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Autonomous Planar Systems of Riccati Type

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Abstract The role of Riccati type systems in the plane along with the related linear, second order differential equation is examined. If x and y are the variables of the Riccati differential equation, then any integrable Riccati system has two independent invariant curves dependent upon these variables whose nature is easily determined from the solution of the linear equation. Each of these curves has the same cofactor. Other invariant curves depend upon x alone and are shown to be less important. The systems have both Liouvillian and non–Liouvillian solutions and are easily transformable to symmetric systems. However, systems derived from them may not be symmetric in their transformed variables. Several systems from the literature are discussed with regard to the forms of the invariant curves presented in the paper. The relation of certain Riccati type systems is considered with respect to Abel differential equations.

Keywords Riccati differential equation, Centre–focus problem, Algebraic invariant curve, Cofactor, Symmetric centres.

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1. Introduction

In this work, we consider differential polynomial systems in the plane having the form

$$\begin{aligned}\frac{dx}{dt} &= -N(x,y),\\ \frac{dy}{dt} &= M(x,y)\end{aligned} \tag{1.1}$$

for polynomials M, N and specifically their relation to Riccati systems for which $M(x,y) = P(x)y^2 + Q(x)y + R(x)$, $N(x,y) = N(x)$. Our primary interest will be for the centre–focus cases

$$\begin{aligned}M(x,y) &= x + q(x,y),\\ N(x,y) &= y + p(x,y),\end{aligned} \tag{1.2}$$

where p, q are homogeneous polynomials of degree $n \geq 2$ or

$$\begin{aligned}M(x,y) &= x + q_2(x,y) + q_3(x,y),\\ N(x,y) &= y + p_2(x,y) + p_3(x,y),\end{aligned} \tag{1.3}$$

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where p_2, q_2 and p_3, q_3 are homogeneous polynomials of degree 2 and 3 respectively and how these systems relate to the Riccati systems. We shall refer to the first of these as homogeneous systems and to the second as cubic systems. We give examples of this type that can be reduced to Riccati type systems. Associated with (1.1) is the ordinary differential equation

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}. \tag{1.4}$$

In these, we assume the variables along with any associated parameters in the differential equations are real, although some of the latter could be complex.

The cubic system (1.3) is a particular case of a more general system of centre–focus type in which the nonlinearity is expressed, as the sum of homogeneous polynomials having degrees n and $2n-1$ for integers $n \geq 2$. The cubic system corresponds to $n = 2$. In [22], the authors use a relation to an Abel differential equation to consider certain centre conditions for the quintic $n = 3$ system.

A point (x_0, y_0) is said to be a critical point of (1.1), if $M(x_0, y_0) = N(x_0, y_0) = 0$. This point is said to be a centre if all trajectories of the system on a neighbourhood of the critical point are closed. In his original work [20], Poincaré developed a method for determining, if the origin is a centre by seeking an analytic solution to (1.4), where M, N satisfy $M(0,0) = N(0,0) = 0$. This is given by

$$U(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{k=3}^{\infty} U_k(x,y), \tag{1.5}$$

where the U_k are homogeneous polynomials of degree k. The solution (1.5) is required to satisfy the condition

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} = \sum_{k=2}^{\infty} V_k (x^2 + y^2)^k.$$

The V_k are called *Lyapunov coefficients* and they are homogeneous polynomials in the coefficients of the system. A necessary and sufficient condition for the critical point $(0,0)$ to be a centre is the vanishing of all the Lyapunov coefficients. One way of finding centre conditions is to compute several of the V_k, and then obtain necessary conditions for them to vanish. From this, one hopes to show that these conditions are sufficient so that all $V_k = 0$. In this case, (1.5) will have a form which is convergent and the solution will be given by $U(x,y) = C$ where C is a constant. Another approach is to show that (1.4) can be solved. The main method for doing this is the *Darboux method*. This approach has been well studied and a great number of general results concerning it are known. It requires the existence of *algebraic invariant curves* which are used to construct integrating factors, and in some cases, solutions. An integrating factor of (1.4) is a function $\mu(x,y)$ which satisfies the partial differential equation

$$-N(x,y)\frac{\partial \mu}{\partial x} + M(x,y)\frac{\partial \mu}{\partial y} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu. \tag{1.6}$$

An *algebraic invariant curve* of a system (1.1) is an expression of the form $f(x,y) = 0$ where f is a polynomial. It is required to satisfy the partial differential equation

$$-N(x,y)\frac{\partial f}{\partial x} + M(x,y)\frac{\partial f}{\partial y} = \lambda(x,y) f. \tag{1.7}$$

The function λ is called the cofactor of f and is a polynomial of degree at most $n-1$ where n is the degree of the system. There are a number of examples of this type of approach in the literature. See, for example, Chavarriga et al., [1–3] and this author [17] in which several such systems are given. Usually, the terminology for a solution of (1.7) is invariant algebraic curve. However, as all the curves we will obtain for Riccati systems are invariant curves which may or may not be reducible to an algebraic form, we shall retain the previous terminology throughout this work.

A *Darbouxian function* is a function of the form

$$\mathcal{D}(x,y) = \exp\left(\frac{g(x,y)}{f(x,y)}\right)\prod_{k=1}^{N} f_k^{\alpha_k}(x,y),$$

where f, g, f_k are polynomials and the exponents α_k can be real or complex. If an equation of the type (1.4) is Liouvillian integrable, it has an integrating factor of Darboux type. See Zhang [23], Theorem 3.11, and the references for Prelle, Singer and Christopher cited therein. The f_k are algebraic invariant curves of the system as is f provided it is not constant. In this case, the term $e^{g/f}$ is called an exponential factor and satisfies (1.7) for a polynomial function (cofactor) λ of degree less than or equal to $n-1$. For our purposes, a Liouvillian function is one obtained from a set of rational functions by a finite sequence of operations including arithmetic operations, exponentiations, differentiations, integrations and the solution of algebraic equations. See Żołądek and Llibre [25], Definition 1.

A major purpose of this work is not to continue with the Darboux approach, but to consider systems (1.1) which arise from Riccati differential equations. A Riccati differential equation is one of the form

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x), \tag{1.8}$$

where P, Q, R are arbitrary functions with $P(x) \neq 0$. Such an equation can always be paired with a corresponding linear, second order differential equation. Therefore, the solvability of one implies the solvability of the other. Suppose a Riccati system has an invariant curve (not necessarily algebraic) $\Phi(x,y) = 0$ for which $\frac{\partial\Phi}{\partial y} \neq 0$. Writing it in the form $y = \phi(x)$, it is straightforward to show that $y = \phi$ defines a particular solution of the Riccati equation. Letting $y \to y + \phi$ in (1.8) yields the Bernoulli equation

$$\frac{dy}{dx} = P(x)y^2 + (Q(x) + 2P(x)\phi(x))y.$$

Solving this and setting the integration constant equal to zero, we obtain a second solution to the Riccatti equation, which can be expressed as $(y-\phi(x))\int P(x)G(x)\,dx + G(x) = 0$, where $G(x) = \exp(\int(Q(x) + 2P(x)\phi(x))\,dx)$. A simple calculation shows that the cofactor of the invariant curve $y - \phi = 0$ is $\lambda = Py + P\phi + Q$, and a little more work shows that the second solution also has the same cofactor. Since it is unlikely that λ is a polynomial, some modifications are necessary in order to obtain one. These basically involve the forms of the particular solutions used; the details of this are discussed in Section 3. However, this simple example illustrates two basic aspects of Riccati systems. One is that there are two independent invariant curves and the other is that both of the curves (when expressed properly) have the same cofactor. Although this method of finding a second solution of a Riccati equation when one solution is given is well-known (see [21]), it is worthwhile, for our purposes, to express it in this manner. Hence, whenever a Riccati equation (system) has an invariant curve of the form $\Phi(x, y) = 0$, both it and the corresponding linear, second order equation are solvable. We shall assume this is true throughout this work. The problem of when (1.8) has an algebraic invariant curve was considered in [14] by Llibre and Vals for the case when P, Q, R are polynomials.

The paper consists of two main parts. In Sections 2 and 3, we present some of the forms of Riccati systems and the main results concerning these. Also, in Sections 3–5, some results from the literature are reviewed with regard to these Riccati forms. In Section 2, we discuss the transformation of Riccati systems to symmetric and more general systems and in Section 3 we present some general properties of Riccati systems. In particular, we show that any such system has two invariant curves having a particular and specific form. Whether or not these are algebraic depends upon the solution of the related linear, second order differential equation. In Section 5, we obtain the reduction of a cubic system (1.3) which is solvable in terms of *Airy functions* to a Riccati system. We also present two more general cubic systems which are also solvable in terms of these functions. Furthermore, we obtain the reduction of a homogeneous system (1.2) to a Bernoulli form. In the final section, we give a partial proof of the results in Section 5. All symbolic computations for the paper were carried out in the Computer Algebra System (CAS) Maple.

2. Transformation of Riccati equations to general systems

The homogeneous systems (1.2) given by

$$\begin{aligned} \frac{dx}{dt} &= -y - ax^{n-1}y, \\ \frac{dy}{dt} &= x + bx^{n-2}y^2 + cx^{n-4}y^4, \end{aligned} \tag{2.1}$$

where $n \geq 4$ is an integer and a, b, c are parameters are generally solvable in terms of special functions, although some do have elementary solutions. Each is reducible to a Riccati differential equation which can be transformed to a linear, second order differential equation and this is what gives rise to the form of the solutions. In addition, each of the systems in (2.1) is symmetric, having trajectories which are symmetric about the x-axis.

