Journal of Nonlinear Modeling and Analysis
Volume 4, Number 2, June 2022, 207–219
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.207

# Existence and Uniqueness of Solutions and Lyapunov-type Inequality for a Mixed Fractional Boundary Value Problem*

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**Abstract** In this paper, a Lyapunov-type inequality for a linear differential equation involving right Riemann-Liouville and left Caputo fractional derivatives under Sturm-Liouville boundary conditions is established. Furthermore, the existence of solutions for the corresponding nonlinear differential equation is obtained by fixed point theorems.

**Keywords** Fractional derivative, Lyapunov-type inequality, Fixed point theorem.

**MSC(2010)** 26A33, 34A40, 47H10.

## 1. Introduction

Fractional calculus is the theory of differential and integral of any order, which is the extension of integer order calculus. Since it can well describe the real world phenomena, it has been gaining popularity among scientists working on different subject. According to the actual needs, mathematicians give a variety of definitions of fractional derivatives and integrals [13]. The most commonly used fractional calculus operators are perhaps Riemann-Liouville and Caputo fractional integrals and derivatives.

Recently, Lyapunov-type inequalities for fractional differential equations have been widely used in various problems, including oscillation, disconjugacy and eigenvalue problems. This work was first done by Ferreira [4], who obtained a Lyapunov-type inequality for the following differential equations with Riemann-Liouville fractional derivative:

$$\begin{cases} D_{a+}^{\alpha}u(t)+q(t)u(t)=0, a<t<b, 1<\alpha<2, \\ u(a)=u(b)=0. \end{cases}$$

In 2018, Ntouyas et al., [16] summarized the development of Lyapunov inequalities in fractional differential equations. Moreover, many authors have obtained Lyapunov-type inequalities for mixed fractional differential equations [5,6,8,11,12].

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*The authors were supported by National Natural Science Foundation of China (No. 11971145).

For example, Khaldi [12] considered the equation:

$$\begin{cases} -{}^C D_{b-}^{\alpha} D_{a+}^{\beta} u(t) + q(t)u(t) = 0, a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

where $0 < \beta \leq \alpha \leq 1$, $1 < \alpha + \beta \leq 2$.

At the same time, more and more attention has been paid to the existence of solutions for fractional boundary value problems [1–3,9,10,14,15,17,20]. In particular, fixed theorems have been extensively used to study the solutions of equations. For example, in [7], the authors investigated the existence of nonnegative solutions of fractional Liouville equation by using Krasnoselskill's fixed point theorem. In 2011, Samet et al., [19] introduced a new concept of $\alpha - \psi-$contractive type mappings and established fixed point theorems for such mappings in complete metric spaces. Motivated by [3, 12], in this paper, we consider the Lyapunov-type inequality for the following fractional differential equation:

$$\begin{cases} {}^C D_{a+}^{\beta} D_{b-}^{\alpha} u(t) + q(t)u(t) = 0, a < t < b, \\ u(b) = 0, pu(a) = \gamma D_{b-}^{\alpha} u(a), \end{cases} \tag{1.1}$$

where $0 < \alpha \leq \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $p\gamma \leq 0$ and $p \neq 0$, ${}^C D_{a+}^{\beta}$ denotes the left Caputo derivative, $D_{b-}^{\alpha}$ denotes the right Riemann-Liouville derivative, $u$ is the unknown function and $q \in C([a, b], \mathbb{R})$.

Furthermore, we obtain the existence of solutions for the corresponding nonlinear problem:

$$\begin{cases} {}^C D_{a+}^{\beta} D_{b-}^{\alpha} u(t) + f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) = 0, a < t < b, \\ u(b) = 0, pu(a) = \gamma D_{b-}^{\alpha} u(a), \end{cases} \tag{1.2}$$

where $\lambda, \tau, \delta > 0$ and $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$.

This paper is organized as follows: In section 2, we introduce some basic concepts. In Section 3, we prove Lyapunov-type inequality and the existence of solutions. Finally, we give two examples to illustrate the theoretical results.

## 2. Preliminaries

In this section, we recall the basic concepts related to our work.

**Definition 2.1** ( [13,18])**.** The left and right Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined respectively by

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},$$

$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}},$$

where $\Gamma$ is the gamma function.

**Definition 2.2** ( [13,18])**.** The left and right Riemann-Liouville fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined respectively by

$$(D_{a+}^{\alpha} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I_{a+}^{n-\alpha} f)(x),$$

$$(D_{b-}^{\alpha}f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I_{b-}^{n-\alpha}f)(x),$$

where $n-1<\alpha<n$.

