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Existence of Viscosity Solutions to a System of Hyperbolic Balance Laws

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Abstract In this paper, the existence of viscous solutions of a hyperbolic equilibrium law system derived from the nonlinear entropy moment closure of a dynamic equation is established. In addition, by using the natural entropy of the system, some higher order estimates of some viscosity solutions are obtained.

Keywords System of hyperbolic balance laws, Viscosity solutions, The invariant region.

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1. Introduction

In this paper, we consider viscosity solutions to the following system of hyperbolic balance laws:

$$\begin{cases} \rho_t + \partial_x J = 0, \\ J_t + \partial_x(\rho\psi(\frac{J}{\rho})) = -J, \\ \rho(x,0) = \rho_0(x), \ J(x,0) = 0, \end{cases} \tag{1.1}$$

where ρ is the density and $u = \frac{J}{\rho}$ is the the velocity. ψ is given by:

$$\begin{aligned} \psi: \ & (-1,1) \to (0,+\infty) \\ & u \mapsto u^2 + \mathbb{G}'(\mathbb{G}^{-1}(u)) = \frac{\mathbb{F}''}{\mathbb{F}}(\mathbb{G}^{-1}(\frac{J}{\rho})). \end{aligned} \tag{1.2}$$

Here, $\mathbb{F}(\beta) = \frac{\sinh\beta}{\beta}$, $\mathbb{G}(\beta) = \coth\beta - \frac{1}{\beta} = \frac{\mathbb{F}'(\beta)}{\mathbb{F}(\beta)}$ and $\mathbb{G}$ is C^∞ diffeomorphism from $\mathbb{R}$ onto $(-1,1)$. From the definition, we know that $\mathbb{F}$, ψ are even functions, while $\mathbb{G}$ is an odd function, and ψ is strictly convex with

$$\mathbb{F}(0) = 1, \ \mathbb{G}(0) = 0, \ \psi(0) = \mathbb{G}'(0) = \frac{1}{3}, \ \psi'(0) = 0, \tag{1.3}$$

$$\lim_{u\to\pm1} \psi(u) = 1, \ \lim_{u\to\pm1} \psi'(u) = \pm 2. \tag{1.4}$$

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Direct calculation implies that the eigenvalues satisfy:

$$\lambda_i(u) = \frac{\psi'(u) \pm \sqrt{[\psi'(u)]^2 - 4u\psi'(u) + 4\psi(u)}}{2}, \ i = 1, 2, \tag{1.5}$$

with $\lambda_1(u) < u < \lambda_2(u)$ and $\lambda_i'(u) > 0$, and the corresponding eigenvectors are given by

$$r_i(u) = \begin{pmatrix} 1 \\ \lambda_i(u) \end{pmatrix}. \tag{1.6}$$

System (1.1) is strictly hyperbolic and genuinely nonlinear, and all the properties mentioned above are given in [1]. Moreover, it is shown that the corresponding homogeneous Riemann problem can be solved without smallness assumption. The existence of global weak solutions with vacuum for the isothermal Euler equations was proved in [4]. Diperna [2] gave the global weak solutions to the isentropic gas dynamics system with the vanishing viscosity method. For the Broadwell model, Lu [5] gave the existence of the viscosity solutions.

Now, we give the structure of the paper as follows: In Section 2, we review the existence theorem of invariant region. In Section 3, we prove the existence of invariant region and obtain the lower bound of the density. In Section 4, we apply entropy-entropy flux pairs to establish higher order estimation of viscosity solutions.

2. Preliminaries

In this section, we review the definition of invariant region and the theorem that we will apply to prove existence of invariant regions. Consider the following system:

$$\begin{cases} \partial_t v = \epsilon D(v,x) v_{xx} + M(v,x) v_x + f(v,t), & (x,t) \in \Omega \times \mathbb{R}_+, \\ v(0,x) = v_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Here, $\epsilon > 0$, Ω is an open interval in $\mathbb{R}$, $D = D(v,x)$, and $M = M(v,x)$ are matrix-valued functions defined on an open subset $U \times V \subset \mathbb{R}^n \times \Omega$, $D \geq 0$. $v = (v_1, v_2 \ldots v_n)$, and f is a smoothing mapping from $U \times \mathbb{R}_+$ into $\mathbb{R}^n$.

