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Existence of Solutions to a Class of Fractional Differential Equations*

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Abstract In this paper, the existence of solutions to a class of fractional differential equations $D^{\alpha}_{0+}u(t) = h(t)f(t, u(t), D^{\theta}_{0+}u(t))$ is obtained by an efficient and simple monotone iteration method. At first, the existence of a solution to the problem above is guaranteed by finding a bounded domain D_M on functions f and g. Then, sufficient conditions for the existence of monotone solution to the problem are established by applying monotone iteration method. Moreover, two efficient iterative schemes are proposed, and the convergence of the iterative process is proved by using the monotonicity assumption on f and g. In particular, a new algorithm which combines Gauss-Kronrod quadrature method with cubic spline interpolation method is adopted to achieve the monotone iteration method in Matlab environment, and the high-precision approximate solution is obtained. Finally, the main results of the paper are illustrated by some numerical simulations, and the approximate solutions graphs are provided by using the iterative method.

Keywords Fractional differential equation, Monotone iteration method, Numerical simulation, Approximate solutions graphs.

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1. Introduction

At present, fractional differential equation (FDE) is a focused issue concerned by many researchers around the world owing to their various applications in physics, chemistry, biology, dynamical control, engineering and medicine, etc. [1, 10, 19, 22, 24, 28]. Although considerable attention has been paid to the solutions of FDEs (see [3, 6, 9, 14, 16–18, 21, 25, 27] and the references), there are few works [3, 9, 18] on numerical methods which are used to compute approximate solutions of the FDEs whose nonlinear term involves the derivative.

For integral order ordinary differential equations (ODEs), there are many efficient numerical methods, such as Euler method, extrapolation method, monotone iteration method, variational iteration method [2, 4, 5, 7, 12, 13, 20, 23, 26], but it

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is very difficult to solve or to compute approximate solutions of the ODEs whose nonlinear term involves the derivative. For FDEs, it is even harder to obtain approximate solutions. Therefore, the study on numerical methods for FDEs whose nonlinear term involves the derivative is of theoretical and practical significance.

In the first place, literature review has been made on some related studies. In [23], Yao studied the following problem

$$\begin{cases} u^{(4)} = f(t, u(t), u'(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous. With the help of the improved monotonically iterative method, Yao dealt with the nonlinear boundary value problem and obtained the existence and iteration of positive solution to the nonlinear problem. He also provided a useful computational method.

In [7], Edson et al., investigated the following two fourth-order problems. The first problem is

$$\begin{cases} u^{(4)} = f\left(t, u(t), u'(t)\right), \quad t \in (0,1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)). \end{cases}$$

The second one is

$$\begin{cases} u^{(4)} = f\left(t, u(t), u'(t)\right), \quad t \in (0,1), \\ u(0) = u'(0) = 0, \\ u'(1) = 0, \quad u'''(1) = g(u(1)). \end{cases}$$

By using monotone iteration method, the authors proposed a numerical method to compute approximate solutions and obtained monotone positive solutions.

In [26], Zhang discussed an elastic beam equation with a corner

$$\begin{cases} u^{(4)} = q(t) f(t, u(t), u'(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u'(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, and $q(t) : (0,1) \to [0,\infty)$ is a continuous function satisfying $0 < \int_0^1 s^2 q(s) ds < \infty$. By applying monotone iterative techniques, Zhang constructed a successive iterative scheme whose starting point is a simple quadratic function or a zero function, and obtained the existence and iteration of positive solutions to the above boundary value problem.

Motivated by the ideas mentioned above, in this paper, we consider the following FDE whose nonlinear term involves the derivative

$$D^{\alpha}_{0+} u(t) = h(t) f(t, u(t), D^{\theta}_{0+} u(t)), \ t \in [0,1] \tag{1.1}$$

with either the boundary value conditions or the initial value conditions

$$\Gamma_1(t, u(t), D^{\beta}_{0+} u(t)) = 0, \cdots, \Gamma_n(t, u(t), D^{\beta}_{0+} u(t)) = 0, \tag{1.2}$$

where constants $0 < \theta < \alpha \le n,\ 0 < \beta < \alpha,\ D_{0+}^{\alpha},\ D_{0+}^{\theta}$ and D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives of orders $\alpha,\ \theta$ and β. We assume that $\Gamma_1, \cdots, \Gamma_n : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying problems (1.1)-(1.2) that can be reduced to the following integral equation

$$u(t) = \int_0^1 G(t,s)h(s)f(s,u(s),D_{0+}^{\theta}u(s))ds + g(\xi,u(\xi),D_{0+}^{\theta}u(\xi))\varphi(t) \tag{1.3}$$

or

$$u(t) = \int_0^1 G(t,s)h(s)f(s,u(s),D_{0+}^{\theta}u(s))ds, \tag{1.4}$$

where constants $\xi \in [0,1]$, and $\theta > 0,\ G(t,s)$ is the corresponding Green's function of the homogeneous linear problem $D_{0+}^{\alpha}u(t) = 0$ with either the boundary value conditions or the initial value conditions. We assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $h(t) : [0,1] \to [0,\infty)$ are continuous, and $G(t,s)$ is a.e. continuous with respect s and continuous with respect t on $[0,1] \times [0,1]$.

First, according to the functions f and g, a specified bounded domain, where the defined operator is completely continuous is discovered. This ensures that problems (1.1)-(1.2) have at least one positive solution generated by the fixed point theorem. Secondly, by applying monotone iteration method on a standard cone of increasing positive functions, sufficient conditions for the existence of monotone solution to problems (1.1)-(1.2) are established. Thirdly, two efficient iterative schemes are proposed, and the convergence of the iterative process is proved by using the monotonicity assumption on f and g. Moreover, the implementation of the algorithm is presented. Finally, the main results of the paper are illustrated by some numerical simulations, and the approximate solutions graphs are provided by using the iterative method.

It is easy to find that (1.4) is the special case when $g(\xi,u(\xi),D_{0+}^{\theta}u(\xi)) = 0$ or $\varphi(t) = 0$ in (1.3). Therefore, we will only discuss integral equation (1.3).

In contrast to the prerequisite that $f(t,u,v)$ meets locally Lipschitz condition in u and v in other iterative methods [4, 5, 13, 20], we will remove the restriction and consider the monotonicity on f only in a bounded domain. It should be noticed that the monotone iteration method to obtain the existence of positive solution by using the method of lower and upper solutions is not easy to implement [12, 13, 20]. Therefore, in this paper, we use a simple and efficient method to overcome the limitation.

It is widely known that numerical integration methods are relatively mature and of high precision. In Matlab environment, there are some classical algorithm functions of numerical integration methods, such as Trapezoidal method, Simpson method, Runge-Kutta method and Gauss method. However, for solving FDEs by computer simulation, only Trapezoidal algorithm function of Matlab is directly utilized to compile the program code, and it has a lower precision. Through theoretical analysis and experiments, we first combine classical numerical integration algorithm functions with cubic spline interpolation method. Then, we design and compile programs with the Matlab. As a result, we achieve the monotone iteration method by means of a new algorithm, which combines Gauss-Kronrod quadrature method of classical numerical integration with cubic spline interpolation method in Matlab environment, and finally some of the approximate solutions graphs are obtained.

This paper is organized as follows. In Section 2, some preliminary definitions and the related lemmas are given to help readers fully understand this paper. Further, the existence criteria of the solution to problems (1.1)-(1.2) is discussed in Section 3. In Section 4, the sufficient conditions for existence of monotone solution to problems (1.1)-(1.2) is established, and the iterative sequence is constructed. In Section 5, two efficient iterative schemes are proposed, the convergence of the iterative process is discussed and the implementation of the algorithm is presented. In Section 6, some numerical simulations with the approximate solutions graphs are provided to illustrate the efficiency of the iterative method.

2. Preliminaries

To fully appreciate this paper, in this section, we will introduce some basic definitions and the related lemmas.

Definition 2.1 ([15]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}y(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

providing that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([15]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+}y(t) = \tfrac{1}{\Gamma(n-\alpha)} \left(\tfrac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)ds,$$

providing that the right side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$.

