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A Retrospective Study on Applications of the Lindley Distribution

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Abstract The need for efficient statistical models has increased with the flow of new data, which makes distribution theory a particularly interesting and attractive field. Here, we provide a thorough study of the applications of the Lindley distribution and its diverse generalizations. More precisely, we review some special applications in various areas, such as time series analysis, stress strength analysis, acceptance sampling plans and data analysis. We also conduct a comparative study between the Lindley distribution and some of its generalizations by using four real-life data sets.

Keywords Lindley distribution, Stress-strength, Time series modeling, Quality control, Astrophysics.

MSC(2010) 60E05, 62H10.

1. Introduction

In recent years, there has been a growing interest in introducing new distributions and their generalizations because of the diversity of the data encountered in practice. Therefore, the statisticians aim to develop different distributions presenting flexible and original properties.

In this spirit, Lindley [46] coined the term "Lindley distribution" to refer to a one-parameter distribution used in fiducial and Bayesian inferences. Its properties and applications in reliability analysis were studied by Ghitany et al., [38], showing that it may provide a better fit than the exponential distribution. The Lindley distribution's simplicity and moderate flexibility paved the way for generalized versions, with the goal of building models and with better goodness of fit to data sets than the well-known basic distributions. Some of these generalizations are the size-biased Poisson-Lindley distribution by Ghitany and Al-Mutairi [34], discrete Poisson-Lindley distribution by Sankaran [57] and zero-truncated Poisson-Lindley distribution by Ghitany et al., [36], two-parameter Lindley distribution by Shanker and Mishra [62], power Lindley distribution by Ghitany et al., [35], inverse Lindley distribution by Sharma et al., [65], exponentiated power Lindley distribution by Ashour and Eltehiwy [13], generalized power Lindley distribution by Liyanage

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and Parai [47], extended Lindley distribution by Bakouch et al., [14], Akash distribution by Shanker [59], quasi Akash distribution by Shanker [60], weighted Akash distribution Shanker and Shukla [64], quasi Lindley distribution by Shanker and Mishra [63], extended power Lindley distribution by Alkarni [4], discrete Lindley distribution by Deniz and Ojeda [30], weighted Lindley distribution by Ghitany et al., [37], discrete Poisson-Akash distribution by Shanker [61], new weighted Lindley distribution by Asgharzadeh [12], transmuted Lindley distribution by Merovci [50], new extended generalized Lindley distribution by Maya and Irshad [48], and transmuted two-parameter Lindley distribution by Kemaloglu and Yilmaz [41].

Some recent works based on the Lindley distribution are the Topp-Leone odd Lindley family of distributions by Reyad et al., [55], wrapped Lindley distribution by Joshi and Jose [40], Marshall-Olkin extended quasi Lindley distribution by Udoudo and Etuk [68], three-parameter generalized Lindley by Ekhosuehi and Opone [33], Lindley Weibull distribution by Cordeiro et al., [29], alpha power transformed power Lindley distribution by Hassan et al., [39], alpha power transformed Lindley distribution by Dey et al., [31], Weibull Marshall-Olkin Lindley distribution by Afify et al., [2], inverted modified Lindley distribution by Chesneau et al., [26], sum and difference of two Lindley distributions by Chesneau et al., [24], modified Lindley distribution by Chesneau et al., [25], wrapped modified Lindley distribution by Chesneau et al., [27], and Lindley-Lindley distribution by Chesneau et al., [28]. Tomy [66] contains a previous review of the Lindley distribution and its generalizations. Trigonometric extensions of the Lindley distribution derived from trigonometric families of distributions (see Chesneau and Artault [23]) are under development. As a first proposal, we may cite the sine-modified Lindley distribution introduced by Tomy et al., [67].

The main motivation behind this study is to expose the diverse applications of the Lindley distribution and its generalizations in various fields, like reliability, time series, quality control, astrophysics and the analysis of various kinds of data as well.

The paper unfolds as follows: In Section 2, we consider some applications of the Lindley distribution and some of its generalizations in time series modeling. Section 3 presents applications of stress-strength analysis. Section 4 contains applications for various acceptance sampling plans. Section 5 discusses applications in real data analysis. Finally, in Section 6, we conclude the paper.

