-1)}{\mu\, a_{01}} + \frac{b_{12}}{\mu},$
$c_{03} = \frac{a_{03}(\omega-1)}{\mu\, a_{01}} + \frac{b_{03}}{\mu}, \qquad d_{20} = \frac{a_{20}}{a_{01}}, \quad d_{11} = \frac{a_{11}}{a_{01}}, \qquad d_{02} = \frac{a_{02}}{a_{01}},$
$d_{30} = \frac{a_{30}}{a_{01}}, \quad d_{21} = \frac{a_{21}}{a_{01}}, \quad d_{12} = \frac{a_{12}}{a_{01}}, \quad d_{03} = \frac{a_{03}}{a_{01}}.$

Furthermore

$F_{XX}|_{(0,0)} = 2c_{02}\mu^2, \qquad F_{XY}|_{(0,0)} = c_{11}a_{01}\mu + 2c_{02}\mu(1-\omega),$
$F_{YY}|_{(0,0)} = 2c_{20}a_{01}^2 + 2c_{11}a_{01}(1-\omega), \qquad F_{XXX}|_{(0,0)} = 6c_{03}\mu^3,$
$F_{XXY}|_{(0,0)} = 2c_{21}a_{01}\mu^2 + 6c_{03}\mu^2(1-\omega),$
$F_{XYY}|_{(0,0)} = 2c_{21}a_{01}^2\mu + 4c_{12}a_{01}\mu(1-\omega) + 6c_{03}\mu(1-\omega)^2,$
$F_{YYY}|_{(0,0)} = 4(1-\omega)^3 + 6c_{30}a_{01}^3 + 4c_{21}a_{01}^2(1-\omega)$
$\qquad + 6c_{12}a_{01}(1-\omega)^2,$
$G_{XX}|_{(0,0)} = 2d_{02}\mu^2, \qquad G_{XY}|_{(0,0)} = d_{11}a_{01}\mu + 2d_{02}\mu(1-\omega),$
$G_{YY}|_{(0,0)} = 2d_{20}a_{01}^2 + 2d_{11}a_{01}(1-\omega), \qquad G_{XXX}|_{(0,0)} = 6d_{03}\mu^3,$
$G_{XXY}|_{(0,0)} = 2d_{21}a_{01}\mu^2 + 6d_{03}\mu^2(1-\omega),$
$G_{XYY}|_{(0,0)} = 2d_{21}a_{01}^2\mu + 4d_{12}a_{01}\mu(1-\omega) + 6d_{03}\mu(1-\omega)^2,$
$G_{YYY}|_{(0,0)} = 4(1-\omega)^3 + 6d_{30}a_{01}^3 + 4d_{21}d_{01}^2(1-\omega)$
$\qquad + 6d_{12}d_{01}(1-\omega)^2.$

The fifth step. To make sure system (3.4) undergoes a Neimark-Sacker bifurcation, we require that the following discriminatory quantity L is not zero [2,9,13,20]:

$$L = -Re\left[\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\eta_{11}\eta_{20}\right] - \frac{1}{2}|\eta_{11}|^2 - |\eta_{02}|^2 + Re(\lambda_2\eta_{21}), \tag{3.6}$$

where

$$\eta_{20} = \frac{1}{2}[(F_{XX} - F_{YY} + 2G_{XY}) + i(G_{XX} - G_{YY} - 2F_{XY})]|_{(0,0)},$$
$$\eta_{11} = \frac{1}{4}[(F_{XX} + F_{YY}) + i(G_{XX} + G_{YY})]|_{(0,0)},$$
$$\eta_{02} = \frac{1}{8}[(F_{XX} - F_{YY} - 2G_{XY}) + i(G_{XX} - G_{YY} + 2F_{XY})]|_{(0,0)},$$
$$\eta_{21} = \frac{1}{16}[(F_{XXX} + F_{XYY} + G_{XXY} + G_{YYY}) + i(G_{XXX} + G_{XYY} - F_{XXY} - F_{YYY})]|_{(0,0)}.$$

Based on above analysis, we obtain the following theorem.