Lemma 2.1 (Lemma 1, [11]). *Assume* $\alpha > 0, \beta > 0, u(t) \in L(0,1) \cap C(0,1)$, *then*

(i) $D^{\beta}_{0+}I^{\alpha}_{0+}u(t) = I^{\alpha-\beta}_{0+}u(t),\ \alpha > \beta$;

(ii) $D^{\alpha}_{0+}I^{\alpha}_{0+}u(t) = u(t)$;

(iii) $I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i},\ n-1 < \alpha \le n,\ c_i \in R,\ i = 1, 2, \dots, n,$ $D^{\alpha}_{0+}u \in L(0,1) \cap C(0,1)$;

(iv) $D^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha},\ \beta > -1,\ \beta > \alpha - 1,\ t > 0.$

Definition 2.3 ([27]). Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone for all $x \in P$ and $\lambda \ge 0,\ \lambda x \in P$, and if $x, -x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \preceq y$, if and only if $y - x \in P$. Moreover, if there exists a constant N such that, for all $x, y \in E,\ 0 \preceq x \preceq y$ implies $\|x\| \preceq N\|y\|$, P is called normal, where N is called normality constant of P. Obviously, $N \ge 1$.

Definition 2.4 ([27]). Let E be a real Banach space. If $x,\ y \in E$, the set $[x, y] = \{u \in E : x \preceq u \preceq y\}$ is called the order interval between x and y.

Lemma 2.2 (Lemma 2, [7] (monotone iteration method)). *Let P be a normal cone of a Banach space E and $x_0 \preceq y_0$. Suppose that*

(H1) $T : [x_0, y_0] \subset E \to E$ *is completely continuous;*

(H2) T *is a monotonous increasing in* $[x_0, y_0]$. *That is,* $x_0 \preceq y_0$. *Then,* $Tx_0 \preceq Ty_0$;

(H3) x_0 *is a subsolution of* T*. That is,* $x_0 \preceq Tx_0$*;*

(H4) y_0 *is a supersolution of* T*. That is,* $Ty_0 \preceq y_0$*.*

Then, the iterative sequences

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0,$$

and converge to respectively, $x, y \in [x_0, y_0]$*, which are fixed points of* T*.*

3. Existence of solution

In this section, we will discuss the existence of solution to problems (1.1)-(1.2).

Let $E = \{u : u \in C[0,1], D_{0+}^{\theta}u(t) \in C[0,1]\}$ be a Banach space endowed with norm

$$\|u\| = \max\left\{\max_{t\in[0,1]}|u(t)|,\ \max_{t\in[0,1]}\left|D_{0+}^{\theta}u(t)\right|\right\}. \tag{3.1}$$

An operator $T : C[0,1] \to C[0,1]$ is defined by

$$Tu(t) = \int_0^1 G(t,s)h(s)f(s,u(s),v(s))ds + g(\xi,u(\xi),v(\xi))\varphi(t),$$

where $v(t) = D_{0+}^{\theta}u(t)$.

In the following, we discuss the existence of the solution to problems (1.1)-(1.2). For the sake of convenience, we introduce the following notations:

$$a_1 = \max_{t\in[0,1]}\int_0^1 |G(t,s)h(s)|ds, \quad b_1 = \max_{t\in[0,1]}|\varphi(t)|,$$

$$a_2 = \max_{t\in[0,1]}\int_0^1 |D_t^{\theta}G(t,s)h(s)|ds, \quad b_2 = \max_{t\in[0,1]}|D_{0+}^{\theta}\varphi(t)|,$$

$$\lambda = \max\{a_1+b_1, a_2+b_2\}.$$

Lemma 3.1. *Given* $\Psi \in C[0,1]$*, for any* $M > 0$*,* $\beta > 0$ *and* $\xi \in [0,1]$*. Let*

$$u(t) = \int_0^1 G(t,s)h(s)\Psi(s)ds + \eta(\xi)\varphi(t),$$

$$v(t) = \int_0^1 D_t^{\theta}G(t,s)h(s)\cdot\Psi(s)ds + \eta(\xi)\cdot D_{0+}^{\theta}\varphi(t).$$

If $|\Psi| \le M$*,* $|\eta| \le M$*, then there exist the estimates*

$$|u| \le (a_1+b_1)M \quad \text{and} \quad |v| \le (a_2+b_2)M.$$

Further, there holds $\|u\| \le \gamma$*, where* $\gamma = \lambda M$*.*

Proof. Since $G(t,s)$ and $D_t^{\theta}G(t,s)$ are a.e. continuous with respect s on $[0,1]$, and $G(t,s)$ and $D_t^{\theta}G(t,s)$ are integrable on $[0,1]$. Thus, $|G(t,s)|$ and $|D_t^{\theta}G(t,s)|$ are also integrable on $[0,1]$.

Therefore, according to the assumption, we have

$$|u(t)| \le \int_0^1 |G(t,s)h(s)|\cdot|\Psi(s)|ds + |\eta(\xi)|\cdot|\varphi(t)|$$

$$\begin{aligned}
&\le \int_0^1 |G(t,s)h(s)|M ds + |\varphi(t)|M \\
&\le M\left[\max_{t\in[0,1]}\int_0^1 |G(t,s)h(s)|ds + \max_{t\in[0,1]}|\varphi(t)|\right] \\
&=(a_1+b_1)M.
\end{aligned}$$

Analogously,

$$\begin{aligned}
|v(t)| &\le \int_0^1 |D_t^\theta G(t,s)h(s)|\cdot|\Psi(s)|ds + |\eta(\xi)|\cdot|D_{0+}^\theta\varphi(t)| \\
&\le \int_0^1 |D_t^\theta G(t,s)h(s)|\cdot M ds + M\cdot|D_{0+}^\theta\varphi(t)| \\
&\le M\left[\max_{t\in[0,1]}\int_0^1 |D_t^\theta G(t,s)h(s)|ds + \max_{t\in[0,1]}|D_{0+}^\theta\varphi(t)|\right] \\
&=(a_2+b_2)M.
\end{aligned}$$

Finally, by (3.1), we have

$$\|u\| \le \max\{(a_1+b_1)M,\ (a_2+b_2)M\} = \lambda M = \gamma.$$

□

For any $M>0$, we denote

$$D_M = \{(t,u,v) : 0\le t\le 1,\ |u|\le (a_1+b_1)M,\ |v|\le (a_2+b_2)M\}$$

and

$$D_M^+ = \{(t,u,v) : 0\le t\le 1,\ 0<u\le (a_1+b_1)M,\ 0<v\le (a_2+b_2)M\}.$$

Theorem 3.1. *Assume that f and g are continuous, and there exists a number $M>0$ such that*

(i) $|f(t,u,v)|\le M$*, for any* $(t,u,v)\in D_M$;

(ii) $|g(t,u,v)|\le M$*, for any* $(t,u,v)\in D_M$.

Then, problems (1.1)-(1.2) *have at least a solution* $u(t)\in C[0,1]$.

Proof. Let

$$B_\gamma = \{u\in E : \|u\|\le\gamma\}.$$

Now, we prove that operator T has a fixed point in B_γ. First, for $t\in[0,1]$, by the means of conditions (i) and (ii), for $u\in B_\gamma$, we have

$$\begin{aligned}
|Tu| &= \left|\int_0^1 G(t,s)h(s)f(s,u(s),v(s))ds + g(\xi,u(\xi),v(\xi))\varphi(t)\right| \qquad (3.2)\\
&\le \int_0^1 |G(t,s)h(s)|\cdot|f(s,u(s),v(s))|ds + |g(\xi,u(\xi),v(\xi))|\cdot|\varphi(t)| \\
&\le M\int_0^1 |G(t,s)h(s)|ds + M|\varphi(t)|
\end{aligned}$$

$$\leq M \max_{t\in[0,1]} \int_0^1 |G(t,s)h(s)|ds + M \max_{t\in[0,1]} |\varphi(t)|$$
$$= M(a_1 + b_1) \leq \gamma$$

and

$$\begin{aligned} &|D_{0+}^\theta Tu(t)| \\ =&\left| \int_0^1 D_t^\theta G(t,s)h(s)f(s,u(s),v(s))ds + g(\xi,u(\xi),v(\xi))D_{0+}^\theta\varphi(t) \right| \\ \leq& \int_0^1 |D_t^\theta G(t,s)h(s)| \cdot |f(s,u(s),v(s))|ds + |g(\xi,u(\xi),v(\xi))| \cdot |D_{0+}^\theta\varphi(t)| \\ \leq& M \max_{t\in[0,1]} \int_0^1 |D_t^\theta G(t,s)h(s)|ds + M \max_{t\in[0,1]} |D_{0+}^\theta\varphi(t)| \\ =& M(a_2 + b_2) \leq \gamma. \end{aligned} \tag{3.3}$$

Thus,

$$\|Tu\| = \max\left\{ \max_{t\in[0,1]} |Tu(t)|, \ \max_{t\in[0,1]} \left|D_{0+}^\theta Tu(t)\right| \right\} \leq \gamma.$$

On the other hand, by the definition of the operator T, we know that $Tu \in C[0,1]$. Then, we infer that $Tu \in E$, which implies $TB_\gamma \subset B_\gamma$. Hence, T maps B_γ into B_γ.

