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The Number of Limit Cycles in a Class of Piecewise Polynomial Systems

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Abstract In this paper, we pay attention to the number of limit cycles for a class of piecewise smooth near-Hamiltonian systems. By using the expression of the first order Melnikov function and some known results about Chebyshev systems, we study upper bound of the number of limit cycles in Hopf bifurcation and Poincaré bifurcation respectively.

Keywords Piecewise smooth system, Melnikov function, ECT-system, Limit cycle.

MSC(2010) 37E05, 34C07.

1. Introduction

As we know, the second part of Hilbert's 16th problem [5] is to estimate the number of limit cycles in a planar system

$$\begin{cases} \dot{x} = P_n(x, y), \\ \dot{y} = Q_n(x, y), \end{cases}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials in (x, y) and to investigate their distributions. In 1977, Arnold [1] first proposed the weakened Hilbert's 16th problem, which is to study the maximum number of zeros of the first order Melnikov function in the following near-Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x, y) + \varepsilon f(x, y), \\ \dot{y} = -H_x(x, y) + \varepsilon g(x, y), \end{cases}$$

where H, f and g are polynomials in (x, y), and $\varepsilon > 0$ is small. This problem is still open. Many mathematicians have done a lot of researches on limit cycle bifurcations.

In recent years, stimulated by non-smooth phenomena in the real world, piecewise smooth systems have been widely investigated (see [2, 3, 6, 7, 9, 12, 13] for example). The authors [9] considered the piecewise near-Hamiltonian system on the

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plane

$$\dot{x} = \begin{cases} H_y^+(x,y) + \varepsilon p^+(x,y,\delta), & x \geq 0, \\ H_y^-(x,y) + \varepsilon p^-(x,y,\delta), & x < 0, \end{cases}$$
$$\dot{y} = \begin{cases} -H_x^+(x,y) + \varepsilon q^+(x,y,\delta), & x \geq 0, \\ -H_x^-(x,y) + \varepsilon q^-(x,y,\delta), & x < 0, \end{cases} \tag{1.1}$$

where $H^\pm, p^\pm$ and $q^\pm$ are C^∞, $\varepsilon > 0$ is small, and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D a compact set. Suppose that $(1.1)|_{\varepsilon=0}$ has a family of periodic orbits around the origin and satisfies the following two assumptions [3,9].

Assumption (I). There exist an interval $J = (\alpha, \beta)$, and two points $A(h) = (0, a(h))$ and $A_1(h) = (0, a_1(h))$ such that for $h \in J$,

$$H^+(A(h)) = H^+(A_1(h)) = h, \; H^-(A(h)) = H^-(A_1(h)),$$

$$H_y^\pm(A(h)) H_y^\pm(A_1(h)) \neq 0, \; a(h) > a_1(h).$$

Assumption (II). The equation $H^+(x,y) = h, \; x \geq 0$, defines an orbital arc L_h^+ starting from $A(h)$ and ending at $A_1(h)$; the equation $H^-(x,y) = H^-(A_1(h)), \; x \leq 0$, defines an orbital arc L_h^- starting from $A_1(h)$ and ending at $A(h)$, such that $(1.1)|_{\varepsilon=0}$ has a family clockwise periodic orbits $L_h = L_h^- \bigcup L_h^+, \; h \in J$.

The first order Melnikov function was defined and its formula was given in [9]. In fact, by Theorem 1.1 in [9] and Lemma 2.2 in [7], the first order Melnikov function of system (1.1) has the form

$$M(h,\delta) = M^+(h,\delta) + \frac{H_y^+(A)}{H_y^-(A)} M^-(h,\delta), \tag{1.2}$$

where
$$M^\pm(h,\delta) = \int_{L_h^\pm} q^\pm dx - p^\pm dy.$$

In [13], the authors studied a piecewise smooth near-Hamiltonian system of the form
$$\begin{cases} \dot{x} = y + \varepsilon p(x,y,\delta), \\ \dot{y} = -g(x) + \varepsilon q(x,y,\delta), \end{cases} \tag{1.3}$$

where
$$g(x) = \begin{cases} a_1 x + a_0, \; x \geq 0, \\ b_1 x + b_0, \; x < 0, \end{cases}$$

$$p(x,y,\delta) = \begin{cases} p^+(x,y,\delta) = \sum_{i+j=0}^n a_{ij}^+ x^i y^j, \; x \geq 0, \\ p^-(x,y,\delta) = \sum_{i+j=0}^n a_{ij}^- x^i y^j, \; x < 0, \end{cases}$$
$$q(x,y,\delta) = \begin{cases} q^+(x,y,\delta) = \sum_{i+j=0}^n b_{ij}^+ x^i y^j, \; x \geq 0, \\ q^-(x,y,\delta) = \sum_{i+j=0}^n b_{ij}^- x^i y^j, \; x < 0, \end{cases}$$

$0 < \varepsilon \ll 1$, and δ is a vector of bounded parameters. Particularly, when system $(1.3)|_{\varepsilon=0}$ satisfies

$$a_1 = 0, \quad a_0 > 0, \quad b_1 > 0, \quad b_0 < 0, \tag{1.4}$$

the origin is an elementary center of system $(1.3)|_{\varepsilon=0}$ according to [8]. Obviously, system $(1.3)|_{\varepsilon=0}$ has a global center under condition (1.4) (see Figure 1). It was

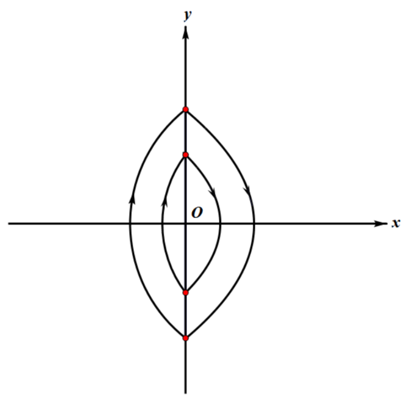


Figure 1. The phase portrait of system $(1.3)|_{\varepsilon=0}$ under condition (1.4)

proved in [13]

$$n + \left[\frac{n+1}{2}\right] \le Z(n) \le n + 2\left[\frac{n+1}{2}\right], \qquad n + \left[\frac{n+1}{2}\right] \le N(n), \tag{1.5}$$

where $Z(n)$ denotes the maximal number of zeros of $M(h,\delta)$ for $h > 0$, $N(n)$ denotes the maximal number of limit cycles of system (1.3) near the origin for $\varepsilon > 0$ sufficiently small, and $[s]$ denotes the integer part of a real number s.

In this paper, we consider the following two piecewise smooth systems

$$(\dot{x}, \dot{y}) = \begin{cases} (y + \varepsilon p^+(x,y,\delta), -a_0 + \varepsilon q^+(x,y,\delta)), & x \ge 0, \\ (y + \varepsilon p^-(x,y,\delta), -x + 1 + \varepsilon q^-(x,y,\delta)), & x < 0, \end{cases} \tag{1.6}$$

and

$$(\dot{x}, \dot{y}) = \begin{cases} (y + \varepsilon p^+(x,y,\delta), -x - 2 + \varepsilon q^+(x,y,\delta)), & x \ge 0, \\ (y + \varepsilon p^-(x,y,\delta), -b_0 + \varepsilon q^-(x,y,\delta)), & x < 0, \end{cases} \tag{1.7}$$

where $a_0 > 0, b_0 < 0$, $p^\pm$ and $q^\pm$ are arbitrary polynomials of degree n.

Note that $H_y^+(A) = H_y^-(A)$ for systems (1.6) and (1.7). By formula (1.2),

$$M(h,\delta) = M^+(h,\delta) + M^-(h,\delta), \quad h \in (0, +\infty). \tag{1.8}$$

Compared with (1.5), we provide a more accurate estimation of the maximal number of positive zeros of $M(h,\delta)$ for systems (1.6) and (1.7) based on the expansion of $M(h,\delta)$ and some known results about Chebyshev systems.