Definition 2.1. [6] A closed subset $\sum \subset \mathbb{R}^n$ is called a (positively) invariant region for the local solution defined by (2.1), if any solution $v(x,t)$ with its boundary and initial values in $\sum$ satisfies $v(x,t) \in \sum$, for all $x \in \Omega$ and $t \in [0, \sigma)$.

We consider the region $\sum$ of the form

$$\Sigma = \bigcap \{v \in V : G_i(v) \leq 0\}, \tag{2.2}$$

where G_i are smooth real-valued functions defined on an open subset of U, and for each i, the DG_i never vanishes.

Theorem 2.1. *[6] Let Σ be defined in* (2.2)*, and suppose that for all $t > 0$ and $v_0 \in \partial\Sigma$ ($G_i(v_0) = 0$ for some i), the following conditions hold:*

(1) DG_i at v_0 is a left eigenvector of $D(v_0,x)$ and $M(v_0,x)$, for all $x \in \mathbb{R}$;

(2) If $DG_i D(v_0,x) = \lambda DG_i$ with $\lambda \neq 0$, then G_i is strongly convex at v_0;

i.e., If $DG_i \cdot \xi = 0$, *then*

$$G_{i\rho\rho}\xi_1^2 + 2G_{i\rho J}\xi_1\xi_2 + G_{iJJ}\xi_2^2 > 0. \tag{2.3}$$

(3) $DG_i f \leq 0$ *at* v_0.
Then, Σ *is invariant for* (2.2) *with every* $\epsilon > 0$.

3. Existence of viscosity solutions

The viscosity system corresponding to system (1.1) satisfies

$$\begin{cases} \rho_t^\epsilon + \partial_x J^\epsilon = \epsilon\rho_{xx}^\epsilon, \\ J_t^\epsilon + \partial_x(\rho^\epsilon\psi(\frac{J^\epsilon}{\rho^\epsilon})) = -J^\epsilon + \epsilon J_{xx}^\epsilon, \\ \rho^\epsilon(x,0) = \rho_0(x), \ J^\epsilon(x,0) = 0, \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that* $\rho_0(x) \leq M$ *and for the system* (3.1), *we define region:*

$$\Sigma = \{(\rho, J) \in (0, \rho_0) \times \mathbb{R} | \ Z_i(\rho, J) \geq Z_i(\rho_0, J_0), i = 1, 2\}, \tag{3.2}$$

which means that $G_i(\rho, J) = Z_i(\rho_0, J_0) - Z_i(\rho, J), \ i = 1, 2,$

$$Z_i = -\ln\rho + \Lambda_i(\frac{J}{\rho}), \tag{3.3}$$

is the Riemann invariant, where $\Lambda_i(u) = \int_0^u \frac{1}{\lambda_1(w) - w} dw$.
Then, $\sum$ *is the invariant region.*

Proof. Now, we check the conditions given in Theorem 2.1.

- **Step 1.** We can rewrite (3.1) as:

$$\begin{cases} U_t + F(U)_x = f + \epsilon U_{xx}, \\ U|_{t=0} = U_0, \end{cases} \tag{3.4}$$

where $U = \binom{\rho}{J}$, $\quad f = \binom{0}{-J}$.
Let mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$F(U) = \begin{pmatrix} J \\ \rho\psi(\frac{J}{\rho}) \end{pmatrix}.$$

Then,

$$DF(U) = \begin{pmatrix} 0 & 1 \\ \psi - \frac{J}{\rho}\psi' & \psi' \end{pmatrix},$$

i.e., $M(U, x) = -DF(U)$, $\quad D(U, x) = I$ in (2.1).
By the definition of Riemann invariant, we have

$$\nabla Z_1 \cdot r_1 = 0, \quad \nabla Z_2 \cdot r_2 = 0. \tag{3.5}$$

Hence,

$$\nabla Z_1 DF(U) = \lambda_2 \nabla Z_1, \quad \nabla Z_2 DF(U) = \lambda_1 \nabla Z_2. \tag{3.6}$$