Finally, we prove that $T(B_\gamma)$ is equicontinuous. We infer from the continuity of $\varphi(t)$, $h(t)$, $f(t,u,v)$ and $g(t,u,v)$, and the almost everywhere continuity of $G(t,s)$ and $D_t^\theta G(t,s)$ with respect s and the continuity of with respect t that the operator T is continuous.

Since $G(t,s)$ and $D_t^\theta G(t,s)$ are continuous with respect t on $[0,1]$, $G(t,s)h(s)$ and $D_t^\theta G(t,s)h(s)$ are uniformly continuous with respect t on $[0,1]$. Thus, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, whenever $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta_1$,

$$|G(t_2,s)h(s) - G(t_1,s)h(s)| < \tfrac{\varepsilon}{2M},$$

$$|\varphi(t_2) - \varphi(t_1)| < \tfrac{\varepsilon}{2M},$$

and there exists $\delta_2 > 0$, whenever $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta_2$,

$$\left|D_t^\theta G(t_2,s)h(s) - D_t^\theta G(t_1,s)h(s)\right| < \tfrac{\varepsilon}{2M},$$

$$\left|D_{0+}^\theta\varphi(t_2) - D_{0+}^\theta\varphi(t_1)\right| < \tfrac{\varepsilon}{2M}.$$

By conditions (i) and (ii), $\forall u \in B_\gamma$, for $\varepsilon > 0$ above, take $\delta = \min\{\delta_1, \delta_2\}$, whenever $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} &|Tu(t_2) - Tu(t_1)| \\ =&\left| \int_0^1 [G(t_2,s)h(s) - G(t_1,s)h(s)]f(s,u,v))ds + g(\xi,u,v)[\varphi(t_2) - \varphi(t_1)] \right| \\ \leq& \int_0^1 |G(t_2,s)h(s) - G(t_1,s)h(s)||f(s,u,v))|ds + |g(\xi,u,v)||\varphi(t_2) - \varphi(t_1)| \end{aligned}$$

$$\leq M\int_0^1 |G(t_2,s)h(s)-G(t_1,s)h(s)|ds + M|\varphi(t_2)-\varphi(t_1)|$$
$$< M\frac{\varepsilon}{2M}+M\frac{\varepsilon}{2M}=\varepsilon$$

and

$$\begin{aligned}&|TD_{0+}^{\theta}u(t_2)-TD_{0+}^{\theta}u(t_1)|\\ \leq&\int_0^1 |D_t^{\theta}G(t_2,s)h(s)-D_t^{\theta}G(t_1,s)h(s)|\cdot|f(s,u(s),v(s))|ds\\ &+|g(\xi,u(\xi),v(\xi))|\cdot|D_{0+}^{\theta}\varphi(t_2)-D_{0+}^{\theta}\varphi(t_1)|\\ \leq& M\int_0^1 |D_t^{\theta}G(t_2,s)h(s)-D_t^{\theta}G(t_1,s)h(s)|ds+M|D_{0+}^{\theta}\varphi(t_2)-D_{0+}^{\theta}\varphi(t_1)|\\ <&\varepsilon,\end{aligned}$$

Thus, $T(B_\gamma)$ is equicontinuous. By Arzela-Ascoli theorem, T is completely continuous on E. Therefore, it has a fixed point by the Schauder's fixed point theorem. □

Remark 3.1. Under all the assumptions of Theorem 3.1, if $G(t,s)h(s)>0,\ \varphi(t)>0$ for $t,s\in[0,1]$, and

(i) $0<f(t,u,v)\leq M$, for any $(t,u,v)\in D_M^+,\ t\in[0,1]$;

(ii) $0<g(t,u,v)\leq M$, for any $(t,u,v)\in D_M^+,\ t\in[0,1]$.

Then, problems (1.1)-(1.2) have at least a positive solution $u(t)\in C[0,1]$.

4. Monotone solutions

In this section, we will further explore existence criteria of monotone solution to problems (1.1)-(1.2) by monotone iteration method.

Now, we define a cone $P\subset E$ by

$$P=\{u\in E: D_{0+}^{\theta}u\in E,\ t\in[0,1]\}.$$

Then, the cone P induces an order relation $\preceq$ in E given by

$$x\preceq y \quad \text{,if and only if} \quad x\leq y \quad \text{and} \quad D_{0+}^{\theta}x\leq D_{0+}^{\theta}y.$$

Theorem 4.1. *Assume f and g are continuous, $G(t,s)h(s)>0,\ \varphi(t)>0$ for $t,s\in[0,1]$, and there exists a number $M>0$ such that*

(C1) $-M\leq f(t,u_1,v_1)\leq f(t,u_2,v_2)\leq M$, *for* $0\leq t\leq 1,\ 0\leq u_1\leq u_2\leq\gamma,\ 0\leq v_1\leq v_2\leq\gamma$;

(C2) $-M\leq g(t,u_1,v_1)\leq g(t,u_2,v_2)\leq M$, *for* $0\leq t\leq 1,\ 0\leq u_1\leq u_2\leq\gamma,\ 0\leq v_1\leq v_2\leq\gamma$;

(C3) $f(t,0,0)\neq 0$ *or* $f(t,\gamma,\gamma)\neq 0$, *for* $0\leq t\leq 1$;

Then, choosing

$$x_0(t) = \int_0^1 G(t,s)h(s)f(s,0,0)ds + g(\xi,0,0)\varphi(t)$$

and

$$y_0(t) = \int_0^1 G(t,s)h(s)f(s,\gamma,\gamma)ds + g(\xi,\gamma,\gamma)\varphi(t),$$

the iterative sequences

$$x_n = Tx_{n-1} \quad \textit{and} \quad y_n = Ty_{n-1}, \quad n = 1,2,3,\cdots \tag{4.1}$$

converge, in norm $\|\cdot\|$, *to monotone positive solutions* x, y *of problems* (1.1)-(1.2) *and satisfy*

$$0 \le x_0 \le x_1 \le \cdots \le x_n \le x \le y \le y_n \le \cdots \le y_1 \le y_0 \le \gamma,$$

$$0 \le D_{0+}^{\theta}x_0 \le D_{0+}^{\theta}x_1 \le \cdots \le D_{0+}^{\theta}x_n \le D_{0+}^{\theta}x \le D_{0+}^{\theta}y \le D_{0+}^{\theta}y_n \le \cdots$$
$$\le D_{0+}^{\theta}y_1 \le D_{0+}^{\theta}y_0 \le \gamma.$$

Proof. It follows from the proof of Theorem 3.1 that the operator T is completely continuous. We infer from (3.2) and (3.3) that $\|x_0\| \le \gamma$ and $\|y_0\| \le \gamma$. Hence, $u \in [x_0, y_0]$ implies $\|u\| \le \gamma$.