Let

$$Z_{1n} = \begin{cases} n+1, & n = 1,2, \\ 5, & n = 3, \end{cases} \text{ and } Z_{2n} = \begin{cases} n+1, & n = 1,2, \\ 6, & n = 3. \end{cases}$$

The main results of the paper are stated in the following theorems.

Theorem 1.1. *Suppose that the first order Melnikov function defined in (1.8) is not zero identically. Then, the maximal number of zeros of $M(h,\delta)$ of system* (1.6) *and system (1.7) for $0<h\ll 1$ is both exactly Z_{1n}, multiplicity taken into account.*

Theorem 1.2. *Suppose that the first order Melnikov function defined in (1.8) is not zero identically. Then, for small $|\varepsilon|>0$ and bounded δ, the maximal number of limit cycles bifurcated from the family of periodic orbits $\{L_h\}$ of system* (1.6) *and system (1.7) is both exactly Z_{2n}, multiplicity taken into account.*

This paper is organized as follows: Several helpful definitions and lemmas will be listed in Section 2. In Section 3, we study Hopf bifurcation and prove Theorem 1.1. In Section 4, we use ECT-systems to get upper bounds for the number of zeros of the first order Melnikov function and prove Theorem 1.2.

2. Preliminary lemmas

In order to prove Theorem 1.1, we first present two lemmas which give concrete expansions of $M^+(h,\delta)$ and $M^-(h,\delta)$ in (1.8) for system (1.6) respectively.

Lemma 2.1 (Lemma 3, [13])**.** *The function $M^+(h,\delta)$ in (1.8) has the following expression*

$$M^+(h,\delta)=h^{1/2}\sum_{i+2k=0}^{n}B^+_{i,2k}h^{i+k},\ \ h>0, \tag{2.1}$$

where

$$B^+_{0,2k}=\frac{2^{k+1+1/2}a^+_{0,2k}}{2k+1},$$
$$B^+_{i,2k}=\frac{2^{k+1+1/2}}{a_0^i}\Big(a^+_{i,2k}+\frac{2k+1}{i}b^+_{i-1,2k+1}\Big)\times\int_0^{\pi/2}\sin^{2k}\theta\cos^{i+1}\theta d\theta,\quad 1\le i\le n.$$

Lemma 2.2 (Lemma 5, [13])**.** *The function $M^-(h,\delta)$ in (1.8) has the following expression for $h>0$*

$$\begin{aligned}M^-(h,\delta)=&\sqrt{h}\left[\sum_{i+2k=0}^{n-1}\bar q^-_{i,2k}\phi^-_{ik}(h)-\sum_{2k=0}^{n}\frac{2^{k+1+1/2}a^-_{0,2k}}{2k+1}h^k\right]\\&+\sum_{i+2k=0}^{n-1}\bar q^-_{i,2k}\left(\sum_{r=0,\ r\ even}^{i}\alpha^-_{irk}\Big(2h+1\Big)^{k+r/2}\bar I^-_{00}(h)\right),\end{aligned}\tag{2.2}$$

where

$$\bar I^-_{00}(h)=\int_{-\sqrt{2h+1}}^{-1}\sqrt{2h+1-v^2}dv,\quad \bar q^-_{ij}=b^-_{i,j+1}+\frac{i+1}{j+1}a^-_{i+1,j},\tag{2.3}$$

each α^-_{irk} is a constant and ϕ^-_{ik} is a polynomial of degree $k+[(i+1)/2]$.

Here, we give a remark related to α^-_{irk} and $\phi^-_{ik}(h)$ in Lemma 2.2.

Remark 2.1. Similar to the formulas (50)-(51), (56)-(57) and (61) in [13], we obtain

(i) α^-_{irk} has an expression of the form

$$\alpha^-_{irk} = 2C_i^r \bar{\alpha}^-_{rk}\bar{\alpha}^-_r, \tag{2.4}$$

where

$$\bar{\alpha}^-_{rk} = \begin{cases} \tilde{\alpha}^-_{rk}, \ k \geq 1, \\ 1, \quad k = 0, \end{cases} \qquad \bar{\alpha}^-_r = \begin{cases} \tilde{\alpha}^-_r, \ r \geq 1, \\ 1, \quad r = 0, \end{cases}$$

with

$$\tilde{\alpha}^-_{rk} = \frac{(2k+1)(2k-1)(2k-3)\times\cdots\times 3}{(2k+2+r)(2k+r)(2k-2+r)\times\cdots\times(4+r)},$$

$$\tilde{\alpha}^-_r = \begin{cases} 0, & r \ odd, \\ \frac{(r-1)(r-3)\times\cdots\times 3\times 1}{(2+r)r(r-2)\times\cdots\times 6\times 4}, & r \ even. \end{cases}$$

(ii) $\phi^-_{ik}(h)$ has an expression of the form

$$\phi^-_{ik}(h) = \sum_{r=0}^{i} 2C_i^r \bar{\psi}^-_{rk}(h), \tag{2.5}$$

where

$$\bar{\psi}^-_{rk}(h) = (-1)^{r+1}\sqrt{2}\bar{\varphi}^-_{rk}(h) - 2\sqrt{2}h\bar{\alpha}^-_{rk}(2h+1)^k\bar{\varphi}^-_r(h),$$

$$\bar{\varphi}^-_{rk}(h) = \begin{cases} \tilde{\varphi}^-_{rk}(h), \ k \geq 1, \\ 0, \quad k = 0, \end{cases} \qquad \bar{\varphi}^-_r(h) = \begin{cases} \tilde{\varphi}^-_r(h), \ r \geq 1, \\ 0, \quad r = 0, \end{cases}$$

with

$$\begin{aligned} \tilde{\varphi}^-_{rk}(h) =& \frac{(2h)^k}{2k+2+r} + \frac{2k+1}{(2k+2+r)(2k+r)} \times (2h+1)(2h)^{k-1} \\ &+ \frac{(2k+1)(k-1)}{(2k+2+r)(2k+r)(2k-2+r)} \times (2h+1)^2(2h)^{k-2} + \cdots \\ &+ \frac{(2k+1)(2k-1)\times\cdots\times 5}{(2k+2+r)(2k+r)(2k-2+r)\times\cdots\times(4+r)} \times (2h+1)^{k-1}2h, \end{aligned}$$

$$\begin{aligned} \tilde{\varphi}^-_r(h) =& \frac{(-1)^{r-1}}{(r+2)} + \frac{(-1)^{r-3}(r-1)}{(2+r)r}(2h+1) + \frac{(-1)^{r-5}(r-1)(r-3)}{(2+r)r(r-2)} \\ &\times (2h+1)^2 + \cdots + \frac{(-1)^{r-1-2[\frac{r-1}{2}]}(r-1)(r-3)\times\cdots\times(r+1-2[\frac{r-1}{2}])}{(2+r)r(r-2)\times\cdots\times(r+2-2[\frac{r-1}{2}])} \\ &\times (2h+1)^{[\frac{r-1}{2}]}. \end{aligned}$$

Similar to the proof of Theorem 2.4.2 in [2], we have further

Lemma 2.3. *Let*

$$M(h,\delta) = h^{\frac{1}{2}} \sum_{j\geq 0} b_j(\delta)h^j,$$

where $\delta \in \mathbb{R}^m$, *and* $0 < h \ll 1$. *Suppose that for an integer* $k \geq 1$,

(i) *rank* $\frac{\partial(b_0,\cdots,b_k)}{\partial(\delta_1,\cdots,\delta_m)}=k+1$, $m\geq k+1$;

(ii) *when* $b_0=b_1=\cdots=b_k=0$, $b_j=0$, $j\geq k+1$.