We consider the linear, second order differential equation

$$\frac{d^2u}{dx^2} - A(x)\frac{du}{dx} + B(x)u = 0, \tag{2.2}$$

where A, B are rational functions such that $A = Q/P$, $B = R/P$, where P, Q and R are polynomials. For the form $Pu'' - Qu' + R = 0$, we assume that $\gcd(P, \gcd(Q, R)) = 1$ and that A, B are fully reduced. By choosing the coefficient functions appropriately, the form (2.2) includes the ${}_2\mathcal{F}_1$ Gaussian hypergeometric functions, the ${}_1\mathcal{F}_1$ confluent hypergeometric functions and the ${}_0\mathcal{F}_1$ hypergeometric functions of which class the Bessel functions are members. All classical orthogonal polynomials are included as well as more general functions such as *Heun functions*.

See [11] for details. Any such equation can be transformed to a Riccati differential equation by the transformation

$$u(x) = e^{-\int \mathcal{F}(x)y(x)\,dx},$$

where $\mathcal{F}$ is an arbitrary differentiable function. The resulting Riccati equation is

$$\frac{dy}{dx} = y^2 + \left(A(x) - \frac{\mathcal{F}'(x)}{\mathcal{F}(x)}\right) y + \frac{B(x)}{\mathcal{F}(x)},$$

and assuming (2.2) is solvable, it has the solution

$$y(x) = -\frac{1}{\mathcal{F}(x)} \frac{C\psi_1'(x) + \psi_2'(x)}{C\psi_1(x) + \psi_2(x)}, \tag{2.3}$$

where ψ_1, ψ_2 are independent solutions of (2.2) and C is a constant. For the systems of this type that we shall consider in this section we can take $\mathcal{F}(x) = 1$. This gives

$$\frac{dy}{dx} = y^2 + A(x)y + B(x). \tag{2.4}$$

The transformation which converts a given Riccati form to a linear, second order equation is given by

$$y(x) = -\frac{1}{\mathcal{G}(x)} \frac{u'(x)}{u(x)}, \tag{2.5}$$

where $\mathcal{G}$ is the coefficient function of the y^2 term in (1.8). To avoid repeated repetition, whenever the functions ψ_1, ψ_2 are referred to in the remainder of the paper (except where necessary in the statement of formal results), it will be assumed that they are linearly independent.

Writing (2.4) as a system and using the assumed forms for A and B, we get

$$\begin{aligned} \frac{dx}{dt} &= P(x), \\ \frac{dy}{dt} &= P(x)y^2 + Q(x)y + R(x). \end{aligned} \tag{2.6}$$

Proposition 2.1. *Let ψ_1 and ψ_2 be linearly independent solutions of* (2.2)*. Then, the solution of the planar system defined by* (2.6) *can be given in terms of ψ_1, ψ_2 and their derivatives.*

Letting $y \to y^2$ in (2.6), we obtain the system

$$\begin{aligned} \frac{dx}{dt} &= 2P(x)y, \\ \frac{dy}{dt} &= P(x)y^4 + Q(x)y^2 + R(x), \end{aligned} \tag{2.7}$$

and letting $y \to -1/y^2$ in (2.6), we obtain the alternate system

$$\begin{aligned} \frac{dx}{dt} &= 2P(x)y, \\ \frac{dy}{dt} &= R(x)y^4 - Q(x)y^2 + P(x). \end{aligned} \tag{2.8}$$

Proposition 2.2. *The trajectories of the systems* (2.7) *and* (2.8) *are symmetric about the x-axis as the systems are invariant under the transformation* $(x, y, t) \to (x, -y, -t)$.

We can now transform (2.7) and (2.8) in any manner we wish to obtain more general systems that may not display the type of symmetry associated with the original systems. In Section 5 we obtain the full transformation which maps a specific member of (2.1) to a cubic system. This is made possible because of the association of these systems with certain Abel differential equations which we discuss in Section 4.

Proposition 2.3. *Let*

$$\begin{aligned} \frac{dx}{dt} &= -y - \mathcal{P}(x, y), \\ \frac{dy}{dt} &= x + \mathcal{Q}(x, y), \end{aligned} \tag{2.9}$$

where $\mathcal{P}$, $\mathcal{Q}$ are polynomials having no constant or linear terms be any system derived from either (2.7) *or* (2.8) *by an invertible transformation. Then the origin of the system is a centre.*

In this case, it does not matter if either (2.7) or (2.8) is solvable, only that (2.9) is transformable to a system which is symmetric. For general choices of P, Q, R in (2.7), it is not obvious how systems like (2.9) can be obtained. However, we can ensure that (2.7) will have a non–degenerate critical point at the origin if we impose the conditions $P(0) \neq 0$, $R(0) = 0$, $R'(0) \neq 0$. Further, this will be of centre-focus type if we take $P(0)R'(0) < 0$. The equation

$$(a_1 x + b_1)\frac{d^2u}{dx^2} - (a_2 x^2 + b_2 x + c_2)\frac{du}{dx} + x(a_3 x^2 + b_3 x + c_3)u = 0, \tag{2.10}$$

where a_1, b_1, $\ldots, c_3$ are parameters such that $b_1 c_3 \neq 0$ satisfies the conditions for a non–degenerate critical point. It is solvable in terms of Heun functions for general non–zero values of the parameters, although there are cases for which it cannot be solved (such as $a_3 = 0$, $a_1 a_2 \neq 0$). However, these do not concern us. For this form, (2.7) becomes the degree 5 system

$$\begin{aligned} \frac{dx}{dt} &= 2b_1 y + 2a_1 xy, \\ \frac{dy}{dt} &= c_3 x + b_3 x^2 + a_3 x^3 + (a_2 x^2 + b_2 x + c_2)y^2 + (a_1 x + b_1)y^4. \end{aligned} \tag{2.11}$$

If $b_1 c_3 < 0$, the linear part has the form of a centre–focus, and if $b_1 c_3 > 0$, it has the form of a hyperbolic saddle. Clearly, there are other possibilities if $b_1 c_3 = 0$. We will show in the next section that unless (2.10) is Liouvillian integrable, (2.11) does not have an algebraic invariant curve.

The solutions of (2.10) and (2.11) tend to be rather cumbersome expressions for general values of the parameters. Denoting the right side of (2.3) by $\Phi(x)$, the solution of (2.11) is given by $y^2 = \Phi(x)$, where ψ_1, ψ_2 are solutions of (2.10). In order to greatly simplify the following discussion, we will proceed using two specific examples. In each of these we will take $P(x) = 1$. Setting $b_1 = 1$, $b_2 = 2, c_2 = -2$, $c_3 = -2$ and the remaining parameters equal to zero, (2.10) becomes

$$\frac{d^2u}{dx^2} - 2(x - 1)\frac{du}{dx} - 2xu = 0, \tag{2.12}$$

and (2.11) is

$$\begin{aligned}\frac{dx}{dt} &= 2y,\\ \frac{dy}{dt} &= -2x + 2(x-1)y^2 + y^4.\end{aligned} \tag{2.13}$$

The general solution of (2.12) is given by $u = C_1\psi_1 + C_2\psi_2$ for arbitrary constants C_1, C_2 and

$$\psi_1(x) = \sqrt{x}e^{x(x-2)/2}I_{1/4}\left(\frac{x^2}{2}\right),$$
$$\psi_2(x) = \sqrt{x}e^{x(x-2)/2}K_{1/4}\left(\frac{x^2}{2}\right),$$

where I, K are the modified Bessel functions of the first and second kinds respectively.

Now, we would like to transform (2.11) in such a manner that the characteristics of the eigenvalues of the linear part are retained. A general transformation that will do this is given by

$$\begin{aligned}x &= \frac{a\xi + b\eta}{(1+f(\xi,\eta))^n},\\ y &= \frac{c\xi + d\eta}{(1+f(\xi,\eta))^n},\end{aligned} \tag{2.14}$$

where $\Delta = ad - bc \neq 0$, f is an arbitrary and suitably differentiable function which satisfies $f(0,0) = 0$, and $n \neq 0$ is an integer. Since we are interested in polynomial systems, we assume f is a polynomial. Using (2.14) to transform the general system (1.1), we can show that

$$\frac{d\xi}{dt}\frac{\partial f}{\partial \xi} + \frac{d\eta}{dt}\frac{\partial f}{\partial \eta} = (1 + f(\xi,\,\eta))\mathcal{P}(\xi,\eta),$$

where the derivatives $\dot{\xi}$ and $\dot{\eta}$ define the transformed system and $\mathcal{P}$ is a polynomial. Hence, from (1.7), it follows that $1+f(\xi,\eta) = 0$ defines an algebraic invariant curve of the transformed system. The actual situation is much more general than this as we can replace the single factor in the denominators for x and y in (2.14) by a product of factors $(1 + f_k(\xi,\eta))^{n_k}$, where the n_k's are non–zero integers and the f_k's are polynomials having no constant terms for $k = 1,\ldots,N$. Each of the factors $1 + f_k(\xi,\,\eta) = 0$ becomes an algebraic invariant curve of the transformed system. Since these curves are induced by the transformation, there are no counterparts to them in the original system. This gives an indication of how difficult it could be to reduce a given system to symmetric form because the appropriate transformation cannot be determined. As an example, we could have a system having N invariant straight lines which is not integrable, because the original system is not integrable.