Theorem 3.2. *Consider the parameters in the space* $\Omega_1 = \{(\alpha, \beta, \delta) \in R^3_+ | \alpha \in (0,1), \beta > 0, \alpha\beta < \delta < \beta, (\beta-\delta)(\beta^2+\delta^2) < \beta^2\}$. *Suppose* $\beta-\delta-(\delta-\alpha\beta)(1-\beta+\delta) = 0$. *Let* $\alpha_0 = \frac{1+\delta}{\beta} - \frac{1}{\beta(1-\beta+\delta)}$, *and* L *be defined as above* (3.6). *If* $L \neq 0$, *system* (1.8) *undergoes a Neimark-Sacker bifurcation at its fixed point* E_3, *when the parameter* α *varies in a small neighborhood of* α_0, *If* $L < (>)0$, *then an attracting (repelling) invariant closed curve bifurcates from the fixed point for* $\alpha > (<)\alpha_0$.

3.2. Bifurcation of system (1.8) at fixed point E_1

From Theorem 2.1(ii.2), one see that when parameter δ passes through the critical value $\delta_0 = \alpha\beta$, the dimensions of unstable manifold and stable manifold change. Therefore, system (1.8) may undergo a bifurcation. Then, what kind of bifurcation is it? The following steps will show the occurrence of transcritical bifurcation.

The first step. Take $u = x - \alpha$ and $v = y$ to transform the fixed point $E_1(\alpha, 0)$ into the origin $O(0,0)$, and system (1.8) into

$$\begin{cases} u \to (u+\alpha)\, e^{(1-u-\alpha)u-v} - \alpha, \\ v \to v\, e^{\beta\,(u+\alpha)-\delta}. \end{cases} \tag{3.7}$$

The second step. We choose the parameter δ as a bifurcation parameter. Given that a small perturbation δ^* of the parameter δ surround $\delta_0 = \alpha\beta$, i.e., $\delta^* = \delta - \delta_0$, with $0 < |\delta^*| \ll 1$, system (3.7) is perturbed into

$$\begin{cases} u \to (u+\alpha)\, e^{(1-u-\alpha)u-v} - \alpha, \\ v \to v\, e^{\beta\,u-\delta^*-\delta_0}. \end{cases} \tag{3.8}$$

The third step. Let $\delta^*_{n+1} = \delta^*_n = \delta^*$, then regard (3.8) as

$$\begin{cases} u \to (u+\alpha)\, e^{(1-u-\alpha)u-v} - \alpha, \\ v \to v\, e^{\beta\,u-\delta^*-\delta_0}, \\ \delta^* \to \delta^*. \end{cases} \tag{3.9}$$

Expanding (3.9) as a Taylor series at $(u, v, \delta_*) = (0,0,0)$ up to terms of order 3 produces the following model

$$\begin{pmatrix} u \\ v \\ \delta^* \end{pmatrix} \to \begin{pmatrix} \lambda_1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ \delta^* \end{pmatrix} + \begin{pmatrix} f(u,v,\delta^*) + O(\rho^4) \\ g(u,v,\delta^*) + O(\rho^4) \\ 0 \end{pmatrix}, \tag{3.10}$$

where $\rho = \sqrt{u^2 + v^2 + (\delta^*)^2}$, $\lambda_1 = 1 + r\alpha(1-\alpha)$,

$$\begin{aligned} f(u, v, \delta^*) &= l_{200}u^2 + l_{110}uv + l_{020}v^2 + l_{300}u^3 + l_{210}u^2v + l_{120}uv^2 + l_{030}v^3, \\ g(u, v, \delta^*) &= k_{200}u^2 + k_{020}v^2 + k_{002}(\delta^*)^2 + k_{110}uv + k_{101}u\delta^* + k_{011}v\delta^* \\ &\quad + k_{300}u^3 + k_{030}v^3 + k_{003}(\delta^*)^3 + k_{210}u^2v + k_{120}uv^2 + k_{021}v^2\delta^* \\ &\quad + k_{012}v(\delta^*)^2 + k_{201}u^2\delta^* + k_{102}u(\delta^*)^2 + k_{111}uv\delta^*, \end{aligned}$$

$l_{200} = 1 - 2\alpha + \frac{1}{2}\alpha(1-\alpha)^2$, $\quad l_{110} = -1 - \alpha(1-\alpha)$, $\quad l_{020} = \frac{\alpha}{2}$,
$l_{300} = -1 + \frac{1}{2}(1-\alpha) - \frac{5}{6}\alpha(1-\alpha)$, $\quad l_{210} = -(1-2\alpha) - \frac{1}{2}\alpha(1-\alpha)^2$,
$l_{120} = \frac{1}{2} + \frac{1}{2}\alpha(1-\alpha)$, $\quad l_{030} = \frac{1}{6}\alpha$, $\quad k_{110} = \beta$, $\quad k_{011} = -1$,
$k_{003} = -\frac{1}{6}$, $\quad k_{210} = \frac{1}{2}\beta^2$, $\quad k_{012} = \frac{1}{2}$, $\quad k_{111} = -\beta$,
$k_{200} = k_{020} = k_{002} = k_{101} = k_{300} = k_{030} = k_{120} = k_{021} = 0$,
$k_{201} = k_{102} = 0$.