Then, the function $M(h,\delta)$ *has at most* k *positive zeros near* $h=0$ *for* $\sum_{j=0}^{k}|b_j|$ *sufficiently small.*

Next, we introduce Chebyshev criterion, which will be used to obtain the maximal number of zeros of the first order Melnikov function. Let $\mathcal{F}=\{f_0,f_1,\cdots,f_{n-1}\}$ be an ordered set of analytic functions defined on an open interval L of $\mathbb{R}$ and Span($\mathcal{F}$) be the set of all linear combinations of elements of $\mathcal{F}$. Then we have

Definition 2.1. The ordered set $\mathcal{F}$ is said to be an extended complete Chebyshev system (for short, an ECT-system) on L if for all $k=1,2,\cdots,n$, any nontrivial function in Span($\mathcal{F}$)

$$\alpha_0f_0(x)+\alpha_1f_1(x)+\cdots+\alpha_{n-1}f_{n-1}(x),$$

where the coefficients $\alpha_0,\cdots,\alpha_{n-1}$ are not all 0, has at most $k-1$ zeros on L, multiplicity taken into account.

Remark 2.2. We remove "isolated" before "zeros" in the definition in [10]. If not, it may lead to misunderstandings. For example, take $\mathcal{F}=\{x,2x\}$. Clearly, any nontrivial function in Span($\mathcal{F}$) has no isolated zero for $x>0$. In fact, the set $\mathcal{F}$ is not an ECT-system.

Now, we recall some relations between the ordered set $\mathcal{F}$ and their Wronskians

$$W_n(x)=W[f_0,f_1,\cdots,f_{n-1}](x)=\begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{n-1}(x)\\ f_0'(x) & f_1'(x) & \cdots & f_{n-1}'(x)\\ \vdots & \vdots & \ddots & \vdots\\ f_0^{(n-1)}(x) & f_1^{(n-1)}(x) & \cdots & f_{n-1}^{(n-1)}(x)\end{vmatrix}.$$

Lemma 2.4 (Lemma 2.3, [10])**.** *The ordered set* $\mathcal{F}$ *is an ECT-system on* L *if and only if for each* $k=1,2,\cdots,n$,

$$W_k(x)\neq 0 \ \textit{for all}\ x\in L.$$

Lemma 2.5 (Corollary 1.4, [11])**.** *Assume that all the Wronskians are nonvanishing except* $W_n(x)$, *which has exactly one zero on* L *and this zero is simple. Then, the maximum number of zeros counting multiplicity of any nontrivial function in Span(*$\mathcal{F}$*) is* $n+1$ *and for any configuration of* $m\leq n+1$ *zeros there exists an element in Span(*$\mathcal{F}$*) realizing it.*

Lemma 2.6 (Theorem 1.1, [11])**.** *Assume that for each* $i=0,\cdots,n$, *the Wronskian* W_i *has* v_i *zeros counting multiplicity. Then, the number of isolated zeros (multiplicity taken into account) for every element of Span(*$\mathcal{F}$*) does not exceed*

$$n-1+v_{n-1}+v_{n-2}+2(v_{n-3}+\cdots+v_0)+\mu_{n-2}+\cdots+\mu_3,$$

where $\mu_k=\min(2v_k,v_{k-3}+\cdots+v_0)$, $k=3,\cdots,n-2$.

3. Proof of Theorem 1.1

First, we compute the expressions of $M^+(h,\delta)$ and $M^-(h,\delta)$ for system (1.6). By (2.1) in Lemma 2.1, we have

$$M^+(h,\delta) = h^{\frac{1}{2}}\left[B_{00}^+ + (B_{10}^+ + B_{02}^+)h + (B_{12}^+ + B_{20}^+)h^2 + B_{30}^+h^3\right], \tag{3.1}$$

where

$$\begin{aligned}B_{00}^+ &= 2\sqrt{2}a_{00}^+,\ B_{10}^+ = \frac{4\sqrt{2}}{3a_0}(a_{10}^+ + b_{01}^+),\ B_{12}^+ = \frac{\sqrt{2}\pi}{4a_0}(a_{12}^+ + 3b_{03}^+),\\ B_{02}^+ &= \frac{4\sqrt{2}}{3}a_{02}^+,\ B_{20}^+ = \frac{16\sqrt{2}}{15a_0^2}\left(a_{20}^+ + \frac{1}{2}b_{11}^+\right),\ B_{30}^+ = \frac{3\sqrt{2}}{8a_0^3}\left(a_{30}^+ + \frac{1}{3}b_{21}^+\right).\end{aligned}$$

From (2.2), (2.4) and (2.5), $M^-(h,\delta)$ has the following expression

$$M^-(h,\delta) = h^{\frac{1}{2}}(-2\sqrt{2}a_{00}^- + P_1 h) + (Q_1 + Q_2 h)\bar{I}_{00}^-, \tag{3.2}$$

where

$$\begin{aligned}P_1 &= -\sqrt{2}\bar{q}_{02}^- - \frac{4\sqrt{2}}{3}\bar{q}_{10}^- - \frac{5\sqrt{2}}{3}\bar{q}_{20}^- - \frac{4\sqrt{2}a_{02}^-}{3},\\ Q_1 &= 2\bar{q}_{00}^- + \frac{3}{2}\bar{q}_{02}^- + 2\bar{q}_{10}^- + \frac{5}{2}\bar{q}_{20}^-,\ Q_2 = 3\bar{q}_{02}^- + \bar{q}_{20}^-.\end{aligned}$$

Take the transformation $u = \sqrt{\frac{v^2}{2h} - \frac{b_0^2}{2hb_1}}$. Then, $\bar{I}_{00}^-(h,\delta)$ in (2.3) becomes

$$\bar{I}_{00}^-(h,\delta) = (2h)^{\frac{3}{2}}\int_0^1 \frac{u\sqrt{1-u^2}}{\sqrt{1+2hu^2}}du.$$

Using the power series expansion of $\frac{1}{\sqrt{1+ax}}$, we obtain that for $0 < h \ll 1$

$$\bar{I}_{00}^-(h,\delta) = (2h)^{\frac{3}{2}}\int_0^1 u\sqrt{1-u^2}\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m(2m-1)!!}{(2m)!!}h^m u^{2m}\right]du. \tag{3.3}$$

Note that

$$\begin{aligned}&\int_0^1 u(1-u^2)^{\frac{1}{2}}du = -\frac{1}{2}\int_0^1 (1-u^2)^{\frac{1}{2}}d(1-u^2) = 0,\\ &\int_0^1 u^{2m+1}(1-u^2)^{\frac{1}{2}}du = \int_0^{\pi/2}\sin^{2m+1}\theta\cos^2\theta d\theta = \frac{(2m)!!}{(2m+3)!!},\quad m = 1,2,\cdots.\end{aligned}$$

Inserting the above formulas into (3.3) gives that

$$\bar{I}_{00}^-(h,\delta) = \sqrt{h}\sum_{m\geq 0}u_{m+1}h^{m+1}, \tag{3.4}$$

where

$$u_{m+1} = \frac{(-1)^m 2^{m+1}\sqrt{2}}{(2m+1)(2m+3)},\ 0 < h \ll 1.$$

Substitutng (3.4) into (3.2), $M^-(h,\delta)$ can be rewritten as

$$\begin{aligned}M^-(h,\delta)=&h^{\frac{1}{2}}\Big[-2\sqrt{2}a_{00}^-+(P_1+u_1Q_1)h+(u_2Q_1+u_1Q_2)h^2\\&+\sum_{m\geq 2}(u_{m+1}Q_1+u_mQ_2)h^{m+1}\Big].\end{aligned}\tag{3.5}$$