The simplest form for (2.14) is

$$\begin{aligned}x &= \frac{a\xi + b\eta}{\alpha\xi + \beta\eta + 1},\\ y &= \frac{c\xi + d\eta}{\alpha\xi + \beta\eta + 1},\end{aligned} \tag{2.15}$$

where α, β are parameters. A transformation of this type was used by Lloyd and Pearson in [15] to investigate centre conditions in certain cubic systems. Using it to transform (2.13), we obtain the system

$$\begin{aligned}\frac{d\xi}{dt} &= -2(ab+cd)\xi - 2(b^2+d^2)\eta + \sum_{k=2}^{5} p_k(\xi,\eta),\\ \frac{d\eta}{dt} &= 2(a^2+c^2)\xi + 2(ab+cd)\eta + \sum_{k=2}^{5} q_k(\xi,\eta),\end{aligned} \tag{2.16}$$

where p_k, q_k are homogeneous polynomials of degree k. The eigenvalues of the linear portion are $\pm 2\Delta i$, and those of the original system are $\pm 2i$. The solution is given by transforming $y^2 = \Phi(x)$ in accordance with (2.15).

Equation (2.2) with $A(x) = -2x^3$, $B(x) = -2x$ is not solvable in Maple. It leads to the system (2.7) given by

$$\begin{aligned}\frac{dx}{dt} &= 2y,\\ \frac{dy}{dt} &= -2x - 2x^3y^2 + y^4,\end{aligned}$$

which by Proposition 2.2 has a centre at the origin. When it is transformed according to (2.15), it produces a system of degree 6 similar in form to (2.16) and which has a centre at $(\xi,\eta) = (0,0)$. We note that neither of these transformed systems is symmetric with respect to the variables ξ, η.

Based on (2.7) and (2.11), we have used simple transformations to produce centre–focus forms of homogeneous systems which have centres at the origin because they are derived from symmetric systems. In Section 5, we carry out this transformation in the opposite way for a cubic system and a particular homogeneous system, both of which are known to have centres at the origin. Each of these systems is shown to be ultimately transformable to a symmetric system and this, once again, raises the question as to whether or not all homogeneous systems with a centre can be transformed to a symmetric form. Ideas such as this are certainly not new. There is, for example, Żołądek's conjecture [24] regarding rationally reversible and Liouvillan integrable cubic systems (also see [9], Conjecture 9.1, by Christopher and Llibre). From the ideas just presented, we think it is highly probable that any polynomial system that is solvable in terms of special functions is derivable from a system such as (2.7).

3. Invariant curves of Riccati equations

Here, we present some basic ideas concerning Riccati type systems such as (2.6), which we shall formalize at the end of the section. In the following, we consider in some detailed two separate cases, although it is clear that the same ideas can be used for other Riccati systems. One case is where the form $Pu'' - Qu' + Ru = 0$ of equation (2.2) can be written having arbitrary polynomial coefficients and the other for the case when the Riccati equation (1.8) has general polynomial coefficients. For the first of these, we have $\mathcal{F}(x) = \mathcal{G}(x) = 1$ and for the second $\mathcal{F}(x) = \mathcal{G}(x) = P(x)$. For a specific pair of corresponding linear, second order and Riccati equations, we can take $\mathcal{F} = \mathcal{G}$.

Solving (2.3) for C, we obtain

$$U(x,y)=\frac{\mathcal{F}(x)\psi_2(x)y+\psi_2'(x)}{\mathcal{F}(x)\psi_1(x)y+\psi_1'(x)}=-C=\frac{1}{K}, \tag{3.1}$$

where K, C are constants and ψ_1, ψ_2 are solutions of (2.2). From this, we can see that no Riccati system can ever have a polynomial first integral. For suitable choices of $\mathcal{F}$, a sufficient condition for the solution to be rational is that ψ_1, ψ_2 be rational. In this solution, there are two particular solutions or invariant curves, which we get by setting one of C or K equal to zero. They are given by

$$\begin{aligned}\mathcal{F}(x)\psi_1(x)y+\psi_1'(x)&=0,\\ \mathcal{F}(x)\psi_2(x)y+\psi_2'(x)&=0\end{aligned} \tag{3.2}$$

and can be obtained by transforming the solutions $u=\psi_1$ and $u=\psi_2$ of (2.2) by (2.5). They can be considered as basic forms for the invariant curves and because they often involve functions which are not elementary, they are not usually algebraic unless they can be reduced in some manner. For example, in the case of system (3.9) below it is necessary to reduce them in order to obtain an algebraic curve which has a polynomial cofactor. The forms (3.2) are the only possibilities for algebraic invariant curves of this type (dependent on y), there can be no others. For if $\Psi(x,y)=0$ defined another such curve, then writing it as $y=-\psi_3'/(\mathcal{F}\psi_3)=-u'/(\mathcal{F}u)$, it would follow that $u=\psi_3$ is another solution of (2.2) which is not possible unless there exist constants α, β such that $\psi_3=\alpha\psi_1+\beta\psi_2$. If this is the case, we obtain $\mathcal{F}\psi_3 y+\psi_3'=(\alpha\mathcal{F}\psi_1+\beta\mathcal{F}\psi_2)y+(\alpha\psi_1'+\beta\psi_2')=\alpha(0)+\beta(0)=0$. Therefore, we get nothing new. Clearly, the same is true for any Riccati system whose solution is given by (3.1). With ψ_3 defined in this way, we can obtain the so–called *cross–ratio solution* of the Riccati equation (see [21]) from (3.1). Planar polynomial systems having non–algebraic invariant curves were considered by García and Giné in [12].

There is another possibility for algebraic invariant curves of Riccati systems. Suppose we have $\dot{x}=\mathcal{P}(x)$ where $\mathcal{P}$ is a non–constant polynomial. Then, writing

$$\mathcal{P}(x)=K\prod_{k=1}^{N}(x-\alpha_k)^{n_k}$$

where $K\neq 0$, the α_k are real or complex constants with $\alpha_i\neq\alpha_j$ unless $i=j$ and the n_k, N are positive integers, we have from (1.7)

$$\begin{aligned}\lambda_k(x,y)&=\frac{1}{x-\alpha_k}\left(\mathcal{P}(x)\frac{d}{dx}(x-\alpha_k)+\frac{dy}{dt}\frac{\partial}{\partial y}(x-\alpha_k)\right)\\ &=\frac{\mathcal{P}(x)}{(x-\alpha_k)^{n_k}}(x-\alpha_k)^{n_k-1}.\end{aligned}$$

Since λ_k is a polynomial, each of the simple factors $x-\alpha_k$ of $\mathcal{P}$ is an irreducible algebraic invariant curve. These are important in the construction of integrating factors of Darboux type, but due to the form (3.5) of the general integrating factor given below, it is clear that we do not need to consider them separately. Any such factor must appear naturally. The expression (3.4) for the Wronskian is valid for the first of the two systems we consider, and for the second system, it is given by $CP(x)e^{\int Q(x)\,dx}$. Hence, the function P appears explicitly in each case. We give a specific example of the appearance of these types of factors later. As given, the invariant curves defined by (3.2) are not necessarily algebraic. The simple Euler form $A(x) = 2(x+2)/x$, $B(x) = (x^2+4x+6)/x^2$ of (2.2) with $\mathcal{F}(x) = x^2$ has the invariant curve $(x^2)(x^2e^x)y + (x^2e^x)' = xe^x(x^3y+x+2) = 0$ which reduces to an algebraic form. More generally, the invariant curves of (2.13) are not algebraic, since the solutions of (2.12) are not Liouvillian.

Expressing the functions A and B of (2.4) in terms of ψ_1, ψ_2 gives

$$\begin{aligned} A(x) &= \frac{Q(x)}{P(x)} = -\frac{\psi_1''(x)\psi_2(x) - \psi_1(x)\psi_2''(x)}{\mathcal{W}(\psi_1(x), \psi_2(x))}, \\ B(x) &= \frac{R(x)}{P(x)} = -\frac{\psi_1''(x)\psi_2'(x) - \psi_1'(x)\psi_2''(x)}{\mathcal{W}(\psi_1(x), \psi_2(x))}, \end{aligned} \tag{3.3}$$

where $\mathcal{W}$ is the Wronskian given by

$$\begin{aligned} \mathcal{W}(\psi_1(x), \psi_2(x)) &= \psi_1(x)\psi_2'(x) - \psi_1'(x)\psi_2(x) \\ &= Ce^{-\int A(x)\,dx} = Ce^{-\int Q(x)/P(x)\,dx} \end{aligned} \tag{3.4}$$

and $C \neq 0$ is a constant. Differentiating the solution (3.1) and its reciprocal with respect to y gives the integrating factors

$$\mu(x,y) = \frac{\mathcal{F}(x)\mathcal{W}(\psi_1(x), \psi_2(x))}{(\mathcal{F}(x)\psi_k(x)y + \psi_k'(x))^2} \tag{3.5}$$

for $k = 1, 2$. They are particular cases of a general solution of (1.6) which we consider at the end of the section. However, they are suitable for our purposes at this time.