**Definition 2.3** ( [13,18]). The left and right Caputo fractional derivatives ${}^CD_{a+}^{\alpha}f$ and ${}^CD_{b-}^{\alpha}f$ of order $\alpha>0$ are defined respectively by

$$({}^CD_{a+}^{\alpha}f)(x) = (I_{a+}^{n-\alpha}D^nf)(x),$$
$$({}^CD_{b-}^{\alpha}f)(x) = (-1)^n(I_{b-}^{n-\alpha}D^nf)(x),$$

where $n-1<\alpha<n$.

**Lemma 2.1** ( [13,18]). *Let* $0<\alpha<1$, *then*

$$I_{a+}^{\alpha}({}^CD_{a+}^{\alpha}f)(x) = f(x)-f(a),$$
$$I_{b-}^{\alpha}({}^CD_{b-}^{\alpha}f)(x) = f(x)-f(b),$$
$$({}^CD_{a+}^{\alpha}f)(x) = D_{a+}^{\alpha}(f(x)-f(a)),$$
$$({}^CD_{b-}^{\alpha}f)(x) = D_{b-}^{\alpha}(f(x)-f(b)).$$

Denote with $\Psi$ the family of nondecreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t>0$, where $\psi^n$ is the $n-$th iterate of $\psi$.

**Definition 2.4** ( [19]). Let $(X,d)$ be a metric space and $T:X\to X$ be a given mapping. We say that $T$ is an $\alpha-\psi-$contraction, if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that $\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y))$ for all $x,y\in X$.

**Definition 2.5** ( [19]). Let $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$. $T$ is called $\alpha-$admissible whenever $\alpha(x,y)\ge1$ implies that $\alpha(Tx,Ty)\ge1$.

**Lemma 2.2** (Theorem 2.2, [19]). *Let $(X,d)$ be a complete metric space and $T:X\to X$ be an $\alpha-\psi-$contractive mapping satisfying the following conditions:*
*(A1) $T$ is $\alpha-$admissible;*
*(A2) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$;*
*(A3) If $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n,x_{n+1})\ge1$ for all $n$ and $x_n\to x\in X$ as $n\to\infty$, then $\alpha(x_n,x)\ge1$ for all $n$.*
*Then, $T$ has a fixed point.*

## 3. Main results

In this section, we first transform the problem (1.1) to an equivalent integral equation.

**Lemma 3.1.** *Assume that $0<\alpha,\beta\le1$. The function $u$ is a solution to the boundary value problem (1.1), if and only if $u$ satisfies the integral equation*

$$u(t) = \int_a^b G(t,r)q(r)u(r)dr, \tag{3.1}$$

*where the Green's function of problem* (3.1) *is given by*

$$G(t,r)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\begin{cases}\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds\\ -\int_t^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds, & a\le r\le t\le b,\\ \frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds\\ -\int_r^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds, & a\le t\le r\le b,\end{cases}\tag{3.2}$$

*with* $\eta=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}-\frac{\gamma}{p}$.

**Proof.** Firstly, we apply the left side fractional integral $I_{a+}^{\beta}$ to equation (1.1), then the right side fractional integral $I_{b-}^{\alpha}$ to the resulting equation and taking into account the properties of Caputo and Riemann-Liouville fractional derivatives and the fact that $u(b)=0$, we get

$$u(t)=\frac{D_{b-}^{\alpha}u(a)}{\Gamma(1+\alpha)}(b-t)^{\alpha}-I_{b-}^{\alpha}I_{a+}^{\beta}q(t)u(t).\tag{3.3}$$

The boundary condition $pu(a)=\gamma D_{b-}^{\alpha}u(a)$ imply that

$$\frac{\gamma}{p}D_{b-}^{\alpha}u(a)=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}D_{b-}^{\alpha}u(a)-I_{b-}^{\alpha}I_{a+}^{\beta}q(t)u(t)|_{t=a}.$$

Thus,

$$D_{b-}^{\alpha}u(a)=\frac{1}{\eta}I_{b-}^{\alpha}I_{a+}^{\beta}q(t)u(t)|_{t=a},$$

with $\eta=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}-\frac{\gamma}{p}$.
Substituting $D_{b-}^{\alpha}u(a)$ into (3.3), it yields

$$\begin{aligned}u(t)=&\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}I_{b-}^{\alpha}I_{a+}^{\beta}q(t)u(t)|_{t=a}-I_{b-}^{\alpha}I_{a+}^{\beta}q(t)u(t)\\ =&\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)\Gamma(\alpha)\Gamma(\beta)}\int_a^b(s-a)^{\alpha-1}\left(\int_a^s(s-r)^{\beta-1}q(r)u(r)dr\right)ds\\ &-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^b(s-t)^{\alpha-1}\left(\int_a^s(s-r)^{\beta-1}q(r)u(r)dr\right)ds.\end{aligned}$$