By (1.8), (3.1) and (3.5), we obtain the following expansion for $h>0$ small

$$M(h,\delta)=\sqrt{h}\sum_{i\geq 0}v_ih^i,\tag{3.6}$$

where

$$\begin{aligned}v_0=&2\sqrt{2}(a_{00}^+-a_{00}^-),\\
v_1=&\frac{4\sqrt{2}}{3a_0}(a_{10}^++b_{01}^+)+\frac{4\sqrt{2}}{3}a_{02}^+-\frac{4\sqrt{2}}{3}a_{02}^-+\frac{4\sqrt{2}}{3}(b_{01}^-+a_{10}^-),\\
v_2=&\frac{\sqrt{2}\pi}{4a_0}(a_{12}^++3b_{03}^+)+\frac{16\sqrt{2}}{15a_0^2}\Big(a_{20}^++\frac{1}{2}b_{11}^+\Big)-\frac{8\sqrt{2}}{15}(b_{01}^-+a_{10}^-)\\
&-\frac{8\sqrt{2}}{15}(b_{11}^-+2a_{20}^-)+\frac{8\sqrt{2}}{5}\Big(b_{03}^-+\frac{1}{3}a_{12}^-\Big),\\
v_3=&\frac{3\sqrt{2}}{8a_0^3}\Big(a_{30}^++\frac{1}{3}b_{21}^+\Big)+\frac{16\sqrt{2}}{35}(b_{01}^-+a_{10}^-)+\frac{16\sqrt{2}}{35}(b_{11}^-+2a_{20}^-)\\
&+\frac{32\sqrt{2}}{105}(b_{21}^-+3a_{30}^-)-\frac{16\sqrt{2}}{35}\Big(b_{03}^-+\frac{1}{3}a_{12}^-\Big),\\
v_4=&-\frac{32\sqrt{2}}{63}(b_{01}^-+a_{10}^-)-\frac{32\sqrt{2}}{63}(b_{11}^-+2a_{20}^-)-\frac{128\sqrt{2}}{315}(b_{21}^-+3a_{30}^-)\\
&+\frac{32\sqrt{2}}{105}\Big(b_{03}^-+\frac{1}{3}a_{12}^-\Big),\\
v_{n+1}=&\frac{(-1)^n2^n\sqrt{2}}{(2n+1)(2n+3)}\Big[4(a_{10}^-+b_{01}^-)-4(b_{11}^-+2a_{20}^-)-\frac{12}{2n-1}\Big(b_{03}^-\\
&+\frac{1}{3}a_{12}^-\Big)+\frac{8n-8}{2n-1}(b_{21}^-+3a_{30}^-)\Big],\ n\geq 4.\end{aligned}\tag{3.7}$$

Let

$$\begin{aligned}\delta_1=&a_{00}^+-a_{00}^-,\ \delta_2=a_{10}^++b_{01}^+,\ \delta_3=b_{21}^-+3a_{30}^-,\ \delta_4=a_{30}^++\frac{1}{3}b_{21}^+,\\
\delta_5=&a_{02}^+-a_{02}^-,\ \delta_6=a_{12}^++3b_{03}^+,\ \delta_7=a_{20}^++\frac{1}{2}b_{11}^+,\ \delta_8=a_{10}^-+b_{01}^-,\\
\delta_9=&b_{11}^-+2a_{20}^-,\ \delta_{10}=b_{03}^-+\frac{1}{3}a_{12}^-.\end{aligned}$$

It is obvious that

$$\text{rank}\frac{\partial(\delta_1,\delta_2,\delta_3,\cdots,\delta_{10})}{\partial(a_{00}^+,a_{10}^+,b_{21}^-,a_{30}^+,a_{02}^+,a_{12}^+,a_{20}^+,a_{10}^-,b_{11}^-,b_{03}^-)}=\text{rank}E_1=10,$$

where E_1 denotes the 10×10 identity matrix. Therefore, $\delta_1,\delta_2,\delta_3,\cdots,\delta_{10}$ can be taken as new free parameters and they are linearly independent. Then, (3.7) can be rewritten as

$$\begin{aligned}
v_0 =& 2\sqrt{2}\delta_1, \\
v_1 =& \frac{4\sqrt{2}}{3a_0}\delta_2 + \frac{4\sqrt{2}}{3}\delta_5 + \frac{4\sqrt{2}}{3}\delta_8, \\
v_2 =& \frac{\sqrt{2}\pi}{4a_0}\delta_6 + \frac{16\sqrt{2}}{15a_0^2}\delta_7 - \frac{8\sqrt{2}}{15}\delta_8 - \frac{8\sqrt{2}}{15}\delta_9 + \frac{8\sqrt{2}}{5}\delta_{10}, \\
v_3 =& \frac{3\sqrt{2}}{8a_0^3}\delta_4 + \frac{16\sqrt{2}}{35}\delta_8 + \frac{16\sqrt{2}}{35}\delta_9 + \frac{32\sqrt{2}}{105}\delta_3 - \frac{16\sqrt{2}}{35}\delta_{10}, \\
v_4 =& -\frac{32\sqrt{2}}{63}\delta_8 - \frac{32\sqrt{2}}{63}\delta_9 - \frac{128\sqrt{2}}{315}\delta_3 + \frac{32\sqrt{2}}{105}\delta_{10}, \\
v_{n+1} =& \frac{(-1)^n 2^{n+1}\sqrt{2}}{(2n+1)(2n+3)}\left[2\delta_8 + 2\delta_9 - \frac{6}{2n-1}\delta_{10} + \frac{4n-4}{2n-1}\delta_3\right],\ n \geq 4.
\end{aligned} \tag{3.8}$$

Next, we prove $v_0, v_1, \cdots, v_5, \bar{v}_{n+1}$ are linearly dependent for $n \geq 5$ and $v_0, v_1, \cdots, v_5$ are linearly independent with each other. One can compute that

$$\frac{\partial(v_0, v_1, v_2, v_3, v_4, v_5, v_{n+1})}{\partial(\delta_1, \delta_2, \delta_7, \delta_4, \delta_8, \delta_9, \delta_{10}, \delta_3, \delta_5, \delta_6)} = \Big(\beta_1, \beta_2, \cdots, \beta_{10}\Big)^T \equiv \bar{A},$$

where

$$\begin{aligned}
&\beta_1 = (2\sqrt{2}, 0, 0, 0, 0, 0, 0), \quad \beta_2 = \Big(0, \frac{4\sqrt{2}}{3a_0}, 0, 0, 0, 0, 0\Big), \\
&\beta_3 = \Big(0, 0, \frac{16\sqrt{2}}{15a_0^2}, 0, 0, 0, 0\Big), \quad \beta_4 = \Big(0, 0, 0, \frac{3\sqrt{2}}{8a_0^3}, 0, 0, 0\Big), \\
&\beta_5 = \Big(0, \frac{4\sqrt{2}}{3}, -\frac{8\sqrt{2}}{15}, \frac{16\sqrt{2}}{35}, -\frac{32\sqrt{2}}{63}, \frac{64\sqrt{2}}{99}, \frac{(-1)^n 2^{n+2}\sqrt{2}}{(2n+1)(2n+3)}\Big), \\
&\beta_6 = \Big(0, 0, -\frac{8\sqrt{2}}{15}, \frac{16\sqrt{2}}{35}, -\frac{32\sqrt{2}}{63}, \frac{64\sqrt{2}}{99}, \frac{(-1)^n 2^{n+2}\sqrt{2}}{(2n+1)(2n+3)}\Big), \\
&\beta_7 = \Big(0, 0, \frac{8\sqrt{2}}{5}, \frac{-16\sqrt{2}}{35}, \frac{32\sqrt{2}}{105}, \frac{-64\sqrt{2}}{231}, \frac{(-1)^{n+1} 2^{n+2}\sqrt{2}}{(2n+1)(2n+3)} \cdot \frac{3}{(2n-1)}\Big), \\
&\beta_8 = \Big(0, 0, 0, \frac{32\sqrt{2}}{105}, \frac{-128\sqrt{2}}{315}, \frac{128\sqrt{2}}{231}, \frac{(-1)^n 2^{n+2}\sqrt{2}}{(2n+1)(2n+3)} \cdot \frac{2n-2}{(2n-1)}\Big), \\
&\beta_9 = \Big(0, \frac{4\sqrt{2}}{3}, 0, 0, 0, 0, 0\Big), \ \beta_{10} = \Big(0, 0, \frac{\sqrt{2}\pi}{4a_0}, 0, 0, 0, 0\Big).
\end{aligned}$$