The integrating factors (3.5) satisfy a form of (1.6) for which $N(x,y) = -1$, but can be easily modified to satisfy other forms of Riccati systems. The appropriate form for (2.4) has $\mathcal{F}(x) = 1$. The invariant curves (3.2) satisfy $y = -\psi'/(\mathcal{F}\psi)$ and ordinarily we would cancel any common factors that appear in this expression. However, in order to obtain a specific form for the cofactor, we will not do this except where indicated. To obtain the first of these, we can use the system derived from (2.6). Since the labelling of the functions ψ_1, ψ_2 is arbitrary, it is obvious that both curves have the same cofactor. Using the first of the curves (3.2) and taking $\mathcal{F}(x) = 1$, we obtain for the quotient with respect to y

$$\begin{aligned} \lambda(x,y) &= \frac{\psi_1'(x)y + \psi_1''(x) + (y^2 + A(x)y + B(x))\psi_1(x)}{\psi_1(x)y + \psi_1'(x)} \\ &= y + A(x) + R(x,y) = y + \frac{Q(x)}{P(x)}. \end{aligned}$$

The numerator of the remainder R is the simple statement that ψ_1 is a solution of (2.2), so it is automatically satisfied. The cofactor

$$\lambda(x,y) = P(x)y + Q(x) \tag{3.6}$$

for (2.6) is then obtained by multiplying the above result (system) by P. The simplest case for an algebraic invariant curve is when $y = -\psi_1'/(\mathcal{F}\psi_1) = \alpha$, a constant. In this case we have $\psi_1(x) = e^{-\int \alpha\mathcal{F}(x)\,dx}$. We shall say more about this later. Another possibility is that there exists a non–zero integer n such that $(\psi_1'/(\mathcal{F}\psi_1))^n$

is a rational function. The form of (2.2) with $A(x) = 1/(2x)$, $B(x) = -a^2/(4x)$ having solutions $e^{\pm a\sqrt{x}}$ generates the curve $4xy^2 - a^2 = 0$.

In [4], Chavarriga and Grau used (2.2) with $A(x) = 2x$ and $B(x) = 2n$, where n is a positive integer to produce the Riccati system

$$\begin{aligned}\frac{dx}{dt} &= 1,\\ \frac{dy}{dt} &= y^2 + 2xy + 2n\end{aligned} \tag{3.7}$$

as an example of a system of low degree which has an algebraic invariant curve of arbitrary degree. This is a polynomial system which also has the form of (2.6). Hence, we can take $\mathcal{F}(x) = P(x) = 1$ in the preceding results. One of the solutions of (2.2) for this case is the Hermite polynomial $\mathcal{H}_n$, which means that (3.7) has an irreducible algebraic invariant curve $\mathcal{H}_n y + \mathcal{H}_n' = 0$ having cofactor (3.6) given by $\lambda(x, y) = y + 2x$. For this system, the other curve in (3.2) is not algebraic. From (3.4) and (3.5), an integrating factor is

$$\mu(x, y) = \frac{e^{x^2}}{(\mathcal{H}_n(x)y + \mathcal{H}_n'(x))^2}.$$

If we set $y = \alpha$, where α is a constant in system (2.6), we obtain the relation

$$P(x)\alpha^2 + Q(x)\alpha + R(x) = 0. \tag{3.8}$$

Hence, if this satisfied, the system has an algebraic invariant curve $y - \alpha = 0$. In [14], Llibre and Valls considered the algebraic invariant curves and algebraic first integrals for polynomial Riccati systems of the form

$$\begin{aligned}\frac{dx}{dt} &= 1,\\ \frac{dy}{dt} &= P(x)y^2 + Q(x)y + R(x).\end{aligned} \tag{3.9}$$

In [14], Theorem 2(b), the relation (3.8) is exactly the condition given for the polynomial system (3.9) to have an algebraic invariant curve. It is also stated that the system has such a curve if and only if (3.8) is satisfied. However, in view of the results for the system (3.7), it is clear that this is not true. As we shall see, this case is not as straightforward as the first one considered.

Transforming (3.9) by $y \to -y'/(Py)$ gives the second order equation

$$\frac{d^2y}{dx^2} - \left(Q(x) + \frac{P'(x)}{P(x)}\right)\frac{dy}{dx} + P(x)R(x)y = 0. \tag{3.10}$$

For the case when (3.8) is satisfied, we have noted that one solution of this equation is $\psi_1(x) = e^{-\int \alpha P(x)\,dx}$. Hence, the corresponding invariant curve (3.2) is

$$P(x)\psi_1(x)y + \psi_1'(x) = P(x)e^{-\int \alpha P(x)\,dx}(y - \alpha) = 0.$$

If we determine the cofactor using the full form of this curve, we obtain

$$\begin{aligned}\lambda(x,y) &= \frac{\frac{\partial}{\partial x}(P(x)E(x)(y-\alpha)) + (P(x)y^2+Q(x)y+R(x))\frac{\partial}{\partial y}(P(x)E(x)(y-\alpha))}{P(x)E(x)(y-\alpha)} \\ &= \frac{E(x)\left[P'(x)+\alpha P^2(x)+(P(x)y-\alpha P(x)-Q(x))P(x)\right]}{P(x)E(x)} \\ &= P(x)y+Q(x)+\frac{P'(x)}{P(x)}\end{aligned} \tag{3.11}$$

where $E(x) = e^{-\int \alpha P(x)\, dx}$. If we used the curve obtained by removing the common term $P(x)$, we would get $\lambda = Py + Q$, and if we used $y - \alpha = 0$ as the invariant curve, we have $\lambda = Py + Q + \alpha P$ which is the form for this case obtained in [14].

For the general form (3.9) and the invariant curves (3.2), (3.11) is exactly the expression we would obtain for the cofactor. Clearly, in order to have a polynomial form for this, we must impose further conditions. The obvious one, $P'(x) = 0$, basically leads back to the previous case given by (3.6). An example for which $P'(x) \neq 0$ is given by

$$\begin{aligned}\frac{dx}{dt} &= 1, \\ \frac{dy}{dt} &= (3x+2n)y^2 + (x+2)y + 1,\end{aligned}$$

where n is an arbitrary parameter which we take to be a non–negative integer. Since the expression for $\dot{y}$ has no factor $y - \alpha$ where α is a constant, the coefficient functions do not satisfy (3.8). The system was developed from a form of (3.10), which has solutions in terms of confluent hypergeometric functions. For these values of n, it will produce polynomial solutions multiplied by a simple exponential factor. For $n = 1$, the system has the irreducible algebraic invariant curve

$$(x-5)(x^4-12x^3+42x^2-40x-9)y + x^4 - 16x^3 + 90x^2 - 208x + 163 = 0. \tag{3.12}$$

For general values of n the degree of the curve is $2n + 4$, and in this regard, it is a result similar to that given by (3.7). The solution of (3.10) that gives rise to (3.12) is $\psi_1(x) = (x-5)(x^4-12x^3+42x^2-40x-9)e^{3x}$. Multiplying (3.12) by e^{3x} gives the cofactor $\lambda(x,y) = P(x)y+Q(x) = (3x+2)y+x+2$. What we can further show and what this example has in common with the previous one from the case (3.8) is that the expression for ψ_1' has a factor of P and this common factor P was removed from the expression (3.2) of the invariant curve before the cofactor was determined. This is also true for several other cases we know of, so we believe that the solutions ψ_1, ψ_2 must be such that their derivatives have a factor of P. Substituting $y' = P\phi$ in (3.10), we get the linear, second order equation

$$\begin{aligned}R(x)\frac{d^2\phi}{dx^2} &- (Q(x)R(x)+R'(x))\frac{d\phi}{dx} \\ &+ \left(P(x)R^2(x)+Q(x)R'(x)-R(x)Q'(x)\right)\phi = 0\end{aligned} \tag{3.13}$$

for the function ϕ. Thus, we have $\psi_1' = P\phi_1$, $\psi_2' = P\phi_2$, where ϕ_1, ϕ_2 are linearly independent solutions of (3.13). This leads to the following result which we will establish later.

Proposition 3.1. *A necessary and sufficient condition for the system* (3.9) *to have the unique polynomial cofactor* $\lambda = Py + Q$ *for its invariant curves is that* $\psi' = P\phi$*, where* ψ *and* ϕ *are linearly independent solutions of* (3.10) *and* (3.13) *respectively. In this case, the invariant curves* (3.2) *with* $\mathcal{F} = P$ *are reduced to the form* $\psi y + \phi = 0$.

Corollary 3.1. *A necessary condition for the system* (3.9) *to have an algebraic invariant curve is that it have an invariant curve* $\psi y + \phi = 0$ *having cofactor* $\lambda = Py + Q$ *where* $\psi' = P\phi$ *with* ψ *and* ϕ *being linearly independent solutions of* (3.10) *and* (3.13) *respectively.*

In this case, the general integrating factor (3.5) with $\mathcal{F} = P$ can be taken as

$$\begin{aligned}\mu(x,y) &= \frac{P(x)\mathcal{W}(\psi_1(x),\psi_2(x))}{(P(x)\psi(x)y + P(x)\phi(x))^2} \\ &= \frac{P(x)P(x)e^{\int Q(x)\,dx}}{(P(x)\psi(x)y + P(x)\phi(x))^2} = \frac{e^{\int Q(x)\,dx}}{(\psi(x)y + \phi(x))^2},\end{aligned} \tag{3.14}$$

so that P does not appear in the final result. For the cofactors defined by (3.6) and Proposition 3.1, we can show that if ψ, ψ' have a common exponential factor $e^{\mathcal{P}(x)}$, where $\mathcal{P}$ is a polynomial and this is removed before the determination of the cofactor is carried out, then the system has the cofactor (cf. (3.11)) $\lambda = Py + Q + \mathcal{P}'$. In the case of (3.6), there are other possibilities. For example, if the solutions ψ_1, ψ_2 are polynomials multiplied respectively by powers x^α, $x^{1-\alpha}$, where α is real. A sufficient condition for system (2.6) to have two algebraic invariant curves is that equation (2.2) should have two polynomial (rational) solutions. To accomplish this, let ψ_1, ψ_2 be two arbitrary polynomials having non–constant Wronskian. Set P equal to this and define Q, R using (3.3). In this case, (2.6) has a rational first integral.