Exchanging the order of integration, we get

$$\begin{aligned}u(t)=&\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)\Gamma(\alpha)\Gamma(\beta)}\int_a^b\left(\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds\right)q(r)u(r)dr\\ &-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\int_t^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds\right)q(r)u(r)dr\\ &-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^b\left(\int_r^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds\right)q(r)u(r)dr.\end{aligned}$$

Thus,

$$u(t)=\int_a^b G(t,r)q(r)u(r)dr,$$

where

$$G(t,r)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\begin{cases}\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds\\ -\int_t^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds, & a\le r\le t\le b,\\ \frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds\\ -\int_r^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds, & a\le t\le r\le b.\end{cases}$$

Conversely, we can verify that if $u$ satisfies the integral equation (3.1). Then, $u$ is a solution to the boundary value problem (1.1). The proof is completed. □

Next, we introduce the properties of Green's function.

**Lemma 3.2.** *Assume that* $0<\alpha\le\beta\le 1$ *and* $1<\alpha+\beta\le 2$, *then the Green function* $G(t,r)$ *given in* (3.2) *satisfies the following property:*

$$|G(t,r)|\le\begin{cases}\frac{(\beta\eta\Gamma(1+\alpha))^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}(b-a)^{(\alpha-1)(\alpha+\beta-1)}}, & (b-a)^{\alpha}>-\frac{\gamma}{p}\beta\Gamma(\alpha),\\ \frac{(b-a)^{\alpha+\beta-1}(\beta\eta\Gamma(1+\alpha)-(\alpha+\beta-1)(b-a)^{\alpha})}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)(\alpha+\beta-1)\eta}, & (b-a)^{\alpha}\le-\frac{\gamma}{p}\beta\Gamma(\alpha),\end{cases}$$

*for* $t,r\in[a,b]\times[a,b]$.

**Proof.** Let us define two functions:

$$g_1(t,r)=\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds-\int_t^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds,$$
$$a\le r\le t\le b,$$
$$g_2(t,r)=\frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds-\int_r^b(s-t)^{\alpha-1}(s-r)^{\beta-1}ds,$$
$$a\le t\le r\le b.$$

We start with the function $g_2(t,r)$, which is easier to treat. We have $g_2(t,r)\le 0$. In fact, let $r\in[a,b]$ be fixed. Differentiating $g_2(t,r)$ with respect to $t$, we obtain

$$\frac{\partial g_2(t,r)}{\partial t}=\frac{-\alpha(b-t)^{\alpha-1}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds$$
$$+(\alpha-1)\int_r^b(s-t)^{\alpha-2}(s-r)^{\beta-1}ds\le 0,$$

which means

$$g_2(r,r)\le g_2(t,r)\le g_2(a,r),\quad a\le t\le r.$$

Moreover, by evaluating $g_2(a,r)$, we get

$$g_2(a,r)=\frac{(b-a)^{\alpha}}{\eta\Gamma(1+\alpha)}\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds-\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds$$
$$=\left(\frac{(b-a)^{\alpha}}{\eta\Gamma(1+\alpha)}-1\right)\int_r^b(s-a)^{\alpha-1}(s-r)^{\beta-1}ds\le 0.$$

Thus, it yields

$$g_2(t,r) \le 0.$$

In addition,

$$\begin{aligned}
-g_2(r,r) &= \int_r^b (s-r)^{\alpha-1}(s-r)^{\beta-1}ds - \frac{(b-r)^\alpha}{\eta\Gamma(1+\alpha)}\int_r^b (s-a)^{\alpha-1}(s-r)^{\beta-1}ds \\
&\le \frac{(b-r)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-r)^\alpha}{\eta\Gamma(1+\alpha)}\int_r^b (b-a)^{\alpha-1}(s-r)^{\beta-1}ds \\
&= \frac{(b-r)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-r)^\alpha(b-a)^{\alpha-1}}{\eta\Gamma(1+\alpha)}\frac{(b-r)^\beta}{\beta} \\
&= \frac{(b-r)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-a)^{\alpha-1}(b-r)^{\alpha+\beta}}{\beta\eta\Gamma(1+\alpha)}.
\end{aligned}$$

Thus, we get

$$0 \le -g_2(t,r) \le h(r),$$

where

$$h(r) = \frac{(b-r)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-a)^{\alpha-1}(b-r)^{\alpha+\beta}}{\beta\eta\Gamma(1+\alpha)}.$$