2. Applications in time series modeling

Over the last decades, there has been increasing interest in developing time series models for real-valued observations by using Gaussian or non-Gaussian distributions. Among the existing time series models, let us evoke the autoregressive models, integer valued models for discrete distributions, stochastic volatility and autoregressive conditional duration models. In this section, we consider autoregressive minification processes, geometric processes and the first order non-negative integer valued autoregressive processes.

2.1. Autoregressive minification process

Udoudo and Etuk [68] proposed different minification processes with a generalized quasi Lindley distribution as a marginal distribution.

More precisely, let us consider the AR(1) structure given by

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p, \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } 1-p, \end{cases}$$

where $p \in (0,1)$ and $\{\varepsilon_n\}$ is a sequence of independent and identically (iid) distributed random variables with the quasi Lindley distribution, and also independent of $\{X_n\}$. Then, the process is a stationary AR(1) minification process with the Marshall-Olkin extended quasi Lindley distribution as a marginal distribution. The converse is also true. That is, if $\{X_n\}$ is a stationary Markovian process with the Marshall-Olkin extended quasi Lindley distribution as the marginal. Then, $\{\varepsilon_n\}$ follows the quasi Lindley distribution.
In addition, they gave a more general minification process, which is specified by

$$X_n = \begin{cases} X_{n-1} & \text{with probability } p_2 \\ \varepsilon_n & \text{with probability } p_1(1-p_2) \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } (1-p_1)(1-p_2) \end{cases}$$

where $p_1, p_2 \in (0,1)$ and $\{\varepsilon_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. Then, the process $\{X_n\}$ is a stationary AR(1) minification process with Marshall-Olkin extended quasi Lindley distribution as marginal, if and only if $\{\varepsilon_n\}$ follows the quasi Lindley distribution.

In the literature, Bakouch et al., [15] considered the Lindley AR(1) model, and studied its applications.

2.2. Geometric process

Lam [44] introduced the Geometric process (GP) for modeling inter-arrival time data with a monotone trend. Bicer [17] has recently proposed a GP with power Lindley distribution as the first arrival time distribution. The GP was defined as follows. A stochastic process $\{X_n\}$ is said to be a GP, if there exists a real number $a > 0$ such that the random variables $Y_n = a^{n-1}X_n,\ n = 1, 2, \ldots$ are valid, where X_n is the inter-arrival time the $(n-1)^{th}$ and n^{th} events of a counting process $\{N(t), t \geq 0\}$, the number a is called the ratio parameter of the GP, and X_1 follows the power Lindley distribution.

Similarly, Demirci Bicer [18] proposed a GP with two-parameter Lindley distribution as the distribution of the first arrival time.

2.3. First order non-negative integer valued autoregressive process

The pioneering work of the first order non-negative integer valued autoregressive (INAR(1)) process was proposed by McKenzie [49], and Al-Osh and Alzaid [10]. It was used as a tool for modeling counting processes consist of dependent random variables. Mohammadpour [51] introduced a discrete stationary time series model based on INAR(1), called Poisson Lindley INAR(1) model by using the binomial thinning operator with a study on its properties. The model is given by

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t, \quad t \geq 1,$$

where $\circ$ is the binomial thinning operator defined by $\alpha \circ X = \sum_{i=0}^{X} W_i$, $\alpha \in [0, 1)$, $\{W_i\}$ is a sequence of iid random variables following the Bernoulli(α) distribution, and $\{\varepsilon_t\}$ is a sequence of iid random variables independent of the Bernoulli counting process $\{W_t\}$ and X_m for all $m \leq t$. If $\{X_t\}$ is a stationary process with the Poisson Lindley distribution, then the innovation process $\{\varepsilon_t\}$ has the following probability generating function:

$$\Phi_\varepsilon(s) = \frac{2+\theta-s}{(1+\theta-s)^2} \frac{[\theta+\alpha(1-s)]^2}{1+\theta+\alpha(1-s)}, \quad s \in \mathbb{R}.$$

Similarly, Rostami [54] proposed a new stationary INAR(1) process based on the power series thinning operator under Poisson-Lindley innovations. Lvio et al., [45] introduced the INAR(1) model for modeling nonnegative integer valued time series with over dispersion using Poisson-Lindley innovations based on the binomial thinning operator.