For the purpose of finding algebraic invariant curves of (3.9), equation (3.13) is a better choice than (3.10) because it has the necessary condition $\psi' = P\phi$ implicit in it. Consequently, we need make no restrictions on P, Q, R beyond those necessary to obtain the desired result. Once these are determined, we can find the necessary solutions from (3.10). One simple choice is to take the coefficient of ϕ to be zero. Then, the equation is Liouvillian integrable, and has one constant solution $\phi = \phi_0$. Solving this implied differential equation, for Q gives

$$Q(x) = \left(\int P(x)\,dx\right)R(x),$$

where P, R are arbitrary polynomials. Then, $\psi' = \phi_0 P$ and the algebraic invariant curve of (3.9) is $y\int P(x)\,dx + 1 = 0$. Another choice is to take $P(x) = Q(x) = x$, $R(x) = 1$ which gives the reducible invariant curve $e^x((x-1)y+1) = 0$ for the resulting system. In view of the fact that these results are dependent upon the solutions of equations (3.10) and (3.13), we feel it is unlikely that all the conditions for algebraic invariant curves of (3.9) can ever be fully characterized. To give an idea of how difficult this might be, consider the transformation $y \to ((1+pq)y+q)/(py+1)$ where p, q are arbitrary polynomials, at least one of which is nonzero. This will transform the polynomial system (3.9) to a new system having polynomial coefficients. If the coefficient functions of the first system satisfy (3.8), then those of the new system do not unless both p and q are constant. If two transformations of this type, say T_1 and T_2, are applied consecutively to (3.9) and then in the opposite order, the results are not generally the same (unless $T_1 = T_2$). That is $T_2 \circ T_1 \neq T_1 \circ T_2$. Hence, any finite sequence of N transformations of this type is capable of producing $N!$ different results. If the original system has an algebraic invariant curve, all the transformed systems have one as well. To completely solve the problem of algebraic invariant curves for a polynomial system, it would be necessary to classify all Liouvillian solutions of either (3.10) or (3.13). The algorithm presented by Kovacic [13] for determining if a linear, second order differential equation has Liouvillian solutions is effective if the form of the equation is known, but it is not clear if it can be readily adapted to the case where the equation contains arbitrary functions. For now we can say that the relation (3.8) is sufficient for these to exist, but it is not necessary.

The solution of (2.2) or (3.10), assuming one can be found, is either Liouvillian or non-Liouvillian. For if ψ_1 is one solution, then a second independent solution for the case (2.2) (similarly for (3.10)) is

$$\psi_2(x) = \psi_1(x) \int \frac{e^{-\int A(x)\,dx}}{\psi_1^2(x)}\, dx,$$

which is of the same type as ψ_1. Also, it is clear that in order for the forms (3.2) to be algebraic, the solutions of (2.2) or (3.10) must be Liouvillian. The next result explains the comment made following equation (2.11).

Proposition 3.2. *If either of the systems* (2.6) *or* (3.9) *has a non-Liouvillian first integral or is not solvable, then the corresponding system does not have an algebraic invariant curve.*

In [10], Christopher and Llibre used a Riccati system to show that a system could have an algebraic invariant curve of arbitrary degree and which has no rational first integrals. The system is given by

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x),\\
\frac{dy}{dt} &= y^2 + \left[\left(a+b-1-\frac{2ab}{c}\right)x + 1 - c\right]y + \frac{ab}{c^2}(c-a)(c-b)x^2
\end{aligned}$$

where a is a non–positive integer, $b \le a$ is a constant and c is taken to be an irrational constant in [10], although this is not necessary for the general system. The basic form which describes this system can be written as $\dot{x} = 1$, $\dot{y} = (1/P)y^2 + (Q/P)y + R/P$. This is somewhat different than the previous two systems. However, we can still use the same ideas to obtain the invariant curves and integrating factor. The linear equation obtained from (2.5) with $\mathcal{G}(x) = 1/P(x) = 1/(x(1-x))$ has solutions $\psi_1(x) = (2x+1)(x-1)^2$, $\psi_2(x) = (x-1)^2[4x^2+2x-27-8(2x+1)\ln x]$ for the values $a = -1$, $b = -2$, $c = 1$. The Wronskian is $8(x-1)^7/x$ and the corresponding integrating factor (3.5) with $\mathcal{F}(x) = 1/(x(1-x))$ can be taken as

$$\mu(x,y) = \frac{(x-1)^4}{(2xy+y-6x^2)^2}.$$

This turns the form $\mu[((1/P)y^2 + (Q/P)y + R/P)\,dx - dy] = 0$ exact. Therefore, writing $(y^2+Qy+R)(\mu/P)\,dx - P(\mu/P)\,dy = 0$ gives the integrating factor for the system as given and this agrees with the form given in [10] for the assumed values of the parameters. In this case, we see the appearance of the algebraic invariant curves $x = 0$ and $x - 1 = 0$ in the integrating factor in spite of the fact that these types of curves were not specifically considered.

In the following and in Section 5 of the paper, we will obtain equations of the form

$$\frac{dy}{dx} = y^2 - \frac{Q(x)}{P(x)} y, \tag{3.15}$$

where P, Q are polynomials. These will arise by reducing integrable systems to the given Bernoulli form. Since it can be considered as a reducible Riccati equation, it has particular solutions of the form (3.2). One of the solutions is $y = 0$, and the second solution can be taken in the form $y = -F'/F$, where

$$F(x) = \int \exp\left(-\int \left(\frac{Q(x)}{P(x)}\,dx\right)\right) dx. \tag{3.16}$$

Clearly, this solution need not be such that the form of invariant curve (3.2) would be algebraic. For the homogeneous case we will consider in Section 5, P, and Q will be such that this solution will define an algebraic invariant curve which is obtained by reducing a corresponding curve from the original homogeneous system.

In [12], Lemma 3.2, the authors consider the Lotka–Volterra system

$$\begin{aligned}\frac{dx}{dt} &= x(ax + by + x^2 + 2xy + 4y^2),\\ \frac{dy}{dt} &= y(ax + by + x^2 + xy + 2y^2),\end{aligned}$$

where a, b are real parameters. The purpose is to demonstrate a system having a non–algebraic invariant curve with a polynomial cofactor. The form (1.4) for the system can be transformed to the Bernoulli form (3.15) by the change of variables $x = -(at + b)/((t + 2)v(t))$, $y(x) = -(at + b)/(t(t + 2)v(t))$. This equation has an invariant curve $Fv + F' = 0$, where $F(t) = (b - 2a)\mathrm{Ei}(t) - (at + 2b)e^{-t}/t$ is given by (3.16) and $\mathrm{Ei}(t) = -\int_{-t}^{\infty} e^{-u}/u\,du$ is the exponential integral. Transforming it back to the original variables, we find that the given system has two invariant curves $(2a - b)x\mathrm{Ei}(x/y)e^{x/y} + (x + 2y)^2 + ax + 2by = 0$ and $x + 2y = 0$ having cofactors $\lambda_1(x, y) = ax + by + 2x^2 + 4xy + 4y^2$ and $\lambda_2(x, y) = ax + by + x^2 + 2xy + 2y^2$ respectively. The results in [12] for this system are a little unusual since parameters are introduced which are not present in the original system. The expression $f = f_1 + f_2$ given in the lemma having the component functions

$$\begin{aligned}f_1(x, y) &= C_2 \left(\frac{y}{x}\right)^{\beta_1} \left(1 + \frac{2y}{x}\right)^{\beta_2+2} e^{\beta_3 x/y}\\ &\quad\times \left((b - 2a)xe^{x/y}\mathrm{Ei}\left(-\frac{x}{y}\right) + \frac{C_1}{C_2} xe^{x/y} + ax + 2by\right),\\ f_2(x, y) &= C_2 x^2 \left(\frac{y}{x}\right)^{\beta_1} \left(1 + \frac{2y}{x}\right)^{\beta_2+4} e^{\beta_3 x/y},\end{aligned}$$

where C_1 and $C_2 \neq 0$ are arbitrary constants and the β_k's are real parameters (which must satisfy certain conditions) is such that each function defines an invariant curve of the system. Of these f_2 has the polynomial cofactor $\bar{\lambda}_2(x, y) = 2ax + 2by + \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2$ where the α_k's are arbitrary real parameters. From the definition of f and the linearity of (1.7), it follows that the cofactors satisfy the relation $\bar{\lambda} f = \bar{\lambda}(f_1 + f_2) = \bar{\lambda}_1 f_1 + \bar{\lambda}_2 f_2$, where $\bar{\lambda}$, $\bar{\lambda}_1$ are the cofactors of f, f_1. That none of the cofactors are equal is easily verifiable. Hence, the relation can be written as either $\bar{\lambda}_2 - \bar{\lambda} = (\bar{\lambda} - \bar{\lambda}_1)(f_1/f_2)$ or $f_1/f_2 = -(\bar{\lambda} - \bar{\lambda}_2)/(\bar{\lambda} - \bar{\lambda}_1)$. Since f_1/f_2 is not a rational function, we have from the first one that if $\bar{\lambda}$ is a polynomial then $\bar{\lambda}_1$ cannot be and from the second if $\bar{\lambda}_1$ is a polynomial then $\bar{\lambda}$ cannot be. In fact, neither of them are polynomials. From what this writer can see, the only way to obtain a polynomial cofactor for f is to impose the conditions $b = 2a$, $C_1 = 0$ which would remove the exponential integral term from the invariant curve f_1. This reduces f_1/f_2 to a rational function, but even then $\bar{\lambda}_1$ is not a polynomial.