Now, we show that T is monotone increasing. Suppose $x \preceq y$. Then,

$$x \le y \quad \text{and} \quad D_{0+}^{\theta}x \le D_{0+}^{\theta}y, \quad t \in [0,1].$$

By assumptions (C1) and (C2), we have

$$f(t,x,D_{0+}^{\theta}x) \le f(t,y,D_{0+}^{\theta}y)$$

and

$$g(\xi,x(\xi),D_{0+}^{\theta}x(\xi)) \le g(\xi,y(\xi),D_{0+}^{\theta}y(\xi)).$$

Thus,

$$\begin{aligned} Tx &= \int_0^1 G(t,s)h(s)f(s,x,D_{0+}^{\theta}x)ds + g(\xi,x(\xi),D_{0+}^{\theta}x(\xi))\varphi(t) \\ &\le \int_0^1 G(t,s)h(s)f(s,y,D_{0+}^{\theta}y)ds + g(\xi,y(\xi),D_{0+}^{\theta}y(\xi))\varphi(t) = Ty. \end{aligned} \tag{4.2}$$

Analogously, we obtain

$$T(D_{0+}^{\theta}x) \le T(D_{0+}^{\theta}y). \tag{4.3}$$

Hence, by (4.2) and (4.3), we have $Tx \preceq Ty$, which implies that T is monotone increasing.

Next, we check that x_0 is a subsolution of T. By using $0 \le x_0 \le x_1$, $0 \le D_{0+}^{\theta}x_0 \le D_{0+}^{\theta}x_1$ and the monotonicity assumption on f and g, we have

$$\begin{aligned} x_1 &= Tx_0 \\ &= \int_0^1 G(t,s)h(s)f(s,x_0(s),D_{0+}^{\theta}x_0(s))ds + g(\xi,x_0(\xi),D_{0+}^{\theta}x_0(\xi))\varphi(t) \end{aligned}$$

$$\geq \int_0^1 G(t,s)h(s)f(s,0,0)ds + g(\xi,0,0)\varphi(t) = x_0$$

and

$$\begin{aligned} &D^\theta_{0+}x_1 = T(D^\theta_{0+}x_0) \\ =&\int_0^1 D^\theta_t G(t,s)h(s)f(s,x_0,D^\theta_{0+}x_0)ds + g(\xi,x_0(\xi),D^\theta_{0+}x_0(\xi))\varphi(t) \\ \geq&\int_0^1 D^\theta_t G(t,s)h(s)f(s,0,0)ds + g(\xi,0,0)D^\theta_{0+}\varphi(t) = D^\theta_{0+}x_0, \end{aligned}$$

which mean that $x_0 \preceq Tx_0$. Thus, by induction, we have $x_n \preceq Tx_n,\ n = 0,1,2,\cdots$.

Analogously, we check that y_0 is a supersolution of T. Again, by the means of $y_1 \leq y_0 \leq \gamma$, $D^\theta_{0+}y_1 \leq D^\theta_{0+}y_0 \leq \gamma$ and the monotonicity assumption on f and g, we have

$$\begin{aligned} y_1 = Ty_0 =& \int_0^1 G(t,s)h(s)f(s,y_0,D^\theta_{0+}y_0)ds + g(\xi,x_0(\xi),D^\theta_{0+}x_0(\xi))\varphi(t) \\ \leq& \int_0^1 G(t,s)h(s)f(s,\gamma,\gamma)ds + g(\xi,\gamma,\gamma)\varphi(t) \\ =& y_0 \end{aligned}$$

and

$$\begin{aligned} y_1 =& T(D^\theta_{0+}y_0) \\ =& \int_0^1 D^\theta_t G(t,s)h(s)f(s,y_0,D^\theta_{0+}y_0)ds + g(\xi,x_0(\xi),D^\theta_{0+}x_0(\xi))\varphi(t) \\ \leq& \int_0^1 D^\theta_t G(t,s)h(s)f(s,\gamma,\gamma)ds + g(\xi,\gamma,\gamma)D^\theta_{0+}\varphi(t) \\ =& D^\theta_{0+}y_0, \end{aligned}$$

which imply that $Ty_0 \preceq y_0$. By induction, we have $Ty_n \preceq y_n,\ n = 0,1,2,\cdots$. Thus, we obtain the iteration sequences (4.1). By Lemma 2.2, these sequences converge to the monotone solutions $x, y \in [x_0, y_0] \subset P$ of problems (1.1)-(1.2), with $x \leq y$, which are fixed points of the operator T. □

Corollary 4.1. *Assume that f and g are continuous, $G(t,s)h(s) > 0,\ \varphi(t) > 0$ for $t,s \in [0,1]$, and there exists a number $M > 0$ such that*

(C1) $-M \leq f(t,u_1,v_1) \leq f(t,u_2,v_2) \leq M$ *for* $0 \leq t \leq 1,\ -\gamma \leq u_1 \leq u_2 \leq 0,\ -\gamma \leq v_1 \leq v_2 \leq 0$;

(C2) $-M \leq g(t,u_1,v_1) \leq g(t,u_2,v_2) \leq M$ *for* $0 \leq t \leq 1,\ -\gamma \leq u_1 \leq u_2 \leq 0,\ -\gamma \leq v_1 \leq v_2 \leq 0$;

(C3) $f(t,0,0) \neq 0$ *or* $f(t,-\gamma,-\gamma) \neq 0$ *for* $0 \leq t \leq 1$.

Then, choosing

$$x_0(t) = \int_0^1 G(t,s)h(s)f(s,0,0)ds + g(\xi,0,0)\varphi(t)$$

and

$$y_0(t) = \int_0^1 G(t,s)h(s)f(s,-\gamma,-\gamma)ds + g(\xi,-\gamma,-\gamma)\varphi(t),$$

the iterative sequences

$$x_n = Tx_{n-1} \quad \textit{and} \quad y_n = Ty_{n-1}, \quad n = 1,2,3,\cdots$$

converge, in norm $\|\cdot\|$, *to monotone negative solutions* x, y *of problems* (1.1)-(1.2) *and satisfy*

$$-\gamma \le y_0 \le y_1 \le \cdots \le y_n \le y \le x \le x_n \le \cdots \le x_1 \le x_0 \le 0,$$

$$\begin{aligned}&-\gamma \le D_{0+}^\theta y_0 \le D_{0+}^\theta y_1 \le \cdots \le D_{0+}^\theta y_n \le D_{0+}^\theta y \le D_{0+}^\theta x \le D_{0+}^\theta x_n \le \cdots \\ &\le D_{0+}^\theta x_1 \le D_{0+}^\theta x_0 \le 0.\end{aligned}$$

5. Two iterative schemes

There exist two iterative schemes as follows. Considering the following iterative process, we construct an increasing iterative sequence.

(A1) Given

$$\begin{aligned}u_0(t) &= \int_0^1 G(t,s)h(s)f(s,0,0)ds + g(\xi,0,0)\varphi(t),\ t \in [0,1],\\ v_0(t) &= \int_0^1 D_t^\theta G(t,s)h(s)f(s,0,0)ds + g(\xi,0,0)D_{0+}^\theta\varphi(t),\ t \in [0,1].\end{aligned}$$

(A2) Update $(k = 0,1,2,\cdots)$

$$\begin{aligned}u_{k+1}(t) =&Tu_k(t)\\ =&\int_0^1 G(t,s)h(s)f(s,u_k,v_k)ds + g(\xi,u_k(\xi),v_k(\xi))\varphi(t),\ t \in [0,1],\\ v_{k+1}(t) =&Tv_k(t)\\ =&\int_0^1 D_t^\theta G(t,s)h(s)f(s,u_k,v_k)ds + g(\xi,u_k(\xi),v_k(\xi))D_{0+}^\theta\varphi(t),\ t \in [0,1].\end{aligned}$$

Suppose problems (1.1)-(1.2) can be reduced to the integral equation of the form as (1.3), and has the positive solution. Then, we construct a decreasing iterative sequence by considering the following iterative process.