Now, we turn our attention to the function $g_1(t,r)$. Considering $(b-r)^{\beta-1} \ge \frac{(b-t)^\beta(b-a)^{\alpha-1}}{\eta\Gamma(1+\alpha)}$ and $\frac{(b-r)^\alpha}{\eta\Gamma(1+\alpha)} \le 1$, we have

$$\begin{aligned}
-g_1(t,r) &= \int_t^b (s-t)^{\alpha-1}(s-r)^{\beta-1}ds - \frac{(b-t)^\alpha}{\eta\Gamma(1+\alpha)}\int_r^b (s-a)^{\alpha-1}(s-r)^{\beta-1}ds \\
&\le \int_t^b (s-t)^{\alpha-1}(s-t)^{\beta-1}ds - \frac{(b-t)^\alpha}{\eta\Gamma(1+\alpha)}\int_r^b (b-a)^{\alpha-1}(s-r)^{\beta-1}ds \\
&= \frac{(b-t)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-a)^{\alpha-1}(b-t)^\alpha}{\eta\Gamma(1+\alpha)}\frac{(b-r)^\beta}{\beta} \\
&\le \frac{(b-t)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-a)^{\alpha-1}(b-t)^\alpha}{\eta\Gamma(1+\alpha)}\frac{(b-t)^\beta}{\beta} \\
&= \frac{(b-t)^{\alpha+\beta-1}}{\alpha+\beta-1} - \frac{(b-a)^{\alpha-1}(b-t)^{\alpha+\beta}}{\beta\eta\Gamma(1+\alpha)} = h(t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
-g_1(t,r) &\ge \int_t^b (s-t)^{\alpha-1}(b-r)^{\beta-1}ds - \frac{(b-t)^\alpha}{\eta\Gamma(1+\alpha)}\int_r^b (s-r)^{\alpha-1}(s-r)^{\beta-1}ds \\
&= (b-r)^{\beta-1}\frac{(b-t)^\alpha}{\alpha} - \frac{(b-t)^\alpha}{\eta\Gamma(1+\alpha)}\frac{(b-r)^{\alpha+\beta-1}}{\alpha+\beta-1} \\
&\ge \frac{(b-t)^\beta(b-a)^{\alpha-1}}{\eta\Gamma(1+\alpha)}\frac{(b-t)^\alpha}{\alpha} - \frac{(b-t)^\alpha}{\alpha+\beta-1}\frac{(b-r)^{\alpha+\beta-1}}{\eta\Gamma(1+\alpha)} \\
&\ge \frac{(b-a)^{\alpha-1}(b-t)^{\alpha+\beta}}{\alpha\eta\Gamma(1+\alpha)} - \frac{(b-t)^\alpha}{\alpha+\beta-1}(b-r)^{\beta-1}
\end{aligned}$$

$$\geq \frac{(b-a)^{\alpha-1}(b-t)^{\alpha+\beta}}{\alpha\eta\Gamma(1+\alpha)} - \frac{(b-t)^{\alpha+\beta-1}}{\alpha+\beta-1}.$$

Since $\alpha \leq \beta$, we get

$$-g_1(t,r) \geq -h(t).$$

Therefore, we have

$$|-g_1(t,r)| \leq h(t).$$

Finally, by differentiating the function $h$, it yields

$$h'(r) = -(b-r)^{\alpha+\beta-2} + \frac{(\alpha+\beta)(b-a)^{\alpha-1}(b-r)^{\alpha+\beta-1}}{\beta\eta\Gamma(1+\alpha)}.$$

Moreover,

$$h''(r) = (\alpha+\beta-2)(b-r)^{\alpha+\beta-3} - \frac{(\alpha+\beta)(\alpha+\beta-1)(b-a)^{\alpha-1}(b-r)^{\alpha+\beta-2}}{\beta\eta\Gamma(1+\alpha)} \leq 0.$$

That is, $h(r)$ is concave. We can see that $h'(r) = 0$ for $r^* = b - \frac{\beta\eta\Gamma(1+\alpha)}{(\alpha+\beta)(b-a)^{\alpha-1}}$. If $(b-a)^\alpha > -\frac{\gamma}{p}\beta\Gamma(\alpha)$, then $r^* \in (a,b)$. Hence, the function $h(r)$ has a unique maximum given by

$$\max_{r\in[a,b]} h(r) = h(r^*) = \frac{(\beta\eta\Gamma(1+\alpha))^{\alpha+\beta-1}}{(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}(b-a)^{(\alpha-1)(\alpha+\beta-1)}}.$$

If $(b-a)^\alpha \leq -\frac{\gamma}{p}\beta\Gamma(\alpha)$, then

$$\max_{r\in[a,b]} h(r) = h(a) = \frac{(b-a)^{\alpha+\beta-1}(\beta\eta\Gamma(1+\alpha) - (\alpha+\beta-1)(b-a)^\alpha)}{(\alpha+\beta-1)\beta\eta\Gamma(1+\alpha)}.$$

Thus, we obtain $|G(t,r)| \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \max_{r\in[a,b]} h(r)$, which finishes the proof. □

Now, we are ready to prove the Lyapunov-type inequality for problem (1.1).