The system given in [12], Lemma 4.9, is of a similar nature. In this case, it is directly transformable to a Riccati equation which is solvable in terms of Bessel functions. The invariant curves given by (3.2) can then be directly converted to the invariant curves of the given system, and the first integral is the ratio of these due to the direct relation with the solution (3.1) of the Riccati equation. Although not of the same type as those discussed in the previous section, both this system and the previous one can be considered as inverse problems for the Riccati system. That is, systems having particular properties that are derivable from the basic Riccati form. Using the method described in [17], the homogeneous system $\dot{x} = -y + xP(x, y)$, $\dot{y} = x + yP(x, y)$ where P is a homogeneous polynomial of degree $n-1$ for $n \geq 2$, discussed in [12], Lemma 4.7, can be transformed to the form (3.15) and the invariant curves determined form that. The system has two irreducible invariant curves $x \pm iy = 0$ and another which is either algebraic or non–algebraic according as n is even, odd. The expression $f(x, y) = (x^2 + y^2)^{n/2} \exp(K \arctan(y/x)) = 0$, where K is an arbitrary constant mentioned in the statement of the lemma is an invariant curve of the system. It can be fully expressed in terms of powers of $x \pm iy$.

In the following, we establish the results of Proposition 3.1. First, assume that we can write $\psi' = P\phi$ where ϕ is a solution of (3.13). Setting $\mathcal{F} = P$ and removing the common factor of P, we take the expression for the invariant curve as $\psi y + \phi = 0$. With this we find that

$$\begin{aligned}\lambda(x, y) &= \frac{\psi'(x)y + \phi'(x) + (P(x)y^2 + Q(x)y + R(x))\psi(x)}{\psi(x)y + \phi(x)} \\ &= P(x)y + Q(x) + \frac{\psi'(x) - P(x)\phi(x)}{\psi(x)y + \phi(x)} y + \frac{\phi'(x) - Q(x)\phi(x) + R(x)\psi(x)}{\psi(x)y + \phi(x)},\end{aligned}$$

which gives the cofactor $\lambda = Py + Q$ when we set $\phi = \psi'/P$ and use the fact that ψ is a solution of (3.10). Now, suppose that the cofactor is $\lambda = Py + Q$. The basic form for an invariant curve is given by (3.2), so we take the more general form $\rho\psi y + \phi = 0$ where ρ and ϕ are functions to be determined in order to deal with possible modifications of the coefficient functions. Substituting for λ and the invariant curve in (1.7) and collecting powers of y, we find that the coefficient of the y^2 term is zero, and the vanishing of the other two is given by

$$\begin{aligned}&\rho(x)\frac{d\psi}{dx} + \psi(x)\frac{d\rho}{dx} - P(x)\phi(x) = 0, \\ &\frac{d\phi}{dx} - Q(x)\phi(x) + R(x)\rho(x)\psi(x) = 0.\end{aligned}$$

Solving this system of equations for ϕ and ρ, we find that ϕ is a solution of (3.13). Therefore, it satisfies $\phi = \psi'/P$. Substituting this in the second equation and using the fact that ψ is a solution of (3.10) reduces it to $\rho(x) = 1$. Hence, the invariant curve is $\psi y + \phi = 0$ as before. It is straightforward to show that this cofactor as well as the one for (2.6) is unique.

One question we might ask is: If a rational Riccati system is Liouvillian integrable, does it have an algebraic invariant curve? It seems that there should be a reasonable expectation that this is true. Any integrating factor for the system would have to depend on y, and also there would be one which is of Darboux type [23]. It would be necessary to establish that this particular integrating factor is dependent upon one or both of the invariant curves (3.2). If this is true, then at least one of them must be reducible to an algebraic form. Basic integrating factors are given by (3.5), but there are more general forms. For the system (3.9) the characteristic equations for equation (1.6) for an integrating factor can be written as

$$\frac{dx}{1} = \frac{dy}{P(x)y^2 + Q(x)y + R(x)} = -\frac{d\mu}{(2P(x)y + Q(x))\mu(x,y)}.$$

This system is fully and explicitly solvable. The solution for the Riccati equation (dy/dx) is given by (2.3) and this can be used to eliminate y from the equation for $d\mu/dx$. Since any particular solution of a Riccati equation leads to a solvable equation, the equation has no singular solutions and every solution must be given by (3.1). That is, it must be representable in terms of the invariant curves (3.2). The resulting equation for $d\mu/dx$ is a linear, first order equation which also has no singular solutions. Expressing Q, R in terms of ψ_1, ψ_2 in a manner similar to (3.3), we find the solution can be given as

$$\mu(x,y) = \frac{P(x)\mathcal{W}(\psi_1(x), \psi_2(x))}{(P(x)\psi_1(x)y + \psi_1'(x))^2}\, G\!\left(-\frac{P(x)\psi_2(x)y + \psi_2'(x)}{P(x)\psi_1(x)y + \psi_1'(x)}\right),$$

where $G = G(x)$ is an arbitrary and continuously differentiable function of one variable and $\mathcal{W}$ is the Wronskian. For this general form, any integrating factor must depend upon at least one of the invariant curves (3.2). Since all the examples given for (3.9) involve only a single algebraic invariant curve, we can take $G(x) = 1$ or $G(x) = 1/x^2$ in order to have an integrating factor with this form. For these choices they reduce to those given by (3.5). A (possibly partially) reduced form for an integrating factor for the system (3.9) is given by (3.14). In order that it be of Darboux type, the invariant curve must be reducible to an algebraic expression. The same type of result is true for the system (2.6).

Theorem 3.1. *The systems* (2.6) *and* (3.9) *have a Liouvillian first integral if and only if they have an algebraic invariant curve.*

This result is almost certainly true for any rational Riccati system, but to establish it would take us beyond the scope of the present work. Now, we formalize the results for the two Riccati systems considered in this section. While we have restricted the discussion to these systems, the concepts would seem extendable to more general systems such as those with rational coefficient functions. Furthermore, although the primary emphasis in this work is with real–valued systems, there would seem to be nothing in the preceding discussion which would indicate that the results would not be valid for complex systems as well.

Theorem 3.2. *Let* (3.9)

$$\frac{dx}{dt} = 1,$$
$$\frac{dy}{dt} = P(x)y^2 + Q(x)y + R(x)$$

and (2.6)

$$\frac{dx}{dt} = P(x),$$
$$\frac{dy}{dt} = P(x)y^2 + Q(x)y + R(x),$$

where P, Q, R *are polynomials with* $P(x) \neq 0$ *be Riccati systems and let* (3.10)

$$P(x)\frac{d^2y}{dx^2} - (P(x)Q(x) + P'(x))\frac{dy}{dx} + P^2(x)R(x)y = 0$$

and (2.2)

$$P(x)\frac{d^2y}{dx^2} - Q(x)\frac{dy}{dx} + R(x)y = 0$$

be the respective linear, second order differential equations obtained from the systems. Assume the latter two equations are solvable. If ψ_1, ψ_2 *are linearly independent solutions of these equations, then the following statements hold for the Riccati systems. For* (3.9) *set* $\mathcal{F}(x) = \mathcal{G}(x) = P(x)$ *and for* (2.6) *set* $\mathcal{F}(x) = \mathcal{G}(x) = 1$ *in the pertinent results.*

1. The systems have two invariant curves given by (3.2)*. A necessary and sufficient condition for one or both of these curves to be algebraic or be reducible to an algebraic form is that* ψ_1, ψ_2 *be Liouvillian.*

2. The unique cofactor for these curves is $\lambda(x,y) = P(x)y + Q(x)$*. They are specific to the form of the invariant curves* (3.2)*. In the case of* (3.9) *the solutions* ψ_1, ψ_2 *must be such that their derivatives have a factor of* $P(x)$*. In the case of* (2.6)*, no non–constant common factor should be removed from the form of* (3.2)*, and in the case of* (3.9)*, only the common factor* $P(x)$ *should be removed.*

3. The Riccati differential equations derived from the systems have integrating factors given by one of the forms (3.5)*.*

4. The systems have no polynomial first integrals.

4. Aspects of the Abel differential equation

The transformations considered in the following section rely heavily on certain properties of Abel differential equations, so we will briefly review some of the pertinent ones. An Abel equation of the first kind has the form

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \tag{4.1}$$

and an Abel equation of the second kind has the form

$$\frac{dy}{dx} = \frac{f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)}{g_1(x)y + g_0(x)} \tag{4.2}$$

where the coefficient functions are assumed to be suitably differentiable functions of x. We assume that the forms (4.1) and (4.2) are actual Abel equations and not some degenerate form (e.g., $f_3(x) = 0$ in (4.1)). The form (4.2) can always be transformed to an Abel equation of the first kind by the variable change

$$y(x) = \frac{1}{g_1(x)u(x)} - \frac{g_0(x)}{g_1(x)}. \tag{4.3}$$

For a general Abel equation of the first kind, it is possible to define recursively an infinite sequence of relative invariants by [6]

$$s_3(x) = f_0(x)f_3^2(x) + \frac{2}{27}f_2^3(x) + \frac{1}{3}\left(f_3(x)f_2'(x) - f_2(x)f_3'(x) - f_1(x)f_2(x)f_3(x)\right) \tag{4.4}$$

and

$$s_{2k+1}(x) = f_3(x)s_{2k-1}'(x) + (2k-1)\left(\frac{1}{3}f_2^2(x) - f_3'(x) - f_1(x)f_3(x)\right)s_{2k-1}(x) \tag{4.5}$$

for $k \geq 2$. The invariants of the second kind form (4.2) are defined as the invariants of the corresponding first kind form (4.1). From (4.4) and (4.5), a sequence of absolute invariants can be formed. The first of these is given by

$$I_1(x) = \frac{s_5^3(x)}{s_3^5(x)}. \tag{4.6}$$

Two Abel equations of the first kind are said to belong to the same *equivalence class* (see [6]), if there exists a transformation of the form

$$\begin{aligned} x &= F(t), \\ y(x) &= P_1(t)u(t) + Q_1(t) \end{aligned} \tag{4.7}$$

such that $F'(t)P_1(t) \neq 0$ which transforms one of the equations into the other. The existence of the function F is the necessary and sufficient condition (see [6] and references therein) for the transformation (4.7) to exist. Once it is known, the determination of P_1, Q_1 is straightforward. Hence, if an equation belonging to a particular class is solvable, then every equation in the class can be solved. Suppose two Abel equations (4.1) having independent variables x and t respectively belong to the same equivalence class. Let $I_1(x)$ and $J_1(t)$ be the first absolute invariants of the two equations. Then, it follows that

$$\tilde{I}_1(t) = I_1(x)|_{x=F(t)} = J_1(t).$$

Although the general process [6] for determining F is somewhat more involved (particularly when parameters are present), for our purposes, it is sufficient to obtain an expression for it simply by factoring the difference $I_1(x) - J_1(t)$. Observe that this technique can only be used with equations whose invariants are non–constant.