3. Stress-strength analysis

When assessing system reliability, satisfactory performance is achieved when the strength applied to the component exceeds stress. It is a branch of reliability that aims to assess system performance. The pioneering work is given by Birnbaum [19] and Birnbaum and McCarty [20]. Al-Mutairi et al. [6] investigated stress-strength reliability inferences from the Lindley distribution. In this case, the reliability coefficient R is given by

$$\begin{aligned} R &= P(Y < X) \\ &= 1 - \frac{\theta_1^2[\theta_1(\theta_1+1)+\theta_2(\theta_1+1)(\theta_1+3)+\theta_2^2(2\theta_2+3)+\theta_2^3]}{(\theta_1+1)(\theta_2+1)(\theta_1+\theta_2)^3}, \end{aligned}$$

where $\theta_1, \theta_2 > 0$ and X and Y are two independent random variables following the Lindley distribution with parameters θ_1 and θ_2 respectively. They provide uniformly minimum variance unbiased estimator, maximum likelihood estimator and Bayesian inference of R, and study their effectiveness.

Furthermore, Khamnei [42] studied the reliability of the Lindley distribution when an outlier is present in the strength component, Krishna and Kumar [43] provided a reliability estimator by using the progressively type II censored sample, Al-Mutairi et al., [6] examined inferences on stress-strength reliability from the weighted Lindley distribution, Sadek et al., [56] discussed estimation of the stress-strength reliability for the quasi Lindley distribution, Pak et al., [53] studied the reliability of a multicomponent stress-strength model by assuming that the components follow the power Lindley distribution, and Akgul et al., [3] derived estimator of system reliability for the generalized inverse Lindley distribution by using several sampling designs.

4. Acceptance sampling plan

An acceptance sampling plan (ASP) is an important inspection and decision making tool, which has been often used by quality assurance managers to determine either to accept or to reject a product based on pre-specified quality standards.

The objective of acceptance sampling is not to estimate the quality of the product, but to decide if the product is likely to be acceptable. There are different types of ASPs. Some of them are the single sampling plan, double sampling plan, multiple sampling plan, time truncated ASP, sequential sampling plan, skip lot sampling plan and continuous sampling plan. If the quality characteristic is related to the product's lifetime, the acceptance sampling problem becomes a life's test.

Al-Omari and Al-Nasser [9] proposed an ASP based on a truncated life test, assuming the product's lifetime follows the two-parameter quasi Lindley distribution. By considering the minimum sample size, time and cost, it encourages practitioners to use this sampling plan. Al-Nasser et al., [8] developed a double ASP based on a truncated life test when the lifetime of the product follows the quasi Lindley distribution. Double sampling is used when the first sample does not give a decision and they recommend it to the researchers, Shahbaz et al., [58] introduced single and double ASP for the power Lindley distribution, and Dhanunjaya et al., [32] studied a continuous ASP for the truncated Lindley distribution.

5. Application in real data analysis

In this section, we perform a comparative study between Lindley, Akash, quasi Akash, two-parameter Lindley, inverse Lindley, quasi Lindley, power Lindley, exponentiated power Lindley, extended power Lindley, three-parameter generalized Lindley and new weighted Lindley distributions. The expressions of the probability density functions (pdfs) are given in the Appendix. These distributions were fitted to four different data sets. We estimate the unknown parameters of each model by the maximum likelihood method of estimation. Also, the statistics of the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are used to compare the eleven models. It is worth noting that $AIC = 2k - 2LogL$ and $BIC = k\log(n) - 2LogL$, where k is the number of parameters, n is the sample size, and $LogL$ is the maximized value of the log-likelihood function under the considered model.

5.1. The carbon fibers data set

This data set was given by Nichols and Padgett [52]. The carbon fibers data set consisting of 63 observations on breaking stress of carbon fibers (in Gba). The data are given below: {0.81, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33,2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03,1.61, 2.12, 3.15, 1.08}.

Table 1 gives the relevant numerical summaries for the fits of the considered distributions based on this data set.