The fourth step. Let matrix

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1-\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad T^{-1} = \begin{pmatrix} 1 & -\frac{1}{1-\alpha} & 0 \\ 0 & \frac{1}{1-\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using transformation

$$\begin{pmatrix} u \\ v \\ \delta^* \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ \delta^* \end{pmatrix},$$

system (3.10) is changed into as follows:

$$\begin{pmatrix} X \\ Y \\ \delta^* \end{pmatrix} \to \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \delta^* \end{pmatrix} + \begin{pmatrix} F_2(X, Y, \delta^*) + O(\rho^4) \\ G_2(X, Y, \delta^*) + O(\rho^4) \\ 0 \end{pmatrix}, \tag{3.11}$$

where $\rho = \sqrt{X^2 + Y^2 + (\delta^*)^2}$,
$F_2(X, Y, \delta^*) = f(X+Y, (1-\alpha)Y, \delta^*) - \frac{g(X+Y,(1-\alpha)Y,\delta^*)}{1-\alpha}$,
$G_2(X, Y, \delta^*) = \frac{1}{(1-\alpha)g(X+Y,(1-\alpha)Y,\delta^*)}$.

The fifth step. Determine the center manifold $W^c(0,0,0)$ of system (3.11) at the fixed point $O(0,0,0)$ in a small neighborhood of $\delta^* = 0$. By the center manifold theorem, we can obtain the representation of center manifold $W^c(0,0,0)$ as follows:

$$W^c(0,0,0) = \left\{(X, Y) : X = -\frac{1}{2(1-\alpha)}Y^2 + O(\rho^3)\right\},$$

where $\rho = \sqrt{Y^2 + (\delta^*)^2}$.
Then, the map restricted to the center manifold $W^c(0,0)$ is read as

$$\begin{aligned} Y \to G^*(Y, \delta^*) &= Y + k_{110}Y^2 + k_{011}Y\delta^* + (k_{110}m_{20} + k_{210})Y^3 \\ &\quad + \frac{1}{(1-\alpha)}k_{003}(\delta^*)^3 + k_{012}Y(\delta^*)^2 + k_{111}Y^2\delta^* \\ &\quad + O(\sqrt{X^2 + (\delta^*)^2}). \end{aligned}$$

Therefore, the following results are obtained:

$$G^*(Y,\delta^*)|_{(0,0)}=0,\qquad \left.\frac{\partial G^*}{\partial Y}\right|_{(0,0)}=1,\qquad \left.\frac{\partial G^*}{\partial \delta^*}\right|_{(0,0)}=0,$$
$$\left.\frac{\partial^2 G^*}{\partial Y\partial \delta^*}\right|_{(0,0)}=-1\neq 0,\qquad \left.\frac{\partial^2 G^*}{\partial Y^2}\right|_{(0,0)}=\beta\neq 0.$$

Acorrding to [20, (21.1.43)-(21.1.46),p503], all conditions are satisfied for an occurrence of transcritical bifurcation.

Based on above analysis, one has the following results.

Theorem 3.3. *Suppose the parameters* $(\alpha,\beta,\delta)\in\Omega_2=\{(\alpha,\beta,\delta)\in R^3_+|\alpha\in(0,1),\beta>0,\delta>0\}$*. Giving a perturbation of the parameter* δ *around* $\delta_0=\alpha\beta$*, there is an occurrence of transcritical bifurcation at fixed point* E_1 *of system* (1.8).

3.3. Bifurcation of system (1.8) at fixed point E_2

According to Theorem 2.1(iii.2), when $\delta=\beta$, the fixed point $E_2(1,0)$ is non-hyperbolic, system (1.8) may undergo a bifurcation (the corresponding eigenvalue are $\lambda_1=\alpha$, $\lambda_2=1$).