The general Abel equation of the second kind

$$\frac{dy}{dx} = \frac{(a_3x + b_3)y^3 + (a_2x + b_2)y^2 + (a_1x + b_1)y + a_0x + b_0}{(c_3x^3 + c_2x^2 + c_1x + c_0)y + d_3x^3 + d_2x^2 + d_1x + d_0} \tag{4.8}$$

is a sixteen parameter equation which has the property that if we interchange the roles of x and y we obtain a new Abel equation having a similar form, but which (usually) defines a different equivalence class. In the terminology of [7], it is of type AIA (Abel, Inverse–Abel). Obviously, if one of these equations is solvable, so is the other. However, while this is true, it is not necessarily so in CAS such as Maple, because both of these independent cases must be coded separately. The AIA class is a useful way of determining new, solvable classes of Abel equations from existing ones and in Section 6 we shall give an example of this. Currently, any solvable equivalence class of Abel equations is known to have a representative which has AIA form and this is believed to be true in general.

The structure inherent in the Abel equation is a useful tool for developing the transformations which will bring one form into another equivalent form. There is also a correspondence between certain Abel and Riccati equations that we can use to facilitate this process. The Abel equation of the second kind

$$\frac{dy}{dx} = \frac{a_3y^3 + a_2y^2 + a_1y + a_0}{(b_2x^2 + b_1x + b_0)y + c_2x^2 + c_1x + c_0} \tag{4.9}$$

where a_3, $a_2, \ldots, c_0$ are parameters is said to be of type AIR (see [7]) which stands for Abel–Inverse Riccati. It is so named because it becomes a Riccati differential equation when the roles of x and y are reversed. This form is responsible for the appearance of special functions in the solutions of certain Abel equations. In order for (2.4) to generate this type of equation, the degrees of the polynomials P, Q, R cannot exceed one. This clearly limits the types of functions that can satisfy both the Abel and Riccati forms. However, there are some possibilities. The Airy functions which are combinations of Bessel functions of orders $\pm 1/3$ are solutions of the equation $y'' - xy = 0$ and do satisfy this condition. Other possibilities would include the Hermite equation, the Laguerre equation, the confluent hypergeometric equation and certain Bessel type equations. AIR is a distinct subclass of AIA.

In [5], equation (21), Cheb–Terrab considers the AIR equation

$$\frac{d\eta}{d\xi} = \frac{1}{(\xi^2 + s_1\xi + (s_1^2 - 4)/4)\eta + 2\tau\xi - 2\sigma + s_1\tau} \tag{4.10}$$

where s_1, τ, σ are parameters as one which occurs in the solution of a Heun type equation. The parameter s_1 can be removed from (4.10) by means of a rather lengthy variable transformation (see [5], equation (28)). One of the author's interests is to separate out the Liouvillian solutions from the non–Liouvillian forms of the Heun functions obtained from the corresponding linear, second order differential equation. This is done by the calculation of symmetries and results in the general condition $\sigma^2 = \tau^2$ for the existence of a Liouvillian solution. In this case, we can do this much more directly, if we assume that condition (3.8) holds. Transforming (4.10) to a Riccati eqution, we obtain the polynomial form

$$\frac{dy}{dx} = xy^2 + (s_1x + 2\tau)y + \frac{1}{4}(s_1^2 - 4)x - 2\sigma + s_1\tau.$$

Applying (3.8) to the coefficient functions of this equation, we obtain the relations $\sigma = \pm\tau$, $s_1 = 2(A \pm 1)$ where corresponding signs are taken. For these the Riccati equation has the form

$$\frac{dy}{dx} = (y + A)((y + A)x \pm 2x + 2\tau)$$

and letting $y \to y - A$ in this produces the corresponding Bernoulli equation. This equation is independent of A (or, equivalently, s_1 in (4.10)) and the original basic condition $\sigma^2 = \tau^2$ is recovered. The Liouvillian forms of the other Heun equations discussed there can be obtained in a similar manner.

5. Reduction of a cubic system and a homogeneous system to Riccati form

In [19], the author gave the system

$$\begin{aligned}\frac{dx}{dt} &= -y - Ax^2 - xy - Ax^3,\\ \frac{dy}{dt} &= x + x^2 + (2A-1)xy - \frac{2}{3}y^2 + 2A(1-5A)x^3 + \frac{1}{3}(2A-1)x^2y\end{aligned}$$

from the work of Cherkas and Romanovski [8], where A is an arbitrary parameter. The form (1.4) for this system which is of Abel type does not seem to be generally integrable. However, for the specific values $A = 0,\ 1/4$ it is solvable in terms of Airy functions. For these values, the resulting equations are

$$\frac{dy}{dx} = -\frac{x + x^2 - xy - \frac{2}{3}y^2 - \frac{1}{3}x^2y}{(x+1)y} \tag{5.1}$$

and

$$\frac{dy}{dx} = -4\frac{x + x^2 - \frac{1}{2}xy - \frac{2}{3}y^2 - \frac{1}{8}x^3 - \frac{1}{6}x^2y}{(x+1)(4y+x^2)} \tag{5.2}$$

respectively. We will obtain the complete mapping of (5.1) to the Airy form of the Riccati equation (2.4). At the end of the section, we will present two, more general cubic systems of which (5.1) and (5.2) are particular members.

Setting $P(x) = 1, Q(x) = 0, R(x) = -x$ in (2.4) from the Airy differential equation $y'' - xy = 0$, we obtain the Riccati equation which can be written as

$$\frac{du}{dt} = u^2 - t. \tag{5.3}$$

This equation belongs to AIR (4.9), so its inverse is of Abel type. Converting it to an Abel equation and transforming that to a first kind form using (4.3) gives

$$\frac{du}{dt} = u^3 - 2tu^2 \tag{5.4}$$

where, for now, we have retained t and u as the variables. Also converting (5.1) to first kind form using (4.3), we have

$$\frac{dy}{dx} = 9x(x+1)^2y^3 - x(x+3)y^2 - \frac{5}{3}\frac{1}{x+1}y \tag{5.5}$$

where we continue to use x and y as the variables. Both of the equations (5.4) and (5.5) belong to the same equivalence class, so there exists a transformation (4.7) which will convert one of the equations into the other. In this case, we will have $Q_1(t) = 0$ since the function f_0 is absent in each equation and this fact will allow for a significant simplification in the calculations. As we observed, the form for F can be obtained by considering the first absolute invariants. Using (4.6), this is given by

$$I_1(t) = 11664\frac{t^6(8t^3+15)^3}{(8t^3+9)^5}. \qquad (5.6)$$

for (5.4) and for (5.5) by

$$J_1(x) = \frac{729}{4}\frac{(x+3)^6(x^3+9x^2+72x+72)^3}{(x^3+9x^2+54x+54)^5}.$$

Factoring the difference $I_1(t) - J_1(x)$, we obtain the expression

$$24(x+1)t^3 - (x+3)^3 = 0.$$

Solving this for t will provide the form of F necessary to convert (5.4) to (5.5) and solving the cubic in x will give the form to carry out the conversion in the opposite order. Clearly, a solution for t is much the simpler of the two choices. We obtain

$$t = F(x) = \frac{\sqrt[3]{9}}{6}\frac{x+3}{(x+1)^{1/3}}.$$

With this, we can now easily determine P_1 and complete the transformation of (5.4) to (5.5). We do not give the details of all the intermediate steps, but simply give the complete transformation of (5.3) to (5.1). It is the composition of 4 separate transformations, and has the form

$$t = \frac{\sqrt[3]{3}}{12}\frac{(x+3)^2}{(x+1)^{2/3}} - \frac{\sqrt[3]{3}}{3}\frac{1}{(x+1)^{2/3}}y(x),$$
$$u(t) = \frac{\sqrt[3]{9}}{6}\frac{x+3}{(x+1)^{1/3}}.$$

Converting (5.1) to (5.3), we can invert this to give

$$x = 2\sqrt[3]{3}\alpha u(t) - 3,$$
$$y(x) = \sqrt[3]{9}\alpha^2(u^2(t) - t),$$

where α is a root of the cubic equation $X^3 - 2\sqrt[3]{3}u(t)X + 2 = 0$. Both of these transformations can be easily extended to include the transformation to or from the Airy form of the system (2.6). In this case, we would have the rather unusual situation of the complete transformation of a homogeneous system of centre–focus of type to a cubic system of the same type.