Table 1. Estimated values, minus log-likelihood ($-LogL$), AIC and BIC for the carbon fibers data set

Distribution	Estimates	$-LogL$	AIC	BIC
Lindley	$\hat{\theta} = 0.5947$	116.568	235.1361	237.2792
Akash	$\hat{\theta} = 0.8894$	110.1611	222.3222	224.4654
Quasi Akash	$\hat{\alpha} = -0.1404$ $\hat{\theta} = 1.1615$	96.4044	196.8089	201.0952
Two-parameter Lindley	$\hat{\alpha} = -0.3721$ $\hat{\theta} = 0.9219$	97.9329	199.8658	204.1521
Inverse Lindley	$\hat{\theta} = 2.8076$	128.4222	258.8445	260.9876
Quasi Lindley	$\hat{\alpha} = -0.3431$ $\hat{\theta} = 0.9220$	97.9329	199.8658	204.1521
Power Lindley	$\hat{\beta} = 2.4048$ $\hat{\theta} = 0.1404$	84.61056	173.2211	177.5074
Exponentiated power Lindley	$\hat{\alpha} = 0.5380$ $\hat{\beta} = 3.2614$ $\hat{\theta} = 0.03651$	83.95305	173.9061	180.3355
Extended power Lindley	$\hat{\alpha} = 2.7625$ $\hat{\beta} = 0.1592$ $\hat{\theta} = 0.0812$	84.06479	174.1296	180.559
Three-parameter generalized Lindley	$\hat{\alpha} = 2.7620$ $\hat{\beta} = 6.2712$ $\hat{\lambda} = 0.0812$	84.06479	174.1296	180.559
New weighted Lindley	$\hat{\alpha} = 0.00017$ $\hat{\lambda} = 0.9767$	102.1854	208.3708	212.6571

Figures 1 and 2 give the graphs of the estimated pdfs and cumulative density functions (cdfs) respectively.

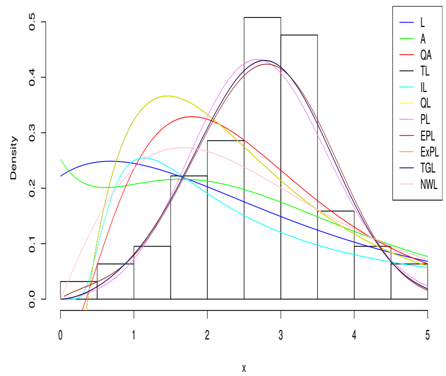

Figure 1. Estimated pdfs of the considered generalized Lindley distributions for the carbon fibers data set

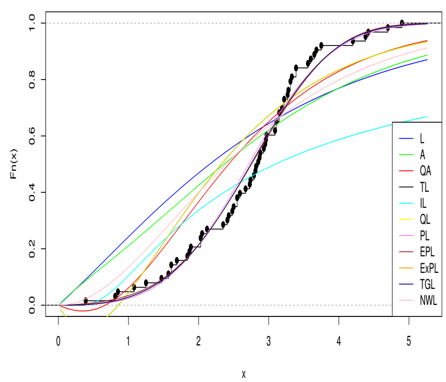

Figure 2. Estimated cdfs of the considered generalized Lindley distributions for the carbon fibers data set

At the end of the section, there will be comments on these results as well as those of the coming applications.

5.2. Guinea pigs data set

This data set was given by Bjerkedal [22]. It is the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. It is given below: {12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376}

Table 2 gives the relevant numerical values for the fits of the considered distributions based on long-axis orientations of guinea pigs data set.

Table 2. Estimated values, $-LogL$, AIC and BIC for the guinea pigs data set

Distribution	Estimates	$-LogL$	AIC	BIC
Lindley	$\hat{\theta} = 0.0198$	394.5197	791.0395	793.3161
Akash	$\hat{\theta} = 0.03004$	397.3508	796.7017	798.9783
Quasi Akash	$\hat{\alpha} = -0.2197$ $\hat{\theta} = 0.0301$	397.3555	798.7109	803.2643
Two-parameter Lindley	$\hat{\alpha} = -10.3902$ $\hat{\theta} = 0.0232$	391.5727	787.1454	791.6987
Inverse Lindley	$\hat{\theta} = 61.05642$	402.6685	807.3371	809.6137
Quasi Lindley	$\hat{\alpha} = -0.2413$ $\hat{\theta} = 0.0232$	391.5727	787.1454	791.6987
Power Lindley	$\hat{\beta} = 0.9951$ $\hat{\theta} = 0.0203$	394.5179	793.0358	797.5891
Exponentiated power Lindley	$\hat{\alpha} = 19.6531$ $\hat{\beta} = 0.4107$ $\hat{\theta} = 0.7839$	390.0966	786.1933	793.0233
Extended power Lindley	$\hat{\alpha} = 0.9863$ $\hat{\beta} = 1.9751$ $\hat{\theta} = 0.02131$	394.3916	794.7832	801.6132
Three-parameter generalized Lindley	$\hat{\alpha} = 0.9943$ $\hat{\beta} = 0.9228$ $\hat{\lambda} = 0.0204$	394.4989	794.9978	801.8278
New weighted Lindley	$\hat{\alpha} = 3.0381$ $\hat{\lambda} = 0.0209$	393.2794	790.5588	795.1121

Figures 3 and 4 give the graphs of the estimated pdfs and cdfs respectively.