(B1) Given

$$\begin{aligned}u_0(t) &= \int_0^1 G(t,s)h(s)f(s,\gamma,\gamma)ds + g(\xi,\gamma,\gamma)\varphi(t),\ t \in [0,1],\\ v_0(t) &= \int_0^1 D_t^\theta G(t,s)h(s)f(s,\gamma,\gamma)ds + g(\xi,\gamma,\gamma)D_{0+}^\theta\varphi(t),\ t \in [0,1].\end{aligned}$$

(B2) Update $(k = 0, 1, 2, \cdots)$

$$\begin{aligned}u_{k+1}(t) =& Tu_k(t)\\=&\int_0^1 G(t,s)h(s)f(s,u_k,v_k)ds + g(\xi,u_k(\xi),v_k(\xi))\varphi(t),\ t\in[0,1],\\v_{k+1}(t) =& Tv_k(t)\\=&\int_0^1 D_t^\theta G(t,s)h(s)f(s,u_k,v_k)ds + g(\xi,u_k(\xi),v_k(\xi))D_{0+}^\theta\varphi(t),\ t\in[0,1].\end{aligned}$$

Remark 5.1. Suppose problems (1.1)-(1.2) can be reduced to the integral equation of the form as (1.3), and has the nonnegative solution. Then, we can construct an increasing iterative sequence by replacing $f(s,\gamma,\gamma)$ and $g(\xi,\gamma,\gamma)$ with $f(s,-\gamma,-\gamma)$ and $g(\xi,-\gamma,-\gamma)$ in (B1) and (B2) respectively.

Remark 5.2. Choosing one of the two iterative schemes above, we can construct a successively iterative sequence and obtain the approximate solution to problems (1.1)-(1.2).

Theorem 5.1. *Under all the assumptions of Theorem* 4.1*, and there exists* $0 < \sigma < \frac{1}{\lambda}$ *such that*

$$\begin{aligned}|f(t,u_n,v_n) - f(t,u_{n-1},v_{n-1})| \le \sigma\|u_n - u_{n-1}\|,\\|g(\xi,u_n,v_n) - g(\xi,u_{n-1},v_{n-1})| \le \sigma\|u_n - u_{n-1}\|,\end{aligned} \tag{5.1}$$

for any $0 \le t \le 1,\ -\gamma \le u_1 \le u_2 \le \gamma,\ -\gamma \le v_1 \le v_2 \le \gamma,\ n = 1,\ 2, \cdots$. *Then, the iterative method above converges with the rate of geometric progression, and there exists the estimate*

$$\|u_n - u_{exact}\| \le \frac{\lambda^n\sigma^n}{1-\lambda\sigma}\|u_1 - u_0\|,$$

where u_{exact} *is the exact solution of problems* (1.1)-(1.2)*.*

Proof. According to (5.1), for any $0 \le t \le 1,\ u_{n-1} \le u_n,\ v_{n-1} \le v_n$, we have

$$\begin{aligned}\max_{t\in[0,1]}|u_{n+1}(t) - u_n(t)| =& \max_{t\in[0,1]}|Tu_n(t) - Tu_{n-1}(t)|\\\le& \max_{t\in[0,1]}\int_0^1 |G(t,s)h(s)|\cdot|f(s,u_n,v_n) - f(s,u_{n-1},v_{n-1})|\,ds\\&+ |g(\xi,u_n(\xi),v_n(\xi)) - g(\xi,u_{n-1}(\xi),v_{n-1}(\xi))|\cdot\max_{t\in[0,1]}|\varphi(t)|\\\le& \max_{t\in[0,1]}\int_0^1 |G(t,s)h(s)|\cdot\sigma\|u_n - u_{n-1}\|ds + \sigma\|u_n - u_{n-1}\|\\&\cdot\max_{t\in[0,1]}|\varphi(t)|\\=&\sigma\|u_n - u_{n-1}\|\left[\max_{t\in[0,1]}\int_0^1|G(t,s)h(s)|ds + \max_{t\in[0,1]}|\varphi(t)|\right]\\=&\sigma\|u_n - u_{n-1}\|(a_1 + b_1)\end{aligned}$$

and

$$\max_{t\in[0,1]}|D_{0+}^\theta u_{n+1} - D_{0+}^\theta u_n|$$

$$
\begin{aligned}
&= \max_{t\in[0,1]} |T(D_{0+}^{\theta}u_n) - T(D_{0+}^{\theta}u_{n-1})| \\
&\le \max_{t\in[0,1]} \int_0^1 |D_t^{\theta}G(t,s)h(s)|\sigma\|u_n - u_{n-1}\|ds + \sigma\|u_n - u_{n-1}\| \max_{t\in[0,1]} |D_{0+}^{\theta}\varphi(t)| \\
&=\sigma\|u_n - u_{n-1}\| \left[\max_{t\in[0,1]} \int_0^1 |D_t^{\theta}G(t,s)h(s)|ds + \max_{t\in[0,1]} |D_{0+}^{\theta}\varphi(t)|\right] \\
&=\sigma\|u_n - u_{n-1}\|(a_2 + b_2).
\end{aligned}
$$

Therefore,

$$\|u_{n+1}-u_n\| \le \max\{a_1+b_1,\ a_2+b_2\}\cdot\sigma\|u_n-u_{n-1}\| = \lambda\sigma\|u_n-u_{n-1}\|,\ (n = 1, 2, \cdots).$$

Moreover,

$$\|u_{n+1} - u_n\| \le \lambda\sigma\|u_n - u_{n-1}\| \le (\lambda\sigma)^2 \|u_{n-1} - u_{n-2}\| \le \cdots \le (\lambda\sigma)^n \|u_1 - u_0\|.$$

By the above estimation, we have

$$
\begin{aligned}
&\|u_{n+m+1} - u_n\| \\
\le&\|u_{n+m+1} - u_{n+m}\| + \|u_{n+m} - u_{n+m-1}\| + \cdots + \|u_{n+1} - u_n\| \\
\le& (\lambda\sigma)^{n+m} \|u_1 - u_0\| + (\lambda\sigma)^{n+m-1} \|u_1 - u_0\| + \cdots + (\lambda\sigma)^n \|u_1 - u_0\| \\
=& (\lambda\sigma)^n \frac{\left[1 - (\lambda\sigma)^{m+1}\right]}{1 - \lambda\sigma} \|u_1 - u_0\|.
\end{aligned}
$$

Hence,

$$\|u_n - u_{exact}\| = \lim_{m\to+\infty} \|u_n - u_{n+m+1}\| \le \frac{\lambda^n\sigma^n}{1 - \lambda\sigma} \|u_1 - u_0\|.$$

□

In the following section, we provide some examples to illustrate the our theoretical result. We choose four examples in which the exact solutions of the problems are known or unknown. The first example is a simple and ordinary initial value problem of FDE for verifying the applicability of our proposed approach. The second one is the FDE problem, whose expression of the solution include a coefficient that needs to be iterated and updated. The third one is a boundary value problem of FDE with p-Laplacian operator. The last one is a boundary value problem for a coupled system of FDE. Besides, the known exact solutions are used to compare with approximation results on the basis of the absolute error for demonstrating the applicability of our approach.

For realization of the iterative method, we use numerical integration method for differential equations. Thus, we can use the classical method such as Trapezoidal method, Simpson method, Runge-Kutta method and Gauss method. To obtain high-precision approximate solution, in this paper, we achieve the monotone iteration method by a new algorithm, which combines Gauss-Kronrod quadrature method of classical numerical integration with cubic spline interpolation method in Matlab environment.

Finally, in all examples, we use the uniform grid with the number of grid points $N = 100$. The simulations experiments are performed till the iteration error $e(k) = \max_{t\in[0,1]} |u_{k+1}-u_k| \le 10^{-16}$. In addition, the absolute error is obtained by $E(k) = \max_{t\in[0,1]} |u(t) - u_{exact}|$, where u_{exact} is the exact solution.

6. Examples

Example 6.1. Consider the following FDE

$$\begin{cases} D_{0+}^{\alpha} u(t) = h(t) f(t, u, u'), \quad t \in (0,1)\,, \\ u(0) = 0, \quad 0 < \alpha \leq 1, \end{cases} \tag{6.1}$$

where

$$h(t) = t^{\mu},$$

and

$$f\,(t, u, u') = \frac{\Gamma(4+\mu)t^{3-\alpha}}{\Gamma(4+\mu-\alpha)} + \frac{2\Gamma(3+\mu)t^{2-\alpha}}{\Gamma(3+\mu-\alpha)} + \frac{\Gamma(2+\mu)t^{1-\alpha}}{\Gamma(2+\mu-\alpha)} - 1.5t^{\mu}(t^3+2t^2+t) + 1.5u.$$

By comparison, we obtain that the exact solution is

$$u(t) = t^{\mu}(t^3 + 2t^2 + t).$$

Below we will find the approximation of the solution for problem (6.1) by applying the proposed method in this paper, and then give a table of the absolute errors between the approximation and the exact solution.