Since the row rank is equal to the column rank for a same matrix, we next obtain the rank of the matrix $\bar{A}$ by doing simple row and column transformations. Let W denote an $n \times m$ matrix, r_i denote the ith row of W, c_j denote the jth column of W, u_{ij} denote the element in the ith row and jth column of W. Take elementary transformations to $\bar{A}$ with the following steps:

$$\begin{aligned}
&S1: \quad r_i/u_{ii}, && i = 1, 2, 3, 4; \\
&S2: \quad c_i + (-u_{2i}) \times c_2, && i = 5, 9; \\
&S3: \quad c_i + (-u_{3i}) \times c_3, && i = 5, 6, 7, 10;
\end{aligned}$$

$$
\begin{aligned}
&S4: && c_i + (-u_{4i}) \times c_4, \qquad i = 5, 6, 7, 8;\\
&S5: && r_5 / \Big(-\frac{32\sqrt{2}}{63} \Big);\\
&S6: && r_6 / \Big(\frac{64\sqrt{2}}{99} \Big);\\
&S7: && r_7 / \Big(\frac{(-1)^n 2^{n+2}\sqrt{2}}{(2n+1)(2n+3)} \Big);\\
&S8: && 5 \times r_5;\ 7 \times r_6;\ (2n-1) \times r_7;\\
&S9: && c_6 - c_5;\ -\frac{1}{3} \times c_7;\ c_8 - c_5;\\
&S10: && c_8 + c_7.
\end{aligned}
$$

Then, $\bar{A}$ becomes

$$
\bar{A}_1 = \begin{pmatrix} E_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{B} & \mathbf{0} \end{pmatrix},
$$

where

$$
\bar{B} = \begin{pmatrix} 5 & 0 & 1 & 0 \\ 7 & 0 & 1 & 0 \\ 2n-1 & 0 & 1 & 0 \end{pmatrix},
$$

E_1 is the 4×4 identity matrix. It is easy to see that

$$
\text{rank}\bar{B} = 2,\ \text{rank}\bar{A} = \text{rank}\bar{A}_1 = \text{rank}\frac{\partial(v_0, v_1, v_2, v_3, v_4, v_5)}{\partial(\delta_1, \delta_2, \delta_7, \delta_4, \delta_8, \delta_9, \delta_{10}, \delta_3, \delta_5, \delta_6)} = 6.
$$

It means that $v_0, v_1, v_2, v_3, v_4, v_5, v_{n+1}$ are linearly dependent and $v_0, v_1, \cdots, v_5$ are linearly independent with each other. That is, when $v_0 = v_1 = \cdots = v_5 = 0$, $\bar{v}_{n+1} = 0$, $n \geq 5$. By Lemma 2.3, $M(h, \delta)$ has at most 5 zeros for $0 < h \ll 1$, multiplicity taken into account. Further, we can vary δ_i such that

$$
0 < |v_0| \ll |v_1| \ll \cdots \ll |v_5| \ll 1,\ v_j v_{j+1} < 0, j = 0, 1, \cdots, 4,
$$

which ensure that $M(h, \delta)$ has 5 isolated positive zeros.

The result for case $n = 2$ ($n = 1$, respectively) follows by taking $\delta_3 = \delta_4 = \delta_6 = \delta_{10} = 0$ ($\delta_3 = \delta_4 = \delta_5 = \delta_6 = \delta_7 = \delta_9 = \delta_{10} = 0$, respectively). Similarly, we obtain the conclusions for system (1.7). The proof of Theorem 1.1 ends.

4. Proof of Theorem 1.2

In this section, we will use ECT-systems and their Wronskians to estimate the upper bound for the number of isolated zeros of the first order Melnikov function for $n = 1, 2, 3$ respectively. Let us consider system (1.6) first.

4.1. Case $n = 1$

By (2.1) in Lemma 2.1, we obtain the expression of $M^+(h, \delta)$

$$
M^+(h, \delta) = h^{\frac{1}{2}} \left(B_{00}^+ + B_{10}^+ h \right), \tag{4.1}
$$

where B_{00}^+ and B_{10}^+ are defined in (3.1). We have further from Lemma 2.2

$$M^-(h,\delta) = -2\sqrt{2}a_{00}^- h^{\frac{1}{2}} + 2\bar{q}_{00}^- \bar{I}_{00}^-. \tag{4.2}$$

Noting that

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}\Big[x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{|a|}\Big] + C,$$

where C is a constant, it follows from (2.3) that

$$\begin{aligned}\bar{I}_{00}^-(h,\delta) &= \int_{-\sqrt{2h+1}}^{-1} \sqrt{2h+1-v^2}dv \\ &= -\frac{\sqrt{2h}}{2} + \frac{1}{2}(2h+1)\Big(\frac{\pi}{2} + \arcsin\frac{-1}{\sqrt{2h+1}}\Big).\end{aligned} \tag{4.3}$$

Substituting the above fomula into (4.2) and combining (1.8) and (4.1), the first order Melnikov function $M(h,\delta)$ can be written as

$$\begin{aligned}M(h) =& h^{\frac{1}{2}}\Big(B_{00}^+ - 2\sqrt{2}a_{00}^- + B_{10}^+ h\Big) + 2\bar{q}_{00}^-\Big[-\frac{\sqrt{2h}}{2} \\ &+ \frac{1}{2}(2h+1)\Big(\frac{\pi}{2} + \arcsin\frac{-1}{\sqrt{2h+1}}\Big)\Big].\end{aligned} \tag{4.4}$$

Let $r = \sqrt{h}$. By (4.4), we have

$$M(r) = c_0 r + c_1(2r^2+1)(\alpha z + 1) + c_2 r^3, \tag{4.5}$$

where

$$\begin{aligned}c_0 =& B_{00}^+ - 2\sqrt{2}a_{00}^- - \sqrt{2}(b_{01}^- + a_{10}^-), c_1 = \frac{\pi}{2}(b_{01}^- + a_{10}^-), \\ c_2 =& B_{10}^+,\ \alpha = \frac{2}{\pi},\ z = \arcsin\frac{-1}{\sqrt{2r^2+1}}.\end{aligned}$$

Through direct calculation, we get

$$\text{rank}\frac{\partial(c_0,c_1,c_2)}{\partial(a_{00}^+, a_{10}^-, a_{10}^+)} = \text{rank}\begin{pmatrix} 2\sqrt{2} & -\sqrt{2} & 0 \\ 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & \frac{4\sqrt{2}}{3a_0}\end{pmatrix} = 3.$$

Therefore, the coefficients c_0, c_1, c_2 in (4.5) are linearly independent.

Let $f_0 = r,\ f_1 = (2r^2+1)(\alpha z+1),\ f_2 = r^3$. Next, we prove that $\{f_0, f_1, f_2\}$ is an ECT-system on $(0,+\infty)$. According to Lemma 2.4, we need to prove that for any $r \in (0,+\infty)$, the following three conclusions hold:

$$W_1 \neq 0,\ W_2 \neq 0,\ W_3 \neq 0.$$

Clearly, we have $W_1 = r \neq 0$ for $r \in (0,+\infty)$. The Wronskian W_2 is provided by

$$W_2 = \begin{vmatrix} r & (2r^2+1)(\alpha z+1) \\ 1 & 4r(\alpha z+1) + \sqrt{2}\alpha \end{vmatrix} = (2r^2-1)(\alpha z+1) + \sqrt{2}\alpha r \equiv g_{12}(r).$$

Note that $g_{12}(0) = 0$ and $z \in (-\frac{\pi}{2}, 0)$. We have

$$g'_{12}(r) = 4r(\alpha z + 1) + \sqrt{2}\alpha(1 - \frac{2}{2r^2+1} + 1) > 4r(\alpha z + 1) > 0.$$

Therefore, $W_2 > 0$ for any $r \in (0, +\infty)$.