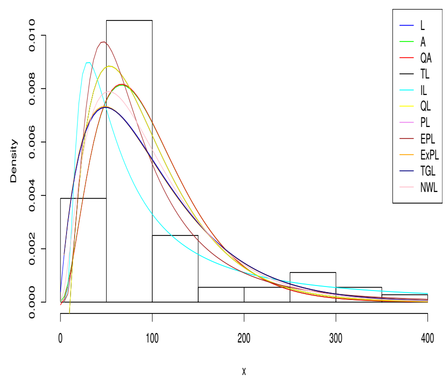

Figure 3. Estimated pdfs of the considered generalized Lindley distributions for the guinea pigs data set

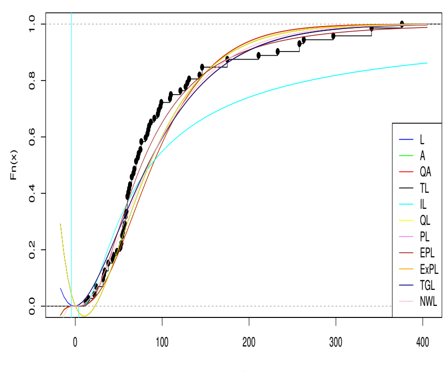

Figure 4. Estimated cdfs of the considered generalized Lindley distributions for the guinea pigs data set

5.3. The vinyl chloride data set

The real data represent 34 observations of the vinyl chloride data (in mg/L) that was obtained from cleaned-up gradient ground-water monitoring wells. The data are obtained from Bhaumik et al., [21] and are presented below. {5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2}
Table 3 gives the relevant numerical values for the fits of the considered distributions based on vinyl chloride data set.

Table 3. Estimated values, $-LogL$, AIC and BIC for vinyl chloride data set

Distribution	Estimates	$-LogL$	AIC	BIC
Lindley	$\hat{\theta} = 0.8238$	56.30364	114.6073	116.1336
Akash	$\hat{\theta} = 1.1657$	57.57463	117.1493	118.6756
Quasi Akash	$\hat{\alpha} = 15.9587$ $\hat{\theta} = 0.6946$	55.34411	114.6882	117.7409
Two-parameter Lindley	$\hat{\alpha} = 176.972$ $\hat{\theta} = 0.5376$	55.45269	114.9054	117.9581
Inverse Lindley	$\hat{\theta} = 0.8774$	61.81358	125.6272	127.1535
Quasi Lindley	$\hat{\alpha} = 538.4519$ $\hat{\theta} = 0.5329$	55.4526	114.9052	117.9579
Power Lindley	$\hat{\beta} = 0.8831$ $\hat{\theta} = 0.9139$	55.75992	115.5198	118.5726
Exponentiated power Lindley	$\hat{\alpha} = 3.7939$ $\hat{\beta} = 0.4988$ $\hat{\theta} = 2.1571$	54.9229	115.8458	120.4249
Extended power Lindley	$\hat{\alpha} = 1.0101$ $\hat{\beta} = 0.00015$ $\hat{\theta} = 0.5264$	55.44962	116.8992	121.4783
Three-parameter generalized Lindley	$\hat{\alpha} = 1.0099$ $\hat{\beta} = 507.164$ $\hat{\lambda} = 0.5283$	55.44963	116.8993	121.4783
New weighted Lindley	$\hat{\alpha} = 29.2320$ $\hat{\lambda} = 0.8348$	55.97061	115.9412	118.9939

Figures 5 and 6 give the graphs of the estimated pdfs and cdfs respectively.