By using the same method as in Section 3.2, we get the following result.

Theorem 3.4. *Consider the parameters in the space* $\Omega_2=\{(\alpha,\beta,\delta)\in R^3_+|\alpha\in(0,1),\beta>0,\delta>0\}$*. When the parameter* δ *goes through the critical value* $\delta_0=\beta$*, system* (1.8) *undergoes a transcritical bifurcation at fixed point* E_2*.*

4. Numerical simulation

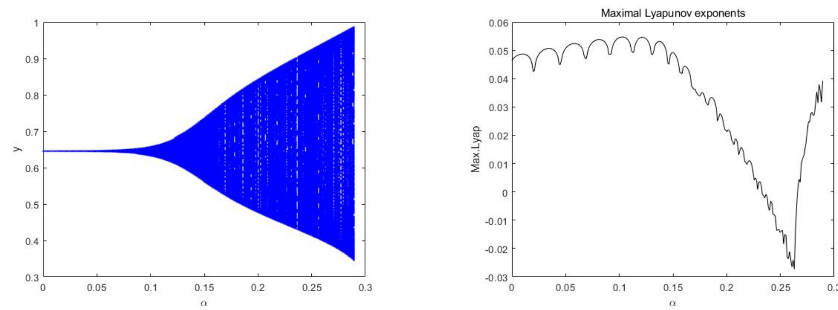
In this section, in order to verify above theoretical analysis, we present the bifurcation diagrams, phase portraits and Lyapunov exponents for some specific parameter values. We consider the following cases of bifurcation parameters.

Vary α in the range (0.01,0.29), and fix $\beta=0.8475$, $\delta=0.5469$ with the initial value $(x_0,y_0)=(0.76,0.06)$. It is easy to get the unique positive fixed point $E_3=(0.6453097345,0.0248542888)$ and $\alpha_0=0.138174668$, and the eigenvalues of $J(E_3)$ are $\lambda_{1,2}=0.9899\pm0.3009i$ with $|\lambda_{1,2}|=1$.

The bifurcation diagram in the (α,x) plane is given in Figure 1(a). It is easy to see that the fixed point E_3 is stable for $\alpha<0.1382$, and that the Neimark-Sacker bifurcation occurs, when $\alpha=0.1382$, and that the fixed point E_3 becomes unstable, when $\alpha>0.1382$. Figure 1(b) depicts the corresponding maximum Lyapunov exponents, which are positive for the parameter $\alpha\in(0.01,0.22)$, which means the chaos occurs in system (1.8).

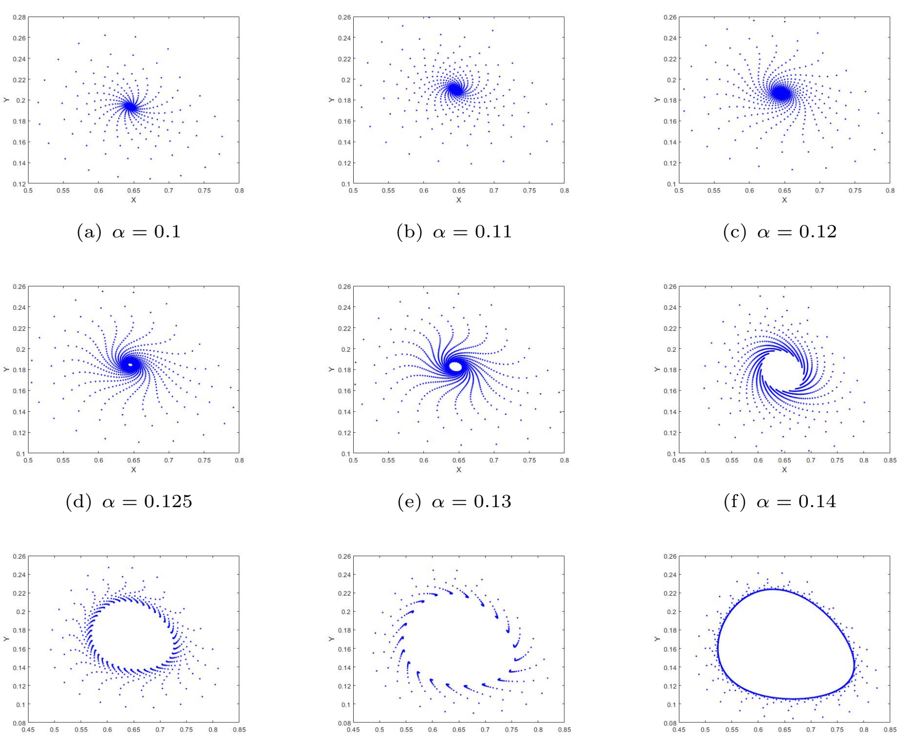
The phase portraits which are associated with Figure 1(a) are displayed in Figure 2. We can see that a smooth invariant circle bifurcates from the fixed point E_3, and its radius becomes big with the increase of α. When α exceeds 0.1382 there appears a circular curve enclosing the fixed point E_3, which depicts the occurrence of Neimark-Sacker bifurcation.