Now, we compute the Wronskian W_3

$$\begin{aligned} W_3 &= \begin{vmatrix} r & (2r^2+1)(\alpha z+1) & r^3 \\ 1 & 4r(\alpha z+1) + \sqrt{2}\alpha & 3r^2 \\ 0 & 4(\alpha z+1) + \frac{4\sqrt{2}\alpha r}{2r^2+1} & 6r \end{vmatrix} \\ &= \frac{2r}{2r^2+1}\left[(2r^2+1)(\alpha z+1)(2r-3) + \sqrt{2}\alpha r(2r^2+3)\right] \\ &\equiv \frac{2r}{2r^2+1} g_{13}(r). \end{aligned}$$

The behavior of function $g_{13}(r)$ on $[0, 2]$ is drawn by Maple (see Figure 2). Obviously, $W_3 > 0$ for $r \in (0, 2]$. For $r \in (2, +\infty)$, we have

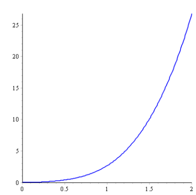


Figure 2. The graphic of function $g_{13}(r)$ for $r \in [0, 2]$

$$g_{13}(r) > (2r^2+1)(\alpha z+1) + \sqrt{2}\alpha r(2r^2+3) > 0.$$

Hence, $W_3 > 0$ for $r > 0$.

Therefore, the ordered set $\{f_0, f_1, f_2\}$ is an ECT-system on $(0, +\infty)$. According to the property of ECT-system, the first order Melnikov function $M(h)$ in (4.4) has at most 2 isolated zeros for $h > 0$, multiplicity taken into account.

4.2. Case $n = 2$

From Lemma 2.1 we have

$$M^+(h, \delta) = h^{\frac{1}{2}}\left[B^+_{00} + (B^+_{10} + B^+_{02})h + +B^+_{20}h^2\right], \tag{4.6}$$

where $B^+_{00}, B^+_{10}, B^+_{02}$ and B^+_{20} are defined in (3.1). The function $M^-(h, \delta)$ has the following expression by (2.2)

$$M^-(h, \delta) = h^{\frac{1}{2}}(-2\sqrt{2}a^-_{00} + P_1 h) + Q_1 \bar{I}^-_{00}, \tag{4.7}$$

where

$$P_1 = -\frac{4\sqrt{2}}{3}\bar{q}^-_{10} - \frac{2\sqrt{2}}{3}a^-_{02}, \quad Q_1 = 2\bar{q}^-_{00} + 2\bar{q}^-_{10}.$$

Inserting (4.3) into (4.7) and combining (4.6), the first order Melnikov function $M(h,\delta)$ can be represented as

$$M(h) = h^{\frac{1}{2}}\left[B_{00}^{+} - 2\sqrt{2}a_{00}^{-} - \frac{Q_1}{\sqrt{2}} + (B_{10}^{+} + B_{02}^{+} + P_1)h + B_{20}^{+}h^2\right] + \frac{\pi Q_1}{4}(2h+1)\Big(1 + \frac{2}{\pi}\arcsin\frac{-1}{\sqrt{2h+1}}\Big). \tag{4.8}$$

Let $r = \sqrt{h}$. We have

$$M(r) = c_0^* r + c_1^*(2r^2+1)(\alpha z+1) + c_2^* r^3 + c_3^* r^5, \tag{4.9}$$

where

$$\alpha = \frac{2}{\pi},\ z = \arcsin\frac{-1}{\sqrt{2r^2+1}},\ c_0^* = B_{00}^{+} - 2\sqrt{2}a_{00}^{-} - \frac{Q_1}{\sqrt{2}},$$
$$c_1^* = \frac{\pi Q_1}{4},\ c_2^* = B_{10}^{+} + B_{02}^{+} + P_1,\ c_3^* = B_{20}^{+}.$$

It is direct that

$$\text{rank}\frac{\partial(c_0^*, c_1^*, c_2^*, c_3^*)}{\partial(a_{00}^{+}, a_{10}^{-}, a_{10}^{+}, a_{20}^{+})} = \text{rank}\begin{pmatrix} 2\sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & \frac{4\sqrt{2}}{3a_0} & 0 \\ 0 & 0 & 0 & \frac{16\sqrt{2}}{15a_0^2} \end{pmatrix} = 4,$$

which implies that the coefficients $c_0^*, c_1^*, c_2^*, c_3^*$ in (4.9) are linearly independent.

Let $f_0 = r,\ f_1 = (2r^2+1)(\alpha z+1),\ f_2 = r^3,\ f_3 = r^5$. Next, we prove that $\{f_0, f_1, f_2, f_3\}$ is an ECT-system on the open interval $(0,+\infty)$.

From the proof in case $n = 1$, we know that $\{f_0, f_1, f_2\}$ is an ECT-system on $(0,+\infty)$. Hence, we only need to prove $W_4 \neq 0$ on $(0,+\infty)$. For the purpose, we calculate the following Wronskian determination

$$W_4 = \begin{vmatrix} r & (2r^2+1)(\alpha z+1) & r^3 & r^5 \\ 1 & 4r(\alpha z+1) + \sqrt{2}\alpha & 3r^2 & 5r^4 \\ 0 & 4(\alpha z+1) + \frac{4\sqrt{2}\alpha r}{2r^2+1} & 6r & 20r^3 \\ 0 & \frac{8\sqrt{2}\alpha}{(2r^2+1)^2} & 6 & 60r^2 \end{vmatrix} = \frac{r^3}{(2r^2+1)^2} g_{24}(r),$$

where

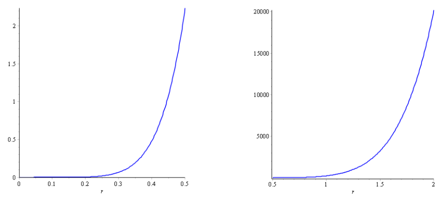
$$g_{24}(r) = 384(\alpha z+1)r^6 + 192\sqrt{2}\alpha r^5 - 576(\alpha z+1)r^4 + 704\sqrt{2}\alpha r^3 - 864(\alpha z+1)r^2 + 240\sqrt{2}\alpha r - 240(\alpha z+1).$$

Part of the image of function $g_{24}(r)$ is shown in the Figure 3. One can see that $W_4 > 0$ for all $r \in (0, 2]$. Next, we consider the determination W_4 for $r \in (2,+\infty)$. Let $u = \alpha z + 1$. It is obvious that $g_{24}(r) > 0$ for all $r \in (2,+\infty)$, since

$$384ur^6 - 576ur^4 = 384ur^4(r^2 - 1.5) > 0,$$

$$
\begin{aligned}
&192\sqrt{2}\alpha r^5 > 0,\\
&704\sqrt{2}\alpha r^3 - 864ur^2 > 633.6r^2(r-1.4) > 0,\\
&240\sqrt{2}\alpha r - 240u > 216(r-1.11) > 0.
\end{aligned}
$$

From Lemma 2.4, $\{f_0, f_1, f_2, f_3\}$ is an ECT-system on the open interval $(0, +\infty)$.



(a) $r \in [0, 0.5]$. (b) $r \in (0.5, 2]$.

Figure 3. The graphic of function $g_{24}(r)$ for $r \in [0, 2]$

Then, the function $M(h)$ in (4.8) has at most 3 isolated zeros for $h > 0$, multiplicity taken into account.