**Theorem 3.1.** *Assume that* $0 < \alpha \leq \beta \leq 1$ *and* $1 < \alpha+\beta \leq 2$. *If the fractional boundary value problem* (1.1) *has a nontrivial continuous solution, then*

$$\int_a^b |q(r)|dr \geq \begin{cases} \frac{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}(b-a)^{(\alpha-1)(\alpha+\beta-1)}}{(\beta\eta\Gamma(1+\alpha))^{\alpha+\beta-1}}, & (b-a)^\alpha > -\frac{\gamma}{p}\beta\Gamma(\alpha), \\ \frac{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)(\alpha+\beta-1)\eta}{(b-a)^{\alpha+\beta-1}(\beta\eta\Gamma(1+\alpha)-(\alpha+\beta-1)(b-a)^\alpha)}, & (b-a)^\alpha \leq -\frac{\gamma}{p}\beta\Gamma(\alpha). \end{cases} \tag{3.4}$$

**Proof.** From Lemma 3.1 and 3.2, if $(b-a)^\alpha > -\frac{\gamma}{p}\beta\Gamma(\alpha)$, we have

$$\begin{aligned} |u(t)| &\leq \int_a^b |G(t,r)||q(r)||u(r)|dr \\ &\leq \|u\| \int_a^b |G(t,r)||q(r)|dr \end{aligned}$$

$$\leq \frac{(\beta\eta\Gamma(1+\alpha))^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}(b-a)^{(\alpha-1)(\alpha+\beta-1)}}\|u\|\int_a^b|q(r)|dr,$$

where $\|u\|=\sup\limits_{t\in[a,b]}u(t)$. Consequently,

$$\|u\|\leq \frac{(\beta\eta\Gamma(1+\alpha))^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}(b-a)^{(\alpha-1)(\alpha+\beta-1)}}\|u\|\int_a^b|q(r)|dr.$$

For the case $(b-a)^\alpha\leq-\frac{\gamma}{p}\beta\Gamma(\alpha)$, we have a similar result. Thus, inequality (3.4) follows. □

**Remark 3.1.** If $(b-a)^\alpha>-\frac{\gamma}{p}\beta\Gamma(\alpha)$ and $\gamma=0$ in Theorem 3.1, we get the corresponding results in [12].

Next, we consider the existence of solutions to problem (1.2).

**Theorem 3.2.** *Let $f:[a,b]\times\mathbb{R}^3\to\mathbb{R}$ be a function, and there exists a function $\xi:\mathbb{R}^2\to\mathbb{R}$. They satisfy the following conditions:*

*$(H1)$ There exists $\psi\in\Psi$ such that*

$$|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)|\leq\frac{1}{\Omega}\psi(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|)$$

*for all $t\in[a,b]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2,3$, where*

$$\frac{(b-a)^{\alpha+\beta}((b-a)^\alpha+(\alpha+\beta)\eta\Gamma(\alpha))}{(\alpha+\beta)\eta\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)}<\Omega.$$

*$(H2)$ There exists $u_0\in C([a,b],\mathbb{R})$ such that $\xi(u_0(t),Au_0(t))\geq0$ for all $t\in[a,b]$, where the operator $A:C([a,b],\mathbb{R})\to C([a,b],\mathbb{R})$ is defined by*

$$\begin{aligned}Au(t)=&\frac{(b-t)^\alpha}{\eta\Gamma(1+\alpha)\Gamma(\alpha)\Gamma(\beta)}\\&\times\int_a^b(s-a)^{\alpha-1}\left(\int_a^s(s-r)^{\beta-1}f(r,\lambda u(r),I_{a+}^\tau u(r),I_{b-}^\delta u(r))dr\right)ds\\&-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^b(s-t)^{\alpha-1}\left(\int_a^s(s-r)^{\beta-1}f(r,\lambda u(r),I_{a+}^\tau u(r),I_{b-}^\delta u(r))dr\right)ds.\end{aligned}$$

*$(H3)$ Assume that for each $t\in[a,b]$ and $u,v\in C([a,b],\mathbb{R})$, $\xi(u(t),v(t))\geq0$ implies $\xi(Au(t),Av(t))\geq0$.*

*$(H4)$ For each $t\in[a,b]$, if $\{u_n\}$ is a sequence in $C([a,b],\mathbb{R})$ such that $u_n\to u$ in $C([a,b],\mathbb{R})$ and $\xi(u_n(t),u_{n+1}(t))\geq0$ for all $n\in\mathbb{N}$. Then, $\xi(u_n(t),u(t))\geq0$.*

*Then, problem (1.2) has at least one solution whenever $\lambda+\frac{(b-a)^\tau}{\Gamma(1+\tau)}+\frac{(b-a)^\delta}{\Gamma(1+\delta)}<1$.*

**Proof.** Let $E$ be the Banach space $C([a,b],\mathbb{R})$ with the metric $d(u,v)=\sup\limits_{t\in[a,b]}|u(t)-v(t)|$.