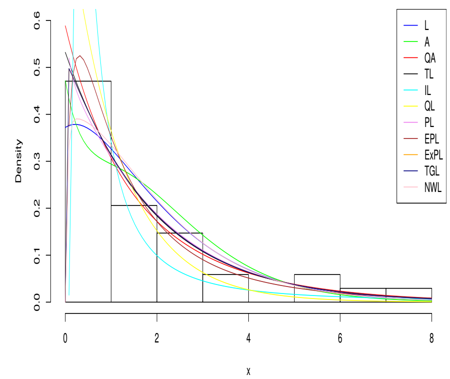

Figure 5. Estimated pdfs of the generalized Lindley distributions for the vinyl chloride data set

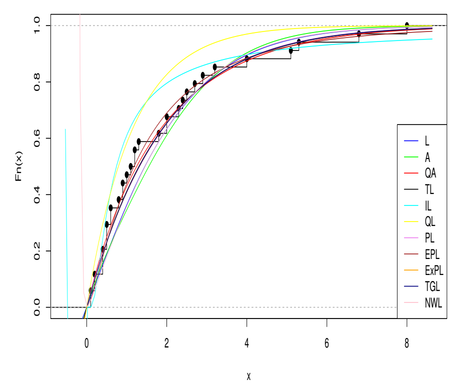

Figure 6. Estimated cdfs of the generalized Lindley distributions for the vinyl chloride data set

5.4. Fatigue fracture data set

The fatigue fracture data set is extracted from Abdul-Moniem and Seham [1], and it has previously been used by Andrews and Herzberg [11] and Barlow et al., [16]. It represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. The data are as follows:0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960}.

Table 4 gives the relevant numerical values for the fits of the distributions based on this data set.

Table 4. Estimated values, $-LogL$, AIC and BIC for the fatigue fracture data set

Distribution	Estimates	$-LogL$	AIC	BIC
Lindley	$\hat{\theta} = 0.7947$	123.6751	249.3503	251.681
Akash	$\hat{\theta} = 1.1324$	124.5755	251.151	253.4817
Quasi Akash	$\hat{\alpha} = 0.3095$ $\hat{\theta} = 1.3542$	122.4782	248.9563	253.6178
Two-parameter Lindley	$\hat{\alpha} = 0.1567$ $\hat{\theta} = 0.9544$	121.6503	247.3006	251.962
Inverse Lindley	$\hat{\theta} = 0.9459$	177.1091	356.2183	358.549
Quasi Lindley	$\hat{\alpha} = 0.1497$ $\hat{\theta} = 0.9543$	121.6503	247.3006	251.962
Power Lindley	$\hat{\beta} = 1.1423$ $\hat{\theta} = 0.7047$	122.4001	248.8001	253.4616
Exponentiated power Lindley	$\hat{\alpha} = 1.5372$ $\hat{\beta} = 0.9496$ $\hat{\theta} = 1.0213$	121.8663	249.7326	256.7248
Extended power Lindley	$\hat{\alpha} = 1.3256$ $\hat{\beta} = 0.00012$ $\hat{\theta} = 0.3666$	122.5247	251.0494	258.0416
Three-parameter generalized Lindley	$\hat{\alpha} = 0.9931$ $\hat{\beta} = 0.1478$ $\hat{\lambda} = 0.9634$	121.6487	249.2973	256.2895
New weighted Lindley	$\hat{\alpha} = 2.4729$ $\hat{\lambda} = 0.9090$	122.8925	249.7851	254.4465

Figures 7 and 8 give the graphs of the estimated pdfs and cdfs respectively.

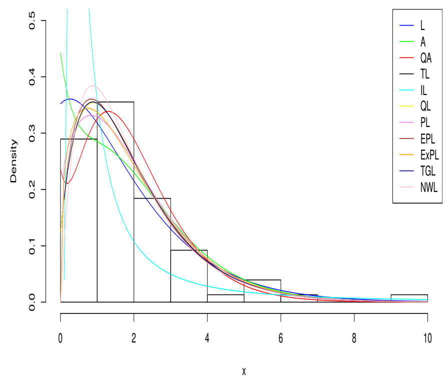

Figure 7. Estimated pdfs of the considered generalized Lindley distributions for the fatigue fracture data set