- **Step 2.** A direct calculation leads to

$$\begin{aligned} Z_{i\rho} &= -\frac{1}{\rho} - \frac{J}{\rho^2}\Lambda_i'(\frac{J}{\rho}),\\ Z_{iJ} &= \frac{1}{\rho}\Lambda_i'(\frac{J}{\rho}),\\ Z_{i\rho\rho} &= \frac{1}{\rho^2} + \frac{J^2}{\rho^4}\Lambda_i''(\frac{J}{\rho}) + 2\frac{J}{\rho^3}\Lambda_i'(\frac{J}{\rho}),\\ Z_{i\rho J} &= -\frac{J}{\rho^3}\Lambda_i''(\frac{J}{\rho}) - \frac{1}{\rho^2}\Lambda_i'(\frac{J}{\rho}),\\ Z_{iJJ} &= \frac{1}{\rho^2}\Lambda_i''(\frac{J}{\rho}). \end{aligned}$$

If $\nabla Z_i \cdot \xi = 0$ with $\xi = (\xi_1, \xi_2)$, then

$$\nabla Z_i \cdot \xi = Z_{i\rho}\xi_1 + Z_{iJ}\xi_2 = \big(-\frac{1}{\rho} - \frac{J}{\rho^2}\Lambda_i'(\frac{J}{\rho})\big)\xi_1 + \big(\frac{1}{\rho}\Lambda_i'(\frac{J}{\rho})\big)\xi_2 = 0,$$

which implies that

$$\frac{\xi_1}{\xi_2} = \frac{\Lambda_i'(\frac{J}{\rho})}{1 + \frac{J}{\rho}\Lambda_i'(\frac{J}{\rho})} = \frac{1}{\lambda_i(\frac{J}{\rho})}.$$

Thus, we have $Z_{i\rho\rho}\xi_1^2 + 2Z_{i\rho J}\xi_1\xi_2 + Z_{iJJ}\xi_2^2 = \frac{-\lambda_i'(\frac{J}{\rho})}{\rho^2} < 0$. Then, G_i is strongly convex at v_0.

- **Step 3.** For every (ρ, J) satisfying $Z_i(\rho, J) = Z_i(\rho_0, 0)$, we have $\int_0^u \frac{1}{\lambda_i(w)-w}dw = \ln\frac{\rho}{\rho_0}$. As $\rho \in (0, \rho_0)$ and $\lambda_1(u) < u < \lambda_2(u)$, we have $\frac{u}{\lambda_i(u)-u} < 0$. That is to say,

$$DG_i f = DG_i \cdot (0, -J) = Z_{iJ} \cdot J = \frac{J}{\rho}\Lambda_i'(\frac{J}{\rho}) = \frac{u}{\lambda_i(u) - u} < 0.$$

Now, we prove that the region Σ defined in (3.2) is invariant region. Thus, we have the uniform L^∞ estimates for $(\rho^\epsilon, J^\epsilon)$, which yields the existence of the viscosity solutions. □

Along the curve $G_i = 0$ on the (ρ, J) plane, direct calculation we leads to

$$\begin{cases} \frac{dJ}{d\rho} = -\frac{G_{i\rho}}{G_{iJ}} = -\frac{Z_{i\rho}}{Z_{iJ}} = \lambda_i(u),\\ \frac{d^2J}{d\rho^2} = \lambda_i'(u)(\frac{1}{\rho}\frac{dJ}{d\rho} - \frac{J}{\rho^2}) = \lambda_i'(u)(\lambda_i(u) - u)\frac{1}{\rho}. \end{cases}$$

Then, the level curve $G_1 = 0$ is convex up, and the curve $G_2 = 0$ is convex down on the (ρ, J) plane. Therefore, we can deduce the Invariant region is behavior like Figure 1.

In order to obtain a positive and lower bound ρ^ϵ, the following lemma is useful.

Lemma 3.1. U*nder the same condition as in Theorem 3.1, the viscosity solutions of the Cauchy problem* (3.1) *have a priori* L^∞ *estimate* $|u^\epsilon| \le 1$.