By Lemma 3.1, $u\in E$ is a solution of (1.2), if and only if it satisfies the integral equation

$$u(t)=\frac{(b-t)^\alpha}{\eta\Gamma(1+\alpha)\Gamma(\alpha)\Gamma(\beta)}$$

$$\times \int_a^b (s-a)^{\alpha-1} \left( \int_a^s (s-r)^{\beta-1} f(r, \lambda u(r), I_{a+}^{\tau} u(r), I_{b-}^{\delta} u(r)) dr \right) ds$$
$$- \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^b (s-t)^{\alpha-1} \left( \int_a^s (s-r)^{\beta-1} f(r, \lambda u(r), I_{a+}^{\tau} u(r), I_{b-}^{\delta} u(r)) dr \right) ds.$$

Then, problem (1.2) is equivalent to finding $u^* \in E$, which is a fixed point of $A$. Let $u, v \in E$, we have

$$\begin{aligned}
&|Au(t) - Av(t)| \\
\leq & \frac{(b-t)^{\alpha}}{\eta\Gamma(1+\alpha)\Gamma(\alpha)\Gamma(\beta)} \int_a^b (s-a)^{\alpha-1} \Bigg( \int_a^s (s-r)^{\beta-1} |f(r, \lambda u(r), I_{a+}^{\tau} u(r), I_{b-}^{\delta} u(r)) \\
& - f(r, \lambda v(r), I_{a+}^{\tau} v(r), I_{b-}^{\delta} v(r))| dr \Bigg) ds \\
& + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^b (s-t)^{\alpha-1} \Bigg( \int_a^s (s-r)^{\beta-1} |f(r, \lambda u(r), I_{a+}^{\tau} u(r), I_{b-}^{\delta} u(r)) \\
& - f(r, \lambda v(r), I_{a+}^{\tau} v(r), I_{b-}^{\delta} v(r))| dr \Bigg) ds \\
\leq & \Bigg( \frac{(b-t)^{\alpha}}{\eta\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)} \int_a^b (s-a)^{\alpha-1} (s-a)^{\beta} ds \\
& + \frac{1}{\Gamma(\alpha)\Gamma(1+\beta)} \int_t^b (s-t)^{\alpha-1} (s-a)^{\beta} ds \Bigg) \\
& \times \sup_{t\in[a,b]} |f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) - f(t, \lambda v(t), I_{a+}^{\tau} v(t), I_{b-}^{\delta} v(t))| \\
\leq & \left( \frac{(b-t)^{\alpha}}{\eta\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)} \int_a^b (s-a)^{\alpha+\beta-1} ds + \frac{(b-a)^{\beta}}{\Gamma(\alpha)\Gamma(1+\beta)} \int_t^b (s-t)^{\alpha-1} ds \right) \\
& \times \sup_{t\in[a,b]} |f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) - f(t, \lambda v(t), I_{a+}^{\tau} v(t), I_{b-}^{\delta} v(t))| \\
\leq & \left( \frac{(b-a)^{2\alpha+\beta}}{\eta(\alpha+\beta)\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)} + \frac{(b-a)^{\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)} \right) \\
& \times \sup_{t\in[a,b]} |f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) - f(t, \lambda v(t), I_{a+}^{\tau} v(t), I_{b-}^{\delta} v(t))| \\
< & \Omega \sup_{t\in[a,b]} |f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) - f(t, \lambda v(t), I_{a+}^{\tau} v(t), I_{b-}^{\delta} v(t))|.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
d(Au, Av) = & \sup_{t\in[a,b]} |Au(t) - Av(t)| \\
\leq & \Omega \sup_{t\in[a,b]} |f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) - f(t, \lambda v(t), I_{a+}^{\tau} v(t), I_{b-}^{\delta} v(t))| \\
\leq & \Omega \sup_{t\in[a,b]} \left( \frac{1}{\Omega} \psi(|\lambda u(t) - \lambda v(t)| + |I_{a+}^{\tau} u(t) - I_{a+}^{\tau} v(t)| + |I_{b-}^{\delta} u(t) - I_{b-}^{\delta} v(t)|) \right) \\
\leq & \psi(\lambda \sup_{t\in[a,b]} |u(t) - v(t)| + \sup_{t\in[a,b]} |I_{a+}^{\tau} u(t) - I_{a+}^{\tau} v(t)| + \sup_{t\in[a,b]} |I_{b-}^{\delta} u(t) - I_{b-}^{\delta} v(t)|) \\
\leq & \psi\Bigg( \lambda \sup_{t\in[a,b]} |u(t) - v(t)| + \frac{1}{\Gamma(\tau)} \sup_{t\in[a,b]} \int_a^t (t-s)^{\tau-1} ds \sup_{t\in[a,b]} |u(t) - v(t)|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\delta)}\sup_{t\in[a,b]}\int_t^b (s-t)^{\delta-1}ds\sup_{t\in[a,b]}|u(t)-v(t)|\Big)\\
\le&\psi\left(\left(\lambda+\frac{(b-a)^{\tau}}{\Gamma(1+\tau)}+\frac{(b-a)^{\delta}}{\Gamma(1+\delta)}\right)\sup_{t\in[a,b]}|u(t)-v(t)|\right)\\
\le&\psi(\sup_{t\in[a,b]}|u(t)-v(t)|).
\end{aligned}
$$