4.3. Case $n = 3$

Combining (3.1)–(3.2) and (4.3), we obtain the expression of first order Melnikov function $M(h, \delta)$ for system (1.6)

$$
\begin{aligned}
M(h) =& h^{\frac{1}{2}}\left[B_{00}^{+} - 2\sqrt{2}a_{00}^{-} - \frac{Q_1}{\sqrt{2}} + \Big(B_{10}^{+} + B_{02}^{+} + P_1 - \frac{Q_2}{\sqrt{2}}\Big)h + (B_{12}^{+} + B_{20}^{+})h^2 + B_{30}^{+}h^3\right]\\
&+ \frac{\pi}{4}\Big[Q_1 + \Big(2Q_1 + Q_2\Big)h + 2Q_2h^2\Big]\Big(1 + \frac{2}{\pi}\arcsin\frac{-1}{\sqrt{2h+1}}\Big).
\end{aligned}
\tag{4.10}
$$

Let $r = \sqrt{h}$. Then, the first order Melnikov function can be rewritten as

$$
M(r) = \tilde{c}_0 r + \tilde{c}_1(2r^2+1)(\alpha z+1) + \tilde{c}_2 r^3 + \tilde{c}_3 r^5 + \tilde{c}_4 r^7 + \tilde{c}_5(\alpha z+1)(r^4 - 0.25), \tag{4.11}
$$

where

$$
\begin{aligned}
\tilde{c}_0 =& B_{00}^{+} - 2\sqrt{2}a_{00}^{-} - \frac{Q_1}{\sqrt{2}},\ \tilde{c}_1 = \frac{\pi}{4}\Big(Q_1 + \frac{1}{2}Q_2\Big),\ \tilde{c}_2 = B_{10}^{+} + B_{02}^{+} + P_1,\\
\tilde{c}_3 =& B_{12}^{+} + B_{20}^{+},\ \tilde{c}_4 = B_{30}^{+},\ \tilde{c}_5 = \frac{\pi}{2}Q_2,\ \alpha = \frac{2}{\pi},\ z = \arcsin\frac{-1}{\sqrt{2r^2+1}}.
\end{aligned}
$$

It is easy to get that

$$
\operatorname{rank}\frac{\partial(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5)}{\partial(a_{00}^{+}, a_{10}^{-}, a_{10}^{+}, a_{20}^{+}, a_{30}^{+}, a_{30}^{-})} = \operatorname{rank}\begin{pmatrix}
2\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0\\
0 & \frac{\pi}{2} & 0 & 0 & 0 & \frac{9\pi}{8}\\
0 & 0 & \frac{4\sqrt{2}}{3a_0} & 0 & 0 & 5\sqrt{2}\\
0 & 0 & 0 & \frac{16\sqrt{2}}{15a_0^2} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{3\sqrt{2}}{8a_0^3} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{3\pi}{2}
\end{pmatrix} = 6.
$$

Hence, $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5$ in (4.11) are linearly independent.

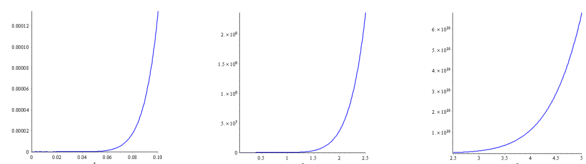
Let $f_0 = r,\ f_1 = (2r^2+1)(\alpha z+1),\ f_2 = r^3,\ f_3 = r^5,\ f_4 = r^7,\ f_5 = (\alpha z + 1)(r^4 - 0.25)$. From the proof in case $n = 2$, we know that $\{f_0, f_1, f_2, f_3\}$ is an ECT-system on $(0, +\infty)$. Hence, we just need to study the properties of W_5 and W_6. The Wronskian W_5 is expressed as follows:

$$W_5 = \begin{vmatrix} r & (2r^2+1)(\alpha z+1) & r^3 & r^5 & r^7 \\ 1 & 4r(\alpha z+1)+\sqrt{2}\alpha & 3r^2 & 5r^4 & 7r^6 \\ 0 & 4(\alpha z+1)+\frac{4\sqrt{2}\alpha r}{2r^2+1} & 6r & 20r^3 & 42r^5 \\ 0 & \frac{8\sqrt{2}\alpha}{(2r^2+1)^2} & 6 & 60r^2 & 210r^4 \\ 0 & -\frac{64\sqrt{2}\alpha r}{(2r^2+1)^3} & 0 & 120r & 840r^3 \end{vmatrix} \equiv \frac{r^6}{(2r^2+1)^3} g_{35}(r),$$

where

$$\begin{aligned} g_{35}(r) =& 184320(\alpha z+1)r^8 + 92160\sqrt{2}\alpha r^7 - 368640(\alpha z+1)r^6 + 586752\sqrt{2}\alpha r^5 \\ &- 829440(\alpha z+1)r^4 + 407040\sqrt{2}\alpha r^3 - 460800(\alpha z+1)r^2 + 80640\sqrt{2}\alpha r \\ &- 80640(\alpha z+1). \end{aligned}$$

The figure of function $g_{35}(r)$ is drawn by Maple as follows. From Figure 4, we know that $g_{35}(r) > 0$ on the interval $(0, 5]$. Next, we prove $g_{35}(r) > 0$ for



(a) $r \in (0, 0.1]$. (b) $r \in (0.1, 2.5]$. (c) $r \in (2.5, 5]$.

Figure 4. The graphic of function $g_{35}(r)$ for $[0, 5]$

$r \in (5, +\infty)$. Let $u = 1 + \alpha z$. Then $g_{35}(r)$ can be rewritten as

$$\begin{aligned} g_{35}(r) =& 96\Big[1920ur^8 + 960\sqrt{2}\alpha r^7 - 3840ur^6 + 6112\sqrt{2}\alpha r^5 - 8640ur^4 + 4240\sqrt{2}\alpha r^3 \\ &- 4800ur^2 + 840\sqrt{2}\alpha r - 840u\Big] \\ \equiv& 96\bar{g}_{35}(r). \end{aligned}$$

Note that $u \in (0.9111, 1)$ for $r \in (5, +\infty)$. For the terms in $\bar{g}_{35}(r)$ we have

$$1920ur^8 - 3840ur^6 > 1749.3r^6(r^2 - 2) > 0 \tag{4.12}$$

$$960\sqrt{2}\alpha r^7 - 8640ur^4 > 960\sqrt{2}\alpha r^4(r^3 - \frac{9\pi}{2\sqrt{2}}) > 0 \tag{4.13}$$

$$6112\sqrt{2}\alpha r^5 - 4800ur^2 > 5500r^2(r^2 - 0.88) > 0 \tag{4.14}$$

$$4240\sqrt{2}\alpha r^3 + 840\sqrt{2}\alpha r - 840u > 3816(r^3 + 0.19r - 0.22) > 0. \tag{4.15}$$

Adding up (4.12)-(4.15) gives that $\bar{g}_{35}(r) > 0$ for all $r \in (5, +\infty)$. Therefore, $W_5 > 0$ for $r > 0$.

Now, we compute Wronskian W_6. For the sake of convenience, let $R^2 = 2r^2 + 1$. Then, W_6 is expressed as follows

$$W_6 = \begin{vmatrix} r & R^2u & r^3 & r^5 & r^7 & u(r^4-0.25) \\ 1 & 4ru+\sqrt{2}\alpha & 3r^2 & 5r^4 & 7r^6 & {f_5}' \\ 0 & 4u+\frac{4\sqrt{2}\alpha r}{R^2} & 6r & 20r^3 & 42r^5 & {f_5}'' \\ 0 & \frac{8\sqrt{2}\alpha}{R^4} & 6 & 60r^2 & 210r^4 & f_5^{(3)} \\ 0 & -\frac{64\sqrt{2}\alpha r}{R^6} & 0 & 120r & 840r^3 & f_5^{(4)} \\ 0 & \frac{64\sqrt{2}\alpha(10r^2-1)}{R^8} & 0 & 120 & 2520r^2 & f_5^{(5)} \end{vmatrix} \equiv \frac{r^2}{R^8}g_{36}(r),$$

where

$$\begin{aligned} {f_5}' =& \frac{\sqrt{2}\alpha(r^4-0.25)}{R^2} + 4r^3u, \\ {f_5}'' =& \frac{4\sqrt{2}\alpha(3r^5+2r^3+0.25r)}{R^4} + 12r^2u, \\ f_5^{(3)} =& \frac{4\sqrt{2}\alpha(9r^4+7r^2+0.25)}{R^4} + 24ru, \\ f_5^{(4)} =& \frac{8\sqrt{2}\alpha(12r^5+16r^3+9r)}{R^6} + 24u, \\ f_5^{(5)} =& \frac{96\sqrt{2}\alpha(-2r^2+1)}{R^8}. \end{aligned}$$