Proof. In the invariant region $\sum$, for $G_i = 0$, we have

$$u^\epsilon = {\Lambda_i}^{-1}(\ln\frac{\rho^\epsilon}{\rho_0^\epsilon}),$$

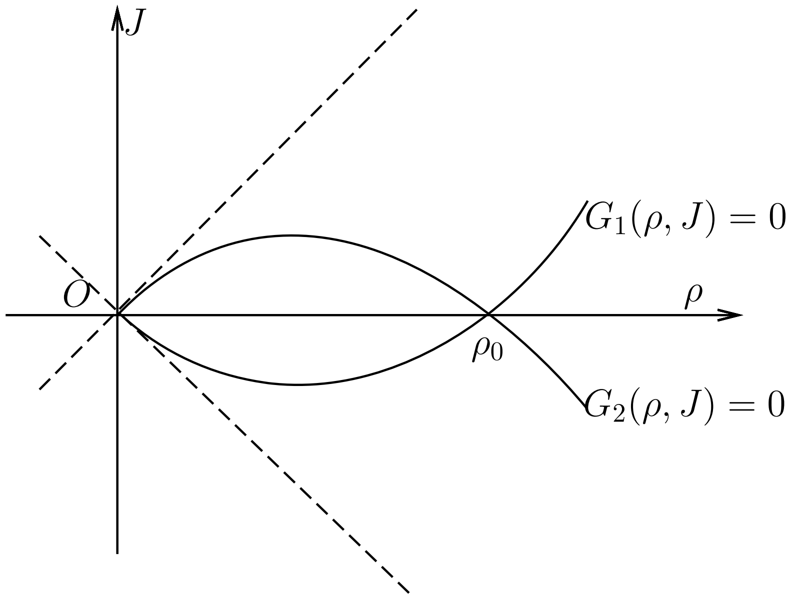

Figure 1. Invariant region

and

$$\Lambda_i'(u^\epsilon) = \frac{1}{\lambda_i(u^\epsilon) - u^\epsilon}.$$

It is obvious that Λ_1 is a decreasing function and Λ_2 is an increasing function. According to [1], one has

$$\Lambda_1(1) = \int_0^1 \frac{1}{\lambda_1(w) - w} dw = -\infty,$$

$$\Lambda_2(-1) = -\int_{-1}^0 \frac{1}{\lambda_2(w) - w} dw = -\infty.$$

Then, u^ϵ satisfies the following relationship in the invariant region:

$$-1 \le {\Lambda_2}^{-1}(\ln \frac{\rho^\epsilon}{\rho_0^\epsilon}) \le u^\epsilon \le {\Lambda_1}^{-1}(\ln \frac{\rho^\epsilon}{\rho_0^\epsilon}) \le 1.$$

□

Lemma 3.2. If *$\rho_0^\epsilon(x) > \delta$, then $\rho^\epsilon(x,t) \ge c(t,\epsilon,\delta) > 0$. Here, δ is a positive constant, and $c(t,\epsilon,\delta)$ tends to zero as time t goes to infinity or ϵ goes to zero.*

Proof. Consider the following system

$$\begin{cases} \rho_t^\epsilon + \partial_x J^\epsilon = \epsilon \rho_{xx}^\epsilon, \\ \rho^\epsilon(x,0) = \rho_0^\epsilon(x). \end{cases}$$

Let $\theta = \ln \rho^\epsilon$, then we can deduce

$$\begin{aligned}\theta_t &= \varepsilon\theta_x^2 + \epsilon\theta_{xx} - \theta_x u^\epsilon - u_x^\epsilon \\ &= \epsilon\theta_{xx} + \epsilon[\theta_x - \frac{u^\epsilon}{2\epsilon}]^2 - \frac{(u^\epsilon)^2}{4\epsilon} - u_x^\epsilon \end{aligned} \tag{3.7}$$

with the initial data $\theta(0,x) = \ln \rho_0^\epsilon$.

The solution θ of (3.7) can be represented by Green function $G(x-y,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\exp\{-\frac{x^2}{4\epsilon t}\}$:

$$\begin{aligned}\theta &= \int_{-\infty}^{\infty} G(x-y,t)\theta_0(y)dy + \int_0^t\int_{-\infty}^{\infty}\{\epsilon[\theta_y - \frac{u^\epsilon}{2\epsilon}]^2 - \frac{(u^\epsilon)^2}{4\epsilon} - u_y^\epsilon\}G(x-y,t-s)dyds \\ &\geq \int_{-\infty}^{\infty} G(x-y,t)\theta_0(y)dy + \int_0^t\int_{-\infty}^{\infty}[-\frac{(u^\epsilon)^2}{4\epsilon} - u_y^\epsilon](y,s)G(x-y,t-s)dyds \\ &= \int_{-\infty}^{\infty} G(x-y,t)\theta_0(y)dy + \int_0^t\int_{-\infty}^{\infty} u^\epsilon(y,s)G_y(x-y,t-s) \\ &\quad - \frac{(u^\epsilon)^2}{4\epsilon}(y,s)G(x-y,t-s)dyds \end{aligned} \tag{3.8}$$