This implies that $d(Au,Av)\le\psi(d(u,v))$ for each $u,v\in E$.
Now, we define $\alpha: E\times E\to[0,\infty)$ by

$$
\alpha(u,v)=\begin{cases}1, & \xi(u(t),v(t))\ge 0,\\ 0, & else.\end{cases}
$$

Hence, we deduce that $\alpha(u,v)d(Au,Av)\le\psi(d(u,v))$ for all $u,v\in E$. Therefore, $A$ is $\alpha-\psi-$contraction. From condition $(H2)$, there exists $u_0\in E$ such that $\alpha(u_0,Au_0)\ge 1$. From condition $(H3)$ and the definition of $\alpha$, it is easy to see that $A$ is $\alpha-$admissible. Finally, by using condition $(H4)$, condition $(A3)$ of Lemma 2.2 is satisfied. By Lemma 2.2, we get that $A$ has a fixed point, which is a solution for the problem (1.2). □

**Theorem 3.3.** *Let $f:[a,b]\times\mathbb{R}^3\to\mathbb{R}$ be a continuous function satisfying the condition:*

*$(H5)$ $|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)|\le L(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|)$ for all $t\in[a,b]$, $u_i,v_i\in\mathbb{R}$, $i=1,2,3$ and $L>0$.*
*Then, problem (1.2) has a unique solution on $[a,b]$ if*

$$
LL_1\phi<1,
$$

*where* $L_1=\lambda+\frac{(b-a)^{\tau}}{\Gamma(1+\tau)}+\frac{(b-a)^{\delta}}{\Gamma(1+\delta)}$,

$$
\phi=\begin{cases}\frac{(\beta\eta\Gamma(1+\alpha))^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}(b-a)^{(\alpha-1)(\alpha+\beta-1)-1}}, & (b-a)^{\alpha}>-\frac{\gamma}{p}\beta\Gamma(\alpha),\\[2ex] \frac{(b-a)^{\alpha+\beta}(\beta\eta\Gamma(1+\alpha)-(\alpha+\beta-1)(b-a)^{\alpha})}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)(\alpha+\beta-1)\eta}, & (b-a)^{\alpha}\le-\frac{\gamma}{p}\beta\Gamma(\alpha).\end{cases}
$$

**Proof.** Let $E$ be the Banach space $C([a,b],\mathbb{R})$ with norm $\|u\|=\sup_{t\in[a,b]}|u(t)|$ and define $\sup_{t\in[a,b]}|f(t,0,0,0)|=M<\infty$.
By Lemma 3.1, $u\in E$ is a solution of (1.2), if and only if it satisfies the integral equation

$$
u(t)=\int_a^b G(t,r)f(r,\lambda u(r),I_{a+}^{\tau}u(r),I_{b-}^{\delta}u(r))dr.
$$

Select $\rho\ge\frac{M\phi}{1-LL_1\phi}$ and define the operator $T:E\to E$ by

$$
Tu(t)=\int_a^b G(t,r)f(r,\lambda u(r),I_{a+}^{\tau}u(r),I_{b-}^{\delta}u(r))dr.
$$

For $u \in B_\rho = \{u \in E : \|u\| \le \rho\}$, we get

$$\begin{aligned}|f(t,\lambda u(t), I^{\tau}_{a+}u(t), I^{\delta}_{b-}u(t))| &\le |f(t,\lambda u(t), I^{\tau}_{a+}u(t), I^{\delta}_{b-}u(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\ &\le L(|\lambda u(t)| + |I^{\tau}_{a+}u(t)| + |I^{\delta}_{b-}u(t)|) + M \\ &\le L\left(\lambda + \frac{(b-a)^\tau}{\Gamma(1+\tau)} + \frac{(b-a)^\delta}{\Gamma(1+\delta)}\right)\|u\| + M \le LL_1\rho + M.\end{aligned}$$

Then,

$$\begin{aligned}\|Tu\| &= \sup_{t\in[a,b]}\left|\int_a^b G(t,r)f(r,\lambda u(r), I^{\tau}_{a+}u(r), I^{\delta}_{b-}u(r))dr\right| \\ &\le \sup_{t\in[a,b]}\int_a^b |G(t,r)||f(r,\lambda u(r), I^{\tau}_{a+}u(r), I^{\delta}_{b-}u(r))|dr \\ &\le \sup_{t\in[a,b]}(LL_1\rho + M)\int_a^b |G(t,r)|dr \\ &= (LL_1\rho + M)\phi \le \rho.\end{aligned}$$