Within the framework of Abel differential equations (5.1) and (5.2) represent the same equation, so there exists a transformation which will convert one of the equations to the other. Since they have the same first absolute invariant I_1, there is no need for a change in the independent variable, and we find that the simple form $y(x) = u(x) + x^2/4$ will convert (5.1) to (5.2). Now, we present the two systems mentioned earlier, both of which are solvable in terms of Airy functions. We do not give the development of these systems, although some of the ideas will be described in the next section. Equation (5.1) is a member of the first system with $a_1 = 1$, $b_1 = -1$ and (5.2) is a member of the second system with $a_2 = 1$, $b_2 = -1/2$.

Theorem 5.1. *Each of the following systems is solvable in terms of Airy functions and defines a centre at the critical point at the origin.*

$$\begin{aligned} \frac{dx}{dt} &= -y - a_1xy, \\ \frac{dy}{dt} &= x + a_1x^2 + b_1xy - \frac{2}{3}a_1y^2 + \frac{1}{3}a_1b_1x^2y \end{aligned}$$

or

$$\begin{aligned} \frac{dx}{dt} &= -y + \frac{1}{2}b_2x^2 - a_2xy + \frac{1}{2}a_2b_2x^3, \\ \frac{dy}{dt} &= x + a_2x^2 + b_2xy - \frac{2}{3}a_2y^2 - \frac{1}{2}b_2^2x^3 + \frac{1}{3}a_2b_2x^2y, \end{aligned}$$

where a_1, b_1, a_2, b_2 *are arbitrary, non–zero parameters.*

Since both systems produce Abel differential equations which can be reduced to the form (5.3) in a manner similar to the one that has just been described, they belong to the same equivalence class. Hence, as with the case of (5.1) and (5.2), there exists a transformation which will convert one of the equations to the other. Factoring the first absolute invariants as was done previously, we obtain the expression

$$\begin{aligned} &\left[4a_1^2a_2^2b_2^2(a_2t+9)t^2 + 27(4a_1^2b_2^2 - a_2^2b_1^2)(a_2t+1)\right](a_1x+1) \\ &- a_1^2a_2^2b_1^2x^2(a_1x+9)(a_2t+1) = 0. \end{aligned}$$

Solving this for x will give the form $x = F(t)$ that will allow conversion of the form (1.4) of the first system to that of the second system in which (x, y) is replaced by (t, u).

Next, we give a reduction to a Bernoulli form for an integrable homogeneous system (1.2). In [17], equation (25), the author gave the system

$$\begin{aligned} p(x,y) &= (n-2)x^{n-2}y^2 - x^n, \\ q(x,y) &= -n(n-2)x^{n-3}y^3 + (3n-2)x^{n-1}y \end{aligned} \tag{5.7}$$

having integrating factor

$$\mu(x,y) = (1 + 2(n-1)x^{n-2}y + 2(n-1)x^{2n-2})^{-(2n-1)/(n-1)} \tag{5.8}$$

where $n \geq 2$ is an integer. It is a particular case of the general systems given in [18]. For the purpose of simplifying the transformation equations we have interchanged the roles of x and y in the result given in [17]. The equation (1.4) for the system (5.7) is first converted to an Abel differential equation [17], and then transformed to its final form. Through this sequence of transformations, (5.7) is converted to the Bernoulli form

$$\frac{dy}{dx} = y^2 - \frac{(n-1)x - n + 3}{(x+2)(nx+5n-5)}y. \tag{5.9}$$

This equation has exactly the form of (3.15) with $P(x) = (x+2)(nx+5n-5)$ and $Q(x) = (n-1)x - n + 3$. From the integrating factor (5.8) of the original system (5.7), we see that the system has the algebraic invariant curve $1 + 2(n-1)x^{n-2}y + 2(n-1)x^{2n-2} = 0$. Transforming this in accordance with each of the transformations used to produce (5.9), we obtain, after simplification, the expression $(nx+5n-x-7)(nx+5n-5)y+(n-1)(x+2)=0$. This is exactly the algebraic invariant curve $Fy+F'=0$ of (3.15) where F is given by (3.16). Although we do not present them, there are other integrable homogeneous systems found in the literature that can be reduced in a similar fashion. The non–symmetric and non–integrable systems given in [16] can also be reduced to symmetric polynomial systems, although these are not of Riccati type.

At present, the most generally solvable class of Abel differential equations other than Bernoulli ($s_3(x)=0$) and constant invariant equations ($I_1'(x)=0$) is the so–called AIR three parameter case. In this, the Abel equation can be reduced to an AIR equation which depends upon at most three independent parameters. Hence, if the equation depends upon four or more parameters and that number cannot be reduced by redefining them or by variable transformations (cf. (4.10)), then it is generally not solvable. It may be possible to impose certain restrictions on the parameters which would reduce the equation to a simpler, integrable case. Since all homogeneous systems can be transformed to an Abel equation [17], this condition will limit the types of systems which would be integrable. For example, symmetric and homogeneous systems of degree greater than three are generally not solvable because they would depend upon $n+1$ parameters. Similarly, the systems described in [16] depend upon $[n/2+1]$ parameters where $[\dots]$ is the greatest integer function.

6. Partial proof of Theorem 5.1

In Theorem 5.1, we gave two cubic systems that are solvable in terms of Airy functions. For this to be true, there must exist transformations which will convert a cubic system of Abel type to an Airy type Abel equation such as (5.4). The most general Abel type cubic system of centre–focus type can be written as

$$\begin{aligned}\frac{dx}{dt} &= -y-a_0x^2-a_1xy-a_2x^3-a_3x^2y,\\ \frac{dy}{dt} &= x+b_0x^2+b_1xy+b_2y^2+b_3x^3+b_4x^2y+b_5xy^2+b_6y^3.\end{aligned} \tag{6.1}$$

To match this with (5.4), we must first convert the form (1.4) of (6.1) to a first kind form using (4.3). This results in an Abel equation (4.1) in which the coefficient functions are rational. In this, we take $b_6=0$, since this condition is required, and it also greatly simplifies the transformed equation. For this choice of b_6, the function f_0 in (4.1) is zero. Therefore, the lowest power of u corresponds to the function f_1 which is given by

$$f_1(x)=\frac{(b_5-2a_3)x+b_2-a_1}{a_3x^2+a_1x+1}. \tag{6.2}$$

The functions f_2, f_3 are much less simple so we don't give them. We need to show that we can transform (5.4) in such a manner that we can match the form of the coefficient functions f_1, f_2, f_3 with those of the transformed equation. From (5.6), we see that the first absolute invariant I_1 of (5.4) is a function of t^3. From this, we obtain the variable transformation $t=\xi^{1/3}$, $u(t)=3\xi^{1/3}\eta(\xi)$, which converts (5.4) to the equation

$$\frac{d\eta}{d\xi}=3\eta^3-2\eta^2-\frac{1}{3\xi}\eta=\frac{9\xi\eta^3-6\xi\eta^2-\eta}{3\xi}. \tag{6.3}$$

This equation is of interest for a couple of reasons. Further, transforming it by (4.7) with $\xi = F(x)$, $\eta(\xi) = P_1(x)u(x)$, $(Q_1(x) = 0)$, we obtain a form of the most general Abel equation of the first kind that can be obtained from (5.4). Carrying this out, we get

$$\frac{du}{dx} = 3F'(x)P_1^2(x)u^3 - 2F'(x)P_1(x)u^2 - \left(\frac{1}{3}\frac{F'(x)}{F(x)} + \frac{P_1'(x)}{P_1(x)}\right)u. \tag{6.4}$$

If it is possible to match this with the transformed form of the cubic system, there must exist functions F, P_1 for which this is true. Comparing (6.4) with the rational coefficient functions f_1, f_2, f_3 obtained by transforming (6.1), we can see that F, P_1, if they exist, must be rational. This explains one of the reasons why it is preferable to use (6.3) rather than (5.4). Since $(3F'P_1^2)/(-2F'P_1) = -(3P_1)/2 = f_3/f_2$, P_1 is rational, and for similar reasons, so is F'. Then, from the expression for f_1, we get that F is also rational. From (6.2) and (6.4), we have

$$\frac{F'(x)}{F(x)} + 3\frac{P_1'(x)}{P_1(x)} = 3\frac{(2a_3 - b_5)x + a_1 - b_2}{a_3x^2 + a_1x + 1},$$

which gives upon integration

$$\ln(F(x)P_1^3(x)) = 3\int \frac{(2a_3 - b_5)x + a_1 - b_2}{a_3x^2 + a_1x + 1}\,dx + \ln A,$$

where A is a constant. Since the integral must produce the logarithm of a rational function, this places some restrictions on the coefficients which appear. We are forced to take $a_3 = b_5 = 0$, which then means that $3(a_1 - b_2)/a_1$ must be an integer. The precise value for this and the value of A are determined later. The remainder of the matching process for f_2, f_3 is both lengthy and uninformative. Therefore, we do not give it. Completing this and redefining the parameters give the results stated in Theorem 5.1.

The second reason why equation (6.3) is useful is that it has AIA form (4.8). It is solved (in Maple), but not its inverse equation

$$\frac{dy}{dx} = \frac{3y}{3(3x-2)x^2y - x}.$$

The AIA nature of this equation is obvious, but if we obtained the equation in the first kind form (with minor changes to (4.3))

$$\frac{dy}{dx} = -3xy^3 + \frac{15x+8}{3x+2}y^2 - \frac{9x+4}{x(3x+2)}y,$$

and this relationship is not evident. The last equation might serve as a representative for this solvable class of Abel equations.

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