owing to the fact that

$$\int_{-\infty}^{\infty} G(x-y,t)dy = 1, \qquad \int_0^t\int_{-\infty}^{\infty}|G_y(x-y,t-s)|dyds = \frac{2\sqrt{t}}{\sqrt{\epsilon\pi}} = \frac{2t^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}},$$

$$\theta_0(y) \geq \ln\delta, \qquad u^\epsilon \in [-1,1].$$

It follows from (3.8) that

$$\begin{aligned}\theta &\geq \ln\delta\int_{-\infty}^{\infty} G(x-y,t)dy - \int_0^t\int_{-\infty}^{\infty} G_y(x-y,t-s) - \frac{1}{4\epsilon}G(x-y,t-s)dyds \\ &= \ln\delta - \frac{2t^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}} - \int_0^t \frac{1}{4\epsilon}ds \\ &= \ln\delta - \frac{2t^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}} - \frac{t}{4\epsilon}. \end{aligned} \tag{3.9}$$

Then, we get

$$\rho^\epsilon \geq \delta e^{-\frac{2t^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}} - \frac{t}{4\epsilon}} = c(t,\epsilon,\delta) > 0.$$

For any $t > 0$, where δ is a positive constant and $c(t,\epsilon,\delta)$ tends to zero, as the time goes to infinity or ϵ goes to zero. □

4. Entropy Estimation

With the help of a nature entropy and the corresponding flux:

$$\eta(\rho,J) = \rho\ln\rho - \rho\ln[\mathbb{F}\circ\mathbb{G}^{-1}(u)] + J\mathbb{G}^{-1}(u), \tag{4.1}$$

$$q(\rho, J) = J \ln \rho - J \ln[\mathbb{F} \circ \mathbb{G}^{-1}(u)] + \rho\psi(\frac{J}{\rho})\mathbb{G}^{-1}(\frac{J}{\rho}), \tag{4.2}$$

we give some higher order estimate of the viscosity solution.

Lemma 4.1. *ρ^ϵ and J^ϵ are given in Section 3, and let η, q be defined by (4.1) and (4.2), then it holds that $\epsilon(\rho^\epsilon_x)^2, \epsilon(J^\epsilon_x)^2, \epsilon u_x^2, \epsilon[u^\epsilon\rho^\epsilon_x - J^\epsilon_x]^2$, and $J^\epsilon\mathbb{G}^{-1}(u^\epsilon)$ all belong to $L^1_{loc}(\mathbb{R} \times [0, T])$.*

Proof. Multiplying the first and second formula of (3.1) by η_ρ, η_J and adding them together, it yields

$$\begin{aligned}\eta(\rho^\epsilon, J^\epsilon)_t + q(\rho^\epsilon, J^\epsilon)_x &= \epsilon\eta(\rho^\epsilon, J^\epsilon)_{xx} - J^\epsilon\eta_{J^\epsilon} + \epsilon[\eta_{\rho^\epsilon}\rho^\epsilon_{xx} + \eta_{J^\epsilon}J^\epsilon_{xx} - \eta_{xx}]\\ &= \epsilon\eta(\rho^\epsilon, J^\epsilon)_{xx} - J^\epsilon\eta_{J^\epsilon}\\ &\quad - \epsilon[\eta_{\rho^\epsilon\rho^\epsilon}(\rho^\epsilon_x)^2 + 2\eta_{\rho^\epsilon J^\epsilon}\rho^\epsilon_x J^\epsilon_x - \eta_{J^\epsilon J^\epsilon}(J^\epsilon_x)^2]. \end{aligned} \tag{4.3}$$