According to Definition 2.1 and Lemma 2.1, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f(s, u, u') ds.$$

Thus, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f(s, u, u') ds - c_1 t^{\alpha-1}.$$

By means of the initial condition $u(0) = 0$, we get $c_1 = 0$. Hence,

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f(s, u, u') ds \\ =& \int_0^1 G(s,t) h(s) f\,(s, u, u')\, ds, \end{aligned}$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 \leq s < t \leq 1, \\ 0, \quad 0 \leq t < s \leq 1, \end{cases}$$

which is the corresponding Green's function.

This means that problem (6.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(s,t) h(s) f\,(s, u, u')\, ds.$$

Thus,

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ 0, \quad 0 \le t \le s \le 1. \end{cases}$$

Now, we take $\alpha = 0.5,\ \mu = 2.5$. Notice that

$$\begin{aligned} a_1 &= \max_{t\in[0,1]} \int_0^1 |G(t,s)h(s)|ds \\ &= \max_{t\in[0,1]} \int_0^t \frac{(t-s)^{\alpha-1}s^{\mu}ds}{\Gamma(\alpha)} \\ &= \max_{t\in[0,1]} \int_0^1 \frac{t^{\alpha+\mu}(1-r)^{\alpha-1}r^{\mu}dr}{\Gamma(\alpha)} \\ &= \max_{t\in[0,1]} \frac{t^{\alpha+\mu}\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} \approx 0.5539 \end{aligned}$$

and

$$\begin{aligned} a_2 &= \max_{t\in[0,1]} \int_0^1 \left| \frac{\partial G(t,s)}{\partial t} h(s) \right| ds \\ &= \max_{t\in[0,1]} \int_0^t \frac{(t-s)^{\alpha-2}s^{\mu}ds}{\Gamma(\alpha-1)} \\ &= \max_{t\in[0,1]} \int_0^1 \frac{t^{\alpha+\mu-1}(1-r)^{\alpha-2}r^{\mu}dr}{\Gamma(\alpha-1)} \\ &= \max_{t\in[0,1]} \frac{t^{\alpha+\mu-1}\Gamma(\mu+1)}{\Gamma(\alpha+\mu)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu)} \approx 1.6617. \end{aligned}$$

Now, we choose a suitable $M > 0$ such that Theorem 3.1 holds in D_M. In view of Lemma **??**, we have $|u| \le a_1 M$. Moreover, by condition (i) of Theorem 3.1, M need satisfied the inequality

$$|f| \le \frac{\Gamma(4+\mu)}{\Gamma(4+\mu-\alpha)} + \frac{2\Gamma(3+\mu)}{\Gamma(3+\mu-\alpha)} + \frac{\Gamma(2+\mu)}{\Gamma(2+\mu-\alpha)} + 1.5a_1 M \le M.$$

Namely,

$$8.6996 + 1.5a_1 M \le M.$$

Choosing $M = 52$, the above inequality holds. Thus,

$$\gamma = \max\{a_1 M, a_2 M\} \approx 86.4084.$$

On the other hand,

$$f(t,0,0) = \frac{\Gamma(4+\mu)t^{3-\alpha}}{\Gamma(4+\mu-\alpha)} + \frac{2\Gamma(3+\mu)t^{2-\alpha}}{\Gamma(3+\mu-\alpha)} + \frac{\Gamma(2+\mu)t^{1-\alpha}}{\Gamma(2+\mu-\alpha)} - 1.5t^{\mu}(t^3+2t^2+t) \ne 0.$$

This implies that all the conditions of Theorem 3.1 are satisfied in D_{52}. Further, it is easy to check that f satisfies the hypotheses of Theorem 4.1. Therefore, problem (6.1) has at least a solution, and the iterative method converges.

Take the initial approximation $u_0^{(1)}(t)$ or $u_0^{(2)}(t)$, namely,

$$\begin{aligned} u_0^{(1)}(t) &= \int_0^1 G(s,t)h(s)f(s,0,0)\,ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}s^{\mu}}{\Gamma(\alpha)}\Big[\frac{\Gamma(4+\mu)s^{3-\alpha}}{\Gamma(4+\mu-\alpha)} + \frac{2\Gamma(3+\mu)s^{2-\alpha}}{\Gamma(3+\mu-\alpha)} + \frac{\Gamma(2+\mu)s^{1-\alpha}}{\Gamma(2+\mu-\alpha)} \\ &\quad -1.5s^{\mu}(s^3+2s^2+s)\Big]\,ds \end{aligned}$$

or

$$\begin{aligned} u_0^{(2)}(t) &= \int_0^1 G(s,t)h(s)f(s,\gamma,\gamma)\,ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}s^{\mu}}{\Gamma(\alpha)}\Big[\frac{\Gamma(4+\mu)s^{3-\alpha}}{\Gamma(4+\mu-\alpha)} + \frac{2\Gamma(3+\mu)s^{2-\alpha}}{\Gamma(3+\mu-\alpha)} + \frac{\Gamma(2+\mu)s^{1-\alpha}}{\Gamma(2+\mu-\alpha)} \\ &\quad - 1.5s^{\mu}(s^3+2s^2+s) + 1.5\gamma\Big]\,ds. \end{aligned}$$

Then, the following iteration formulation is obtained:

$$\begin{aligned} u_{k+1}(t) =& Tu_k(t) \\ =& \int_0^1 G(t,s)h(s)f(s,u_k,v_k)\,ds \\ =& \int_0^t \frac{(t-s)^{\alpha-1}s^{\mu}}{\Gamma(\alpha)}\Big[\frac{\Gamma(4+\mu)s^{3-\alpha}}{\Gamma(4+\mu-\alpha)} + \frac{2\Gamma(3+\mu)s^{2-\alpha}}{\Gamma(3+\mu-\alpha)} + \frac{\Gamma(2+\mu)s^{1-\alpha}}{\Gamma(2+\mu-\alpha)} \\ & -1.5s^{\mu}(s^3+2s^2+s) + 1.5u_k\Big]\,ds, \quad (k=1,2,\cdots). \end{aligned}$$

In the simulation experiment, we apply the new algorithm, which combines Gauss-Kronrod quadrature method of classical numerical integration with cubic spline interpolation method in order to achieve the monotone iteration method in Matlab environment. The simulations experiment shows that for $N = 100$, the iterative method achieves the accuracy $E_{29} = 8.423179 \times 10^{-9}$, after $k = 29$ iterations. Some approximations of the solution with different initial approximations are depicted in Figure 1. Comparison of the exact solution with two convergent solutions is depicted in Figure 2. The absolute errors of the iterative method for problem (6.1) are given in Table 1. From Figure 1, it is easy to see that the approximation u_{29}, and the exact solution u_{exact} almost coincide. It implies that iterative method converges very fast. This demonstrates the applicability of the proposed approach and iterative method.