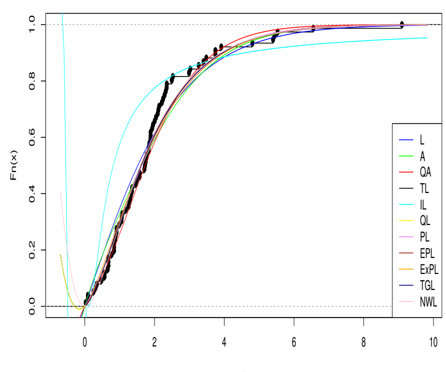

Figure 8. Estimated cdfs of the considered generalized Lindley distributions for the fatigue fracture data set

In Tables 1, 2, 3 and 4, the maximum likelihood estimates of the parameters for the fitted distributions along with the $-LogL$, AIC and BIC values are presented for the four different data sets. It is observed that the power Lindley distribution is appropriate for modeling carbon data, exponentiated power Lindley distribution is appropriate for modeling guinea pigs data, Lindley and quasi Akash distributions are appropriate for modeling vinyl chloride data, and quasi Lindley distribution is appropriate for modeling fatigue fracture data. These conclusions can also be drawn visually from Figures 1, 3, 5, and 7 for the estimated pdfs, and Figures 2, 4, 6 and 8 for the estimated cdfs.

In full generality, the Lindley distribution and its generalized versions are used for analyzing different types of data. In this regard, for circular type data, Joshi and Jose [40] introduced the wrapped Lindley distribution, Chesneau et al., [27] studied the wrapped modified Lindley distribution, and Al-khazaleh and Alkhazaleh [5] suggested the wrapped quasi Lindley distribution. Zaninetti [69] applied Lindley and truncated Lindley distribution to model initial mass function in stars. This work also introduced the Lindley luminosity function and truncated Lindley luminosity function for galaxies. Similarly, Zaninetti [70] studied a three-parameter double truncated Lindley distribution and applied it to model initial mass function in stars. Therefore, we can say that the Lindley distribution and its generalizations have promising applications in Astrophysics.

6. Conclusion

In this article, we provide an overview on applications of the Lindley distribution and its generalizations. This review is based on applications in time series analysis, stress-strength analysis, ASP and various kinds of datas. Modeling of four real life data sets shows the suitability of the Lindley distribution and its generalizations for fitting real lifetime data. For researchers and practitioners, we hope that this review will give a summary of applications of the Lindley distribution and its generalizations as well as references for further study in the theory and applications of statistical distributions.

Appendix

Here, we provide the pdfs of the distributions, which are used in comparative study.

- Lindley distribution

$$f(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\ ;\quad x>0,\ \theta>0$$

- Akash distribution

$$f(x,\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}\ ;\ x>0,\ \theta>0$$

- Quasi Akash distribution

$$f(x;\alpha,\theta)=\frac{\theta^2}{\alpha\theta+2}(\alpha+\theta x^2)e^{-\theta x};\ x>0,\ \alpha>0,\ \theta>0$$

- Two-parameter Lindley distribution

$$f(x;\alpha,\theta)=\frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}\ ;\ x>0,\ \theta>0,\ \alpha\theta>-1$$

- Inverse Lindley distribution

$$f(x;\theta)=\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\ ;\ x>0,\ \theta\ >0$$

- Quasi Lindley distribution

$$f(x;\alpha,\theta)=\frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x}\ ;\ x>0,\alpha>-1,\ \theta>0$$

- Power Lindley distribution

$$f(x;\alpha,\theta) = \frac{\alpha\theta^2}{\theta+1}(1+x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}\ ;\ x>0,\ \alpha>0,\ \theta>0$$

- Exponentiated power Lindley distribution

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\theta^2\beta x^{\beta-1}}{\theta+1}(1+x^\beta)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}$$
$$x>0,\ \alpha>0,\ \beta>0,\ \theta>0$$

- Extended power Lindley distribution

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\theta^2}{\theta+\beta}(1+\beta x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}\ ;\ x>0,\ \alpha>0,\ \beta>0,\ \theta>0$$

- Three-parameter generalized Lindley distribution

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\lambda^2(\beta+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}}{1+\lambda\beta}\ ;\ x>0,\ \alpha>0,\ \beta>0,\ \lambda>0$$

- New weighted Lindley distribution

$$f(x;\alpha,\lambda) = \frac{\lambda^2(1+\alpha)^2}{\alpha\lambda(1+\alpha)+\alpha(2+\alpha)}(1+x)(1-e^{-\lambda\alpha x})e^{-\lambda x};\ x>0,\ \alpha>0,\ \lambda>0$$

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