Now, choosing a different initial value $(x_0,y_0)=(0.65,0.16)$, the correspending phase portraits are plotted in Figure 3. Figures 2(e)-(i) depict that the closed circle is stable outside, while Figure 3 implies that the closed circle is stable inside. This agrees with Theorem 3.2.



(a) $\alpha \in (0.01, 0.29)$ (b) $\alpha \in (0.01, 0.29)$

Figure 1. Bifurcation of system (1.8) in (α, x)–plane and Maximal Lyapunov exponents

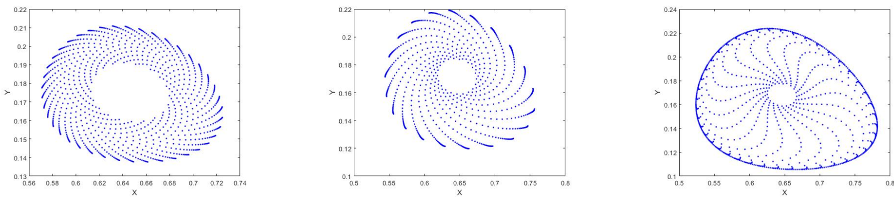


(a) $\alpha = 0.1$ (b) $\alpha = 0.11$ (c) $\alpha = 0.12$

(d) $\alpha = 0.125$ (e) $\alpha = 0.13$ (f) $\alpha = 0.14$

(g) $\alpha = 0.15$ (h) $\alpha = 0.16$ (i) $\alpha = 0.17$

Figure 2. Phase portraits for system (1.8) with $\beta = 0.8475$, $\delta = 0.5469$ and different α with the initial value $(x_0, y_0) = (0.76, 0.06)$ outside the closed orbit



(a) $\alpha = 0.15$ (b) $\alpha = 0.16$ (c) $\alpha = 0.17$

Figure 3. Phase portraits for system (1.8) with $\beta = 0.8475$, $\delta = 0.5469$ and different α with the initial value $(x_0, y_0) = (0.65, 0.16)$ inside the closed orbit

5. Conclusion and discussion

In this paper, we revisit a discrete predator-prey model (1.1) with Allee effect and Holling type-I functional response. By re-scaling, we get a simpler equivalent form (1.5) for its continuous model (1.3). Considering that the forward Euler method used in [22] cannot satisfy the requirement of accuracy, we apply an alternate discretization method to model (1.5) get a new discrete system (1.8). Under given parameter conditions, both system (1.8) and system (1.4) always have three nonnegative fixed points $E_0(0,0)$, $E_1(\alpha,0)$, $E_2(1,0)$. System (1.8) has a positive fixed point $E_3(\frac{\delta}{\beta}, \frac{(\beta-\delta)(\delta-\alpha\beta)}{\beta^2})$, when $\alpha\beta < \delta < \beta$. To our surprise, system (1.8) can not undergo the flip bifurcation at the fixed point E_3, whereas system (1.4) has, which due to the different discrete method we apply. We not only completely formulate the existence and stability of these fixed points, which are more complete results than Proposition 2 in [22], but also study flip bifurcation and Neimark-Sacker bifurcation at the fixed point E_3, and the transcritical bifurcations at the fixed points E_1 and E_2 respectively. However, only the bifurcations at the fixed point E_3 was considered in [22]. Hence, our results complement the corresponding ones in [22]. Our results sufficiently show that different discrete methods to the same continuous model may lead to different conclusions. Finally, we obtain some interesting dynamical properties for Neimark-Sacker bifurcation through numerical simulations.

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