Table 1. The absolute errors in Example 6.1

k	$E(k)$ for $u_0^{(1)}$	$E(k)$ for $u_0^{(2)}$	k	$E(k)$ for $u_0^{(1)}$	$E(k)$ for $u_0^{(2)}$
0	2.161494e+00	1.440907e+02	15	1.510522e-08	9.955413e-06
1	9.918582e-01	1.121701e+02	16	9.727701e-09	2.140874e-06
2	4.024119e-01	6.490936e+01	17	8.671249e-09	4.483431e-07
3	1.479056e-01	3.122171e+01	18	8.469188e-09	9.280373e-08
4	5.005708e-02	1.312655e+01	19	8.431512e-09	2.023788e-08
5	1.578279e-02	4.964196e+00	20	8.424655e-09	6.555607e-09
6	4.676833e-03	1.720364e+00	21	8.423436e-09	8.082831e-09
7	1.311419e-03	5.535028e-01	22	8.423224e-09	8.362568e-09
8	3.498916e-04	1.669293e-01	23	8.423187e-09	8.412621e-09
9	8.922243e-05	4.754241e-02	24	8.423181e-09	8.421379e-09
10	2.182583e-05	1.286248e-02	25	8.423181e-09	8.422879e-09
11	5.137024e-06	3.321573e-03	26	8.423180e-09	8.423131e-09
12	1.165373e-06	8.219867e-04	27	8.423179e-09	8.423173e-09
13	2.542763e-07	1.955889e-04	28	8.423179e-09	8.423179e-09
14	5.229353e-08	4.487812e-05	29	8.423179e-09	8.423179e-09

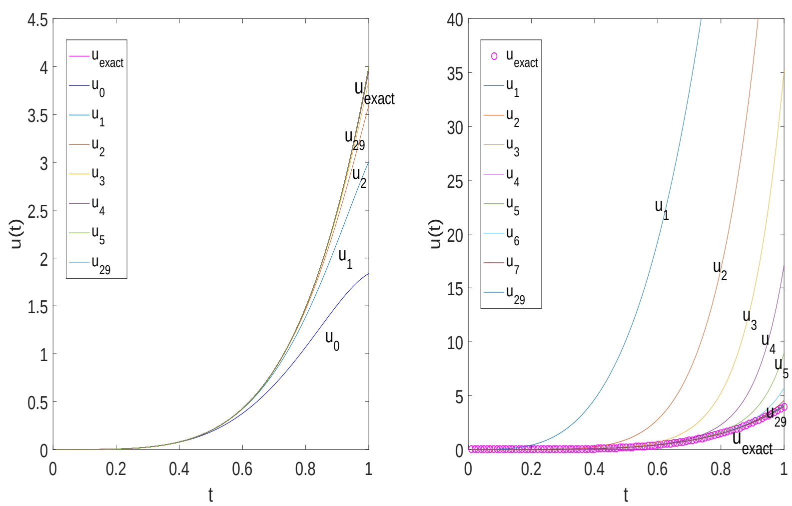

Figure 1. Some approximations of the solution with $u_0^{(1)}$ and $u_0^{(2)}$

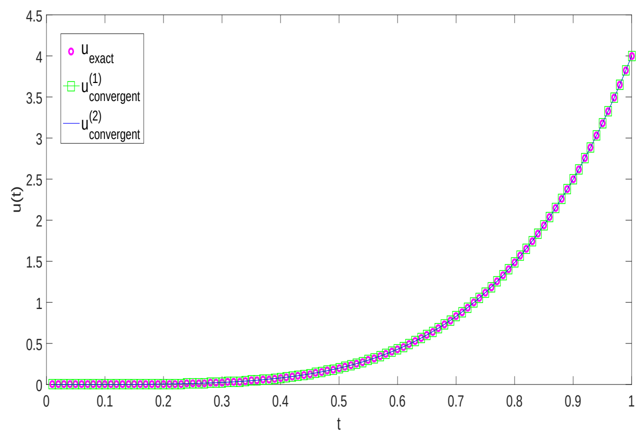

Figure 2. Comparison of exact solution with two convergent solutions

Example 6.2. Consider the following problem whose solution includes variable coefficient

$$\begin{cases} D^{\alpha}_{0+}u(t) + f\left(t, u(t), u'(t)\right) = 0, \quad t \in (0,1)\,, \\ u(0) = u'(0) = u''(0) = 0, \ \left[D^{\beta}_{0+}u(t)\right]_{t=1} = g(u'(1)), \end{cases} \tag{6.2}$$

whose exact solution is given by

$$u(t) = 2t^{\alpha+1} + t^4,$$

where $3 < \alpha \leq 4$, $1 < \beta \leq 2$,

$$f(t,u,u') = 2\Gamma(\alpha+2)t + \frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} - \sqrt[3]{[2(\alpha+1)t^{\alpha} - 4t^3]^2} - 2t^{\alpha+1} - t^4 + u + \sqrt[3]{(u')^2}$$

and

$$g(u'(t)) = \left[\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha-\beta+2)} + \frac{24}{\Gamma(5-\beta)}\right]\frac{30+u'(t)}{2\alpha+36}.$$

Now, by using the proposed method in this paper, we are to find the approximate solution for problem (6.2). First, according to Definition 2.1 and Lemma 2.1, we have

$$\begin{aligned} I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) &= u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + c_4t^{\alpha-4} \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u'(s)\right) ds. \end{aligned}$$

Thus, we have

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u'(s)\right) ds - c_1t^{\alpha-1} - c_2t^{\alpha-2} - c_3t^{\alpha-3} - c_4t^{\alpha-4}.$$

By means of the boundary condition $u(0) = u'(0) = u''(0) = 0$, we obtain $c_2 = c_3 = c_4 = 0$. Hence,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u'(s))\, ds - c_1 t^{\alpha-1}.$$

Considering the boundary condition $\left[D_{0+}^{\beta} u(t)\right]_{t=1} = g(u'(1))$, by Lemma 2.1, we have

$$D_{0+}^{\beta} u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s, u(s), u'(s))\, ds - c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.$$

Let $t = 1$, then

$$-\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s, u(s), u'(s))\, ds - \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} = g(u'(1)).$$

By the above equality, we have

$$c_1 = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s, u(s), u'(s))\, ds - \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} g(u'(1)).$$

Therefore,

$$\begin{aligned} u(t) =& -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u'(s))\, ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} g(u'(1)) t^{\alpha-1} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-\beta-1} f(s, u(s), u'(s))\, ds \\ =& \int_0^1 G(t,s) f(s, u(s), u'(s))\, ds + g(u'(1)) \varphi(t), \end{aligned}$$

where

$$\varphi(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1},$$

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases}$$

This means that problem (6.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s), u'(s))\, ds + g(u'(1)) \varphi(t).$$

Then,

$$\varphi'(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)} t^{\alpha-2},$$

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} \frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to check that $G(t,s) \geq 0$, $\frac{\partial G(t,s)}{\partial t} \geq 0$ for each $(t,s) \in [0,1] \times [0,1]$. Choose $\alpha = 3.5$, $\beta = 1.8$. Then, by calculation, we obtain

$$\begin{aligned}
a_1 &= \max_{t\in[0,1]} \int_0^1 |G(t,s)| ds \\
&= \max_{t\in[0,1]} \left[\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&= \max_{t\in[0,1]} \left[\frac{t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right] \\
&= \frac{1}{(\alpha-\beta)\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha+1)} \approx 0.0910, \\
a_2 &= \max_{t\in[0,1]} \int_0^1 \left| \frac{\partial G(t,s)}{\partial t} G(t,s) \right| ds \\
&= \max_{t\in[0,1]} \left[\int_0^1 \frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-1)} ds - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \right] \\
&= \max_{t\in[0,1]} \left[\frac{t^{\alpha-2}}{(\alpha-\beta)\Gamma(\alpha-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right] \\
&= \frac{1}{(\alpha-\beta)\Gamma(\alpha-1)} - \frac{1}{\Gamma(\alpha)} \approx 0.1416, \\
b_1 &= \max_{t\in[0,1]} |\varphi(t)| = \max_{t\in[0,1]} \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \approx 0.2734, \\
b_2 &= \max_{t\in[0,1]} |\varphi'(t)| = \max_{t\in[0,1]} \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)} t^{\alpha-2} = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)} \approx 0.6835.
\end{aligned}$$

Now, we choose a suitable $M > 0$ such that Theorem 3.1 holds in D_M. In view of Lemma **??**, we have $|u| \leq (a_1 + b_1)M$ and $|v| \leq (a_2 + b_2)M$. Moreover, by conditions (i) and (ii) of Theorem 3.1, M needs satisfied the inequalities

$$\begin{aligned}
|f| =& \left| 2\Gamma(\alpha+2)t + \frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} - \sqrt[3]{[2(\alpha+1)t^{\alpha} + 4t^3]^2} - 2t^{\alpha+1} - t^4 + u + \sqrt[3]{(u')^2} \right| \\
\leq& M.
\end{aligned}$$

That is,

$$2\Gamma(\alpha+2) + \frac{24}{\Gamma(5-\alpha)} + \sqrt[3]{(2\alpha+6)^2} + 3 + (a_1+b_1)M + \sqrt[3]{[(a_2+b_2)M]^2} \leq M$$

and

$$|g| \leq \left[\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha-\beta+2)} + \frac{24}{\Gamma(5-\beta)} \right] \frac{30 + (a_2+b_2)M}{2\alpha+36} \leq M.$$

By the two inequalities above, we choose $M = 280$. Thus, $\gamma = \max\{(a_1+b_1)M, (a_2+b_2)M\} \approx 231.0354$.