With the help of Maple, we obtain that

$$\begin{aligned} g_{36}(r) =& -768\sqrt{2}\alpha r\Big[{f_5}'\varphi_1 + {f_5}''\varphi_2 + f_5^{(3)}\varphi_3 + f_5^{(4)}\varphi_4 + f_5^{(5)}R^2\varphi_5 + u(r^4-0.25) \\ & \times (6300R^6r + \frac{6300}{\sqrt{2}\alpha}uR^8 - 12600R^4r - 40320R^2r^3 - 67200r^5 + 6720r^3)\Big] \\ \equiv& \frac{-768\sqrt{2}\alpha r}{R^8}\bar{g}_{36}(r), \end{aligned}$$

where

$$\begin{aligned} \varphi_1 =& -\frac{6300}{\sqrt{2}\alpha}uR^8r - 6300R^6r^2 + 12600R^4r^2 + 40320R^2r^4 + 67200r^6 - 6720r^4, \\ \varphi_2 =& \frac{1575}{\sqrt{2}\alpha}uR^8(2r^2-1) + 1575R^8r - 4200R^4r^3 - 16320R^2r^5 - 28800r^7 + 2880r^5, \\ \varphi_3 =& \frac{525}{\sqrt{2}\alpha}uR^8r(-2r^2+3) - 1575R^8r + 2100R^6r^4 + 2880R^2r^6 + 6400r^8 - 640r^6, \\ \varphi_4 =& \frac{30}{\sqrt{2}\alpha}uR^8r^2(8r^2-21) + 630R^8r^3 - 1020R^6r^5 + 360R^4r^5 - 640r^9 + 64r^7, \\ \varphi_5 =& \frac{15}{\sqrt{2}\alpha}uR^6r^3(-2r^2+7) - 105R^6r^4 + 180R^4r^6 - 80R^2r^6 - 64r^8. \end{aligned}$$

The figure of function $\bar{g}_{36}(r)$ is drawn by Maple as follows. From Figure 5, we can see W_6 has precisely 1 isolated zero on the interval $(0, +\infty)$. It is easy to calculate that the zero r_0 of W_6 is close to 0.70639773 and $W_6'(r_0) \neq 0$.

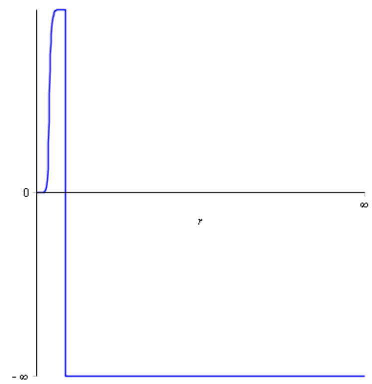


Figure 5. The diagram of function $\bar{g}_{36}(r)$

By Lemma 2.5, the function $M(h)$ in (4.10) has at most 6 isolated zeros on the interval $L = (0, +\infty)$ and 6 zeros can appear for some suitable choice of parameters (ε, δ), multiplicity taken into account.

From above discussion, we can obtain the results for system (1.6) following the Theorem 4.4 in [4]. Make change of variables $(x, t) \to (-x, -t)$, system (1.7) can be transformed into system (1.3) satisfying the condition (1.4). Using the same method as system (1.6), we can obtain the similar conclusions for system (1.7). This ends the proof. □

We remark that different order of set $\mathcal{F}$ leads to different upper bound estimations of zeros of nontrivial function in Span($\mathcal{F}$). For example, consider system (1.6) for case $n = 3$. Let $f_0 = r$, $f_1 = (2r^2+1)(\alpha z+1)$, $f_2 = r^3$, $f_3 = r^5$, $f_4 = (\alpha z+1)(r^4 - 0.25)$, $f_5 = r^7$. Through previous proof, the set $\{f_0, f_1, f_2, f_3\}$ is an ECT-system on $(0, +\infty)$. However, direct calculations show that the Wronskian W_5 has one isolated positive zero in this order. In fact,

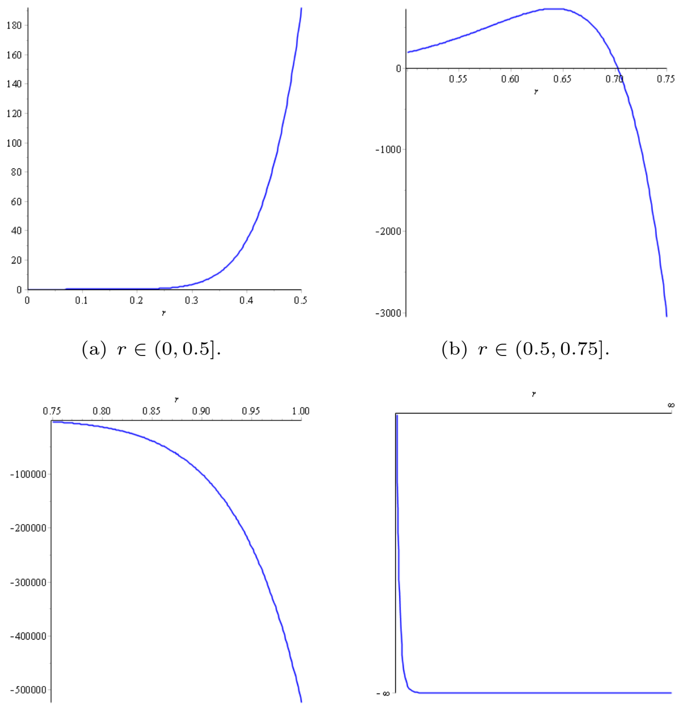
$$
\begin{aligned}
W_5 = &\begin{vmatrix} r & R^2u & r^3 & r^5 & u(r^4-0.25) \\ 1 & 4ru+\sqrt{2}\alpha & 3r^2 & 5r^4 & {f_5}' \\ 0 & 4u+\frac{4\sqrt{2}\alpha r}{R^2} & 6r & 20r^3 & {f_5}'' \\ 0 & \frac{8\sqrt{2}\alpha}{R^4} & 6 & 60r^2 & f_5^{(3)} \\ 0 & -\frac{64\sqrt{2}\alpha r}{R^6} & 0 & 120r & f_5^{(4)} \end{vmatrix} \\
=&\frac{r}{R^6}\sqrt{2}\alpha\Big[{f_5}'\phi_1 + {f_5}''\phi_2 + f_5^{(3)}\phi_3 + f_5^{(4)}\phi_4 + u(r^4-0.25) \\
&\Big(2880R^4r + \frac{2880}{\sqrt{2}\alpha}uR^6 - 5760R^2r - 15360r^3\Big)\Big] \\
\equiv&\frac{r}{R^{12}}\bar{g}_{35}(r),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1 &= -\frac{2880}{\sqrt{2}\alpha}R^6ru - 2880R^4r^2 + 5760R^2r^2 + 15360r^4, \\
\phi_2 &= -\frac{720}{\sqrt{2}\alpha}R^6u(-2r^2+1) + 720R^6r - 1920R^2r^3 - 6144r^5,
\end{aligned}
$$

$$\phi_3 = \frac{240}{\sqrt{2}\alpha} R^6 r u(-2r^2+3) - 720R^6r^2 + 960R^4r^4 + 1024r^6,$$
$$\phi_4 = \frac{240}{\sqrt{2}\alpha} R^6 r^2 u(0.4r^2-1) + 240R^6r^3 - 384R^4r^5 + 128R^2r^5.$$

The figure of function $\bar{g}_{35}(r)$ is drawn in Figure 6.



(a) $r \in (0, 0.5]$. (b) $r \in (0.5, 0.75]$.

(c) $r \in (0.75, 1]$. (d) $r \in (1, \infty)$.

Figure 6. The graphic of function $\bar{g}_{35}(r)$

By Maple, one can see that the Wronskian W_5 has precisely 1 isolated zero r_1 near 0.702581 and $W_5'(r_1) \neq 0$. Then, from Lemma 2.6, we only obtain that the number of zeros of $M(h)$ in (4.10) of system (1.6) does not exceed 7.

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