For η, q, a direct calculation concludes

$$\begin{aligned}\eta_\rho &= \ln\rho + 1 - \ln[\mathbb{F} \circ \mathbb{G}^{-1}(u)] - \rho\frac{\mathbb{F}' \circ \mathbb{G}^{-1}(u)}{\mathbb{F} \circ \mathbb{G}^{-1}(u)}\frac{1}{\mathbb{G}'(\mathbb{G}^{-1}(u))}(-\frac{J}{\rho^2}) + \frac{J}{\mathbb{G}'(\frac{J}{\rho})}(-\frac{J}{\rho^2})\\ &= \ln\rho + 1 - \ln[\mathbb{F} \circ \mathbb{G}^{-1}(u)],\\ \eta_{\rho\rho} &= \frac{1}{\rho} + \frac{u^2}{\rho\mathbb{G}^{-1}(u)},\\ \eta_J &= -\rho\frac{\mathbb{F}' \circ \mathbb{G}^{-1}(u)}{\mathbb{F} \circ \mathbb{G}^{-1}(u)}\frac{1}{\mathbb{G}'(\mathbb{G}^{-1}(u))}(\frac{1}{\rho}) + J\frac{1}{\mathbb{G}'(u)}(\frac{1}{\rho}) + \mathbb{G}^{-1}(u) = \mathbb{G}^{-1}(u),\\ \eta_{JJ} &= \frac{1}{\rho\mathbb{G}^{-1}(u)},\\ \eta_{J\rho} &= -\frac{u}{\rho\mathbb{G}^{-1}(u)}.\end{aligned}$$

Then, one has

$$\begin{aligned}\eta(\rho^\epsilon, J^\epsilon)_t + q(\rho^\epsilon, J^\epsilon)_x &= \epsilon\eta(\rho^\epsilon, J^\epsilon)_{xx} - \epsilon\frac{1}{\rho^\epsilon\mathbb{G}' \circ \mathbb{G}^{-1}(u^\epsilon)}[\psi(\frac{J^\epsilon}{\rho^\epsilon})(\rho^\epsilon_x)^2 - 2u^\epsilon\rho^\epsilon_x J^\epsilon_x + (J^\epsilon_x)^2]\\ &\quad - J^\epsilon\mathbb{G}^{-1}(\frac{J^\epsilon}{\rho^\epsilon}).\end{aligned}$$

For any $\varphi \in C_c^\infty((0, T) \times \mathbb{R})$, we have

$$\begin{aligned}&\int_0^T\int_{-\infty}^{\infty}\{\frac{\epsilon}{\rho^\epsilon\mathbb{G}' \circ \mathbb{G}^{-1}(u^\epsilon)}[\psi(u^\epsilon)(\rho^\epsilon_x)^2 - 2u\epsilon\rho^\epsilon_x J^\epsilon_x + (J^\epsilon_x)^2] + J^\epsilon\mathbb{G}^{-1}(u^\epsilon)\}\varphi dxdt\\ =& \int_0^T\int_{-\infty}^{\infty}\{\frac{\epsilon}{\rho^\epsilon\mathbb{G}' \circ \mathbb{G}^{-1}(u^\epsilon)}[(u^\epsilon)^2(\rho^\epsilon_x)^2 + \mathbb{G}' \circ \mathbb{G}^{-1}(u^\epsilon)(\rho^\epsilon_x)^2 - 2u^\epsilon\rho^\epsilon_x J^\epsilon_x + J^\epsilon_x]^2 dxdt\\ &+ J^\epsilon\mathbb{G}^{-1}(u^\epsilon)\}\varphi dxdt\\ =& \int_0^T\int_{-\infty}^{\infty}\frac{\epsilon}{\rho^\epsilon\mathbb{G}' \circ \mathbb{G}^{-1}(u^\epsilon)}[u^\epsilon\rho^\epsilon_x - J^\epsilon_x]^2\varphi dxdt + \frac{\epsilon}{\rho^\epsilon}(\rho^\epsilon_x)^2\varphi + J^\epsilon\mathbb{G}^{-1}(u^\epsilon)\varphi dxdt\\ =& \int_0^T\int_{-\infty}^{\infty}[\epsilon\eta\varphi_{xx} + \eta\varphi_t + q\varphi_x]dxdt \le C,\end{aligned}$$

where the constant C depends on ϵ.