Thus, $TB_\rho \subset B_\rho$. Next, we show that $T$ is a contraction mapping.
For any $u, v \in B_\rho$, we have

$$\begin{aligned}|Tu(t) - Tv(t)| \le& \int_a^b |G(t,r)||f(r,\lambda u(r), I^{\tau}_{a+}u(r), I^{\delta}_{b-}u(r))dr \\ &- f(r,\lambda v(r), I^{\tau}_{a+}v(r), I^{\delta}_{b-}v(r))|dr \\ \le& LL_1\int_a^b |G(t,r)|dr\|u-v\| \\ \le& LL_1\phi\|u-v\|,\end{aligned}$$

which in view of the given condition $LL_1\phi < 1$, implies that $T$ is a contraction mapping. Thus, this completes the proof. □

## 4. Examples

**Example 4.1.** Consider the following fractional differential equations:

$$\begin{cases} {}^C D^{\frac{3}{4}}_{0+} D^{\frac{1}{2}}_{1-} u(t) + f(t, \frac{1}{125}u(t), I^{\frac{9}{4}}_{0+}u(t), I^{\frac{11}{5}}_{1-}u(t)) = 0, 0 < t < 1, \\ u(1) = 0, 2u(0) = -3D^{\frac{1}{2}}_{1-}u(0), \end{cases} \tag{4.1}$$

where $a = 0$, $b = 1$, $\alpha = \frac{1}{2}$, $\beta = \frac{3}{4}$, $\lambda = \frac{1}{125}$, $\tau = \frac{9}{4}$, $\delta = \frac{11}{5}$, $p = 2$, $\gamma = -3$, and

$$f(t,u_1,u_2,u_3) = \frac{2}{3}t\left(\frac{t}{5}\arctan u_1 + \frac{1}{10}\sin^2 u_2 - \frac{1}{10(1+t)}\frac{|u_3|}{1+|u_3|}\right).$$

We have $|f(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)| \le \frac{2}{3}t(\frac{1}{5}(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|)) \le \frac{2}{3}\psi(t)$ for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}$, $i = 1,2,3$. Here, $\Omega = \frac{3}{2}$, $\psi(t) = \frac{t}{5} \in \Psi$, where $\frac{(b-a)^{\alpha+\beta}((b-a)^\alpha + (\alpha+\beta)\eta\Gamma(\alpha))}{(\alpha+\beta)\eta\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)} \approx 1.4386 < \frac{3}{2}$ and $\lambda + \frac{(b-a)^\tau}{\Gamma(1+\tau)} + \frac{(b-a)^\delta}{\Gamma(1+\delta)} \approx 0.8128 < 1$. In addition, let $\xi(u,v) = |u|$, so conditions $(H2) - (H4)$ are satisfied. Then, problem (4.1) has at least one solution by Theorem 3.2.

**Example 4.2.** Consider the following fractional differential equations:

$$\begin{cases} {}^C D_{0+}^{\frac{4}{5}} D_{1-}^{\frac{21}{50}} u(t) + f(t, \frac{1}{150} u(t), I_{0+}^{\frac{14}{5}} u(t), I_{1-}^{\frac{13}{5}} u(t)) = 0, 0 < t < 1, \\ u(1) = 0, \frac{1}{10} u(0) = -11 D_{1-}^{\frac{21}{50}} u(0), \end{cases} \tag{4.2}$$

where $a = 0$, $b = 1$, $\alpha = \frac{21}{50}$, $\beta = \frac{4}{5}$, $\lambda = \frac{1}{150}$, $\tau = \frac{14}{5}$, $\delta = \frac{13}{5}$, $p = \frac{1}{10}$, $\gamma = -11$, and

$$f(t, u_1, u_2, u_3) = \frac{t^2}{1+t^3} - \frac{3}{5} \cos^2 u_1 + \frac{6}{\sqrt{25+t^2}} u_2 + \frac{2}{5} \arctan u_3.$$

Let $L = \frac{6}{5} > 0$, for all $t \in [0, 1]$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2, 3$. We have $|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \le \frac{6}{5}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|)$. By simple computation, we get $L_1 \approx 0.4887$, $-\frac{\gamma}{p}\beta\Gamma(\alpha) \approx 1.8571$, $\phi = \frac{(b-a)^{\alpha+\beta}(\beta\eta\Gamma(1+\alpha)-(\alpha+\beta-1)(b-a)^{\alpha})}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)(\alpha+\beta-1)\eta} \approx 1.5924$ and $LL_1\phi \approx 0.9338 < 1$. Then, problem (4.2) has a unique solution by Theorem 3.3.

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