In fact, η, q are smooth functions without the possible singular point $\rho^\epsilon = 0$ and $u^\epsilon = \pm 1$. From Theorem 3.1 and Lemma 3.2, we know

$$0 < \delta e^{-\frac{2t^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}} - \frac{t}{4\epsilon}} \le \rho^\epsilon \le M$$

and

$$|J^\epsilon| \le \rho^\epsilon \le M.$$

Since $\ln(\frac{c(T,\epsilon,\delta)}{M}) \le \ln(\frac{\delta e^{-\frac{2t^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}} - \frac{t}{4\epsilon}}}{M}) \le \ln(\frac{\rho^\epsilon}{\rho_0^\epsilon}) \le \ln(\frac{M}{\delta})$, $t \in [0,T]$, we have

$$-1 < r_2(T,\epsilon,\delta) = {\Lambda_2}^{-1}(\ln \frac{c(T,\epsilon,\delta)}{M}) \le {\Lambda_2}^{-1}(\ln \frac{\rho^\epsilon}{\rho_0^\epsilon}) \le u^\epsilon \le {\Lambda_1}^{-1}(\ln \frac{\rho^\epsilon}{\rho_0^\epsilon}) dxdt$$
$$\le {\Lambda_1}^{-1}(\ln \frac{c(T,\epsilon,\delta)}{M}) = r_1 < 1.$$

It is obvious that $J^\epsilon \mathbb{G}^{-1}(u^\epsilon) \in L^1_{loc}(\mathbb{R} \times [0,T])$.

Since

$$\mathbb{G}'(\beta) \in (0, \frac{1}{3}],$$

we have

$$\frac{3\epsilon}{M} \int_0^T \int_{-\infty}^{\infty} [u^\epsilon \rho_x^\epsilon - J_x^\epsilon]^2 \varphi dxdt \le \int_0^T \int_{-\infty}^{\infty} \frac{\epsilon}{\rho^\epsilon \mathbb{G}' \circ \mathbb{G}^{-1}(u^\epsilon)} [u^\epsilon \rho_x^\epsilon - J_x^\epsilon]^2 \varphi dxdt \le C.$$

Hence, $\epsilon[u^\epsilon \rho_x^\epsilon - J_x^\epsilon]^2 \in L^1_{loc}(\mathbb{R} \times [0,T])$.

Notice that

$$\frac{1}{M} \int_0^T \int_{-\infty}^{\infty} \epsilon(\rho_x^\epsilon)^2 \varphi dxdt \le \int_0^T \int_{-\infty}^{\infty} \frac{\epsilon}{\rho^\epsilon} (\rho_x^\epsilon)^2 \varphi dxdt < C.$$

Then, we obtain $(\rho_x^\epsilon)^2 \in L^1_{loc}(\mathbb{R} \times [0,T])$ and

$$\frac{1}{2} \int_0^T \int_{-\infty}^{\infty} \epsilon(J_x^\epsilon)^2 \varphi dxdt = \frac{1}{2} \int_0^T \int_{-\infty}^{\infty} \epsilon[(J_x^\epsilon - u^\epsilon \rho_x^\epsilon) + u^\epsilon \rho_x^\epsilon]^2 \varphi dxdt$$
$$\le \epsilon \int_0^T \int_{-\infty}^{\infty} [(J_x^\epsilon - u^\epsilon \rho_x^\epsilon)^2 + (u \rho_x^\epsilon)^2] \varphi dxdt$$
$$\le \epsilon \int_0^T \int_{-\infty}^{\infty} [(J_x^\epsilon - u^\epsilon \rho_x^\epsilon)^2 + (\rho_x^\epsilon)^2] \varphi dxdt \le C.$$

Then, $\epsilon(J_x^\epsilon)^2 \in L^1_{loc}(\mathbb{R} \times [0,T])$.

Since

$$c(T,\epsilon,\delta) \int_0^T \int_{-\infty}^{\infty} \epsilon(u_x^\epsilon)^2 \varphi dxdt \le \int_0^T \int_{-\infty}^{\infty} \epsilon(\rho^\epsilon)^2 (u_x^\epsilon)^2 \varphi dxdt$$
$$= \int_0^T \int_{-\infty}^{\infty} \epsilon(J_x^\epsilon - \rho_x^\epsilon u^\varepsilon)^2 \varphi dxdt \le C,$$

we obtain $\epsilon(u_x^\epsilon)^2 \in L^1_{loc}(\mathbb{R} \times [0,T])$. □

Remark 4.1. For the system (1.1), a nature entropy (4.1) and the corresponding flux (4.2) can be given by the corresponding kinetic equation, but it is not enough to obtain the convergence of viscous solutions.

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