On the other hand,

$$f(t,0,0) = 2\Gamma(\alpha+2)t + \frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} - \sqrt[3]{[2(\alpha+1)t^{\alpha} + 4t^3]^2} - 2t^{\alpha+1} - t^4 \neq 0.$$

This implies that all the conditions of Theorem 3.1 are satisfied in D_{280}. Further, it is easy to check that f and g satisfy the hypotheses of Theorem 4.1. Therefore, problem (6.2) has at least a solution, and the iterative method converges.

Let

$$\Delta(s) = 2\Gamma(\alpha+2)s + \frac{24s^{4-\alpha}}{\Gamma(5-\alpha)} - \sqrt[3]{[2(\alpha+1)s^\alpha + 4s^3]^2} - 2s^{\alpha+1} - s^4.$$

Take the initial approximation $u_0^{(1)}(t)$ or $u_0^{(2)}(t)$, namely,

$$\begin{aligned} u_0^{(1)}(t) &= \int_0^1 G(s,t)f\,(s,0,0)\,ds + g(0)\varphi(t) \\ &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\Delta(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Delta(s)ds \\ &\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\left[\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha-\beta+2)} + \frac{24}{\Gamma(5-\beta)}\right]\frac{30}{2\alpha+36} \end{aligned}$$

or

$$\begin{aligned} u_0^{(2)}(t) &= \int_0^1 G(s,t)f\,(s,\gamma,\gamma)\,ds + g(\gamma)\varphi(t) \\ &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}(\Delta(s)+\gamma+\sqrt[3]{\gamma^2})ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\Delta(s)+\gamma \\ &\quad + \sqrt[3]{\gamma^2})ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\left[\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha-\beta+2)} + \frac{24}{\Gamma(5-\beta)}\right]\frac{30+\gamma}{2\alpha+36}. \end{aligned}$$

Then, the following iteration formulation is obtained:

$$\begin{aligned} u_{k+1} =& Tu_k \\ =& \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}(\Delta(s)+u_k+\sqrt[3]{(u_k')^2})ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\Delta(s)+ \\ & + u_k + \sqrt[3]{(u_k')^2})ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\left[\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha-\beta+2)} + \frac{24}{\Gamma(5-\beta)}\right]\frac{30+u_k'(1)}{2\alpha+36} \end{aligned}$$

and

$$\begin{aligned} u_{k+1}' =& Tu_k' \\ =& \int_0^1 \frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-1)}(\Delta(s)+u_k+\sqrt[3]{(u_k')^2})ds - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\Delta(s)+ \\ & + u_k + \sqrt[3]{(u_k')^2})ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)}t^{\alpha-2}\left[\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha-\beta+2)} + \frac{24}{\Gamma(5-\beta)}\right]\frac{30+u_k'(1)}{2\alpha+36}, \\ & (k=1,2,\cdots). \end{aligned}$$

In the simulation experiment, we apply the new algorithm, which combines Gauss-Kronrod quadrature method of classical numerical integration with cubic spline interpolation method in order to achieve the monotone iteration method in Matlab environment. The simulations experiment shows that for $N = 100$, the iterative method achieves the accuracy $E_{27} = 4.98125 \times 10^{-7}$ after $k = 27$ iterations. Some approximations of the solution with different initial approximations are depicted in Figure 3. Comparison of exact solution with two convergent solutions is depicted in Figure 4. The absolute errors of the iterative method for problem (6.2) are given in Table 2. From Figure 3, it is easy to see that the approximation u_{27}, and the exact solution u_{exact} almost coincide. It implies that iterative method converges very fast. This demonstrates the applicability of the proposed approach and iterative method.

Table 2. The absolute errors in Example 6.2

k	$E(k)$ for $u_0^{(1)}$	$E(k)$ for $u_0^{(2)}$	k	$E(k)$ for $u_0^{(1)}$	$E(k)$ for $u_0^{(2)}$
0	2.710624e+00	2.424800e+01	14	2.281705e-04	3.021772e-03
1	1.415456e+00	1.765198e+01	15	1.167301e-04	1.545929e-03
3	3.650915e-01	4.710969e+00	17	3.055077e-05	4.046171e-04
4	1.861925e-01	2.431922e+00	18	1.562902e-05	2.069990e-04
5	9.513056e-02	1.251085e+00	19	7.995132e-06	1.059001e-04
6	4.864292e-02	6.420402e-01	20	4.089683e-06	5.417846e-05
7	2.487809e-02	3.289900e-01	21	2.091676e-06	2.771797e-05
8	1.272658e-02	1.684386e-01	22	1.069507e-06	1.418094e-05
9	6.510774e-03	8.620227e-02	23	5.465715e-07	7.255474e-06
10	3.330831e-03	4.410730e-02	24	2.790404e-07	3.712443e-06
12	8.717753e-04	1.154522e-02	26	7.215234e-08	9.725345e-07
13	4.459949e-04	5.906543e-03	27	3.633017e-08	4.981259e-07

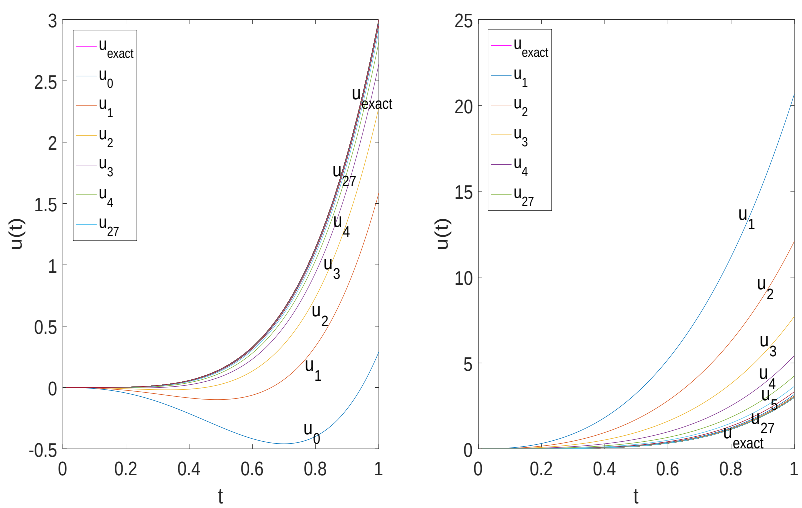

Figure 3. Some approximations of the solution with $u_0^{(1)}$ and $u_0^{(2)}$

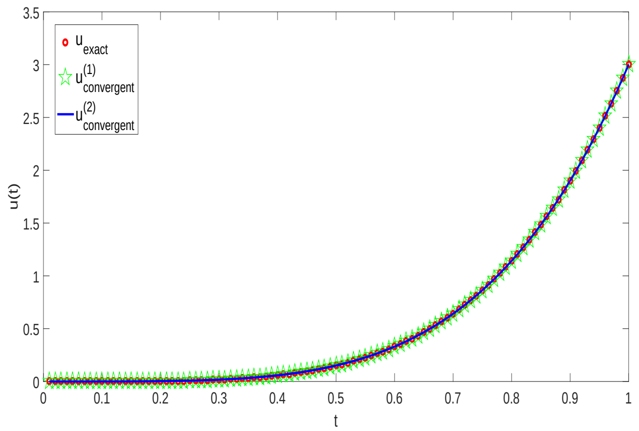

Figure 4. Comparison of convergent solutions with exact solution

Example 6.3. Consider the following FDE with p-Laplacian operator (see [27])

$$\begin{cases} D_{0+}^{\alpha}(\psi_p(D_{0+}^{\alpha}u(t))) = f(t,u(t),D_{0+}^{\alpha}u(t)), \quad 1<\alpha\le 2,\ t\in(0,1)\,, \\ u(0)=0, \quad \psi_p(D_{0+}^{\alpha}u)(0)=0, \\ \Big[D_{0+}^{\beta}u(t)\Big]_{t=1}=0, \quad \Big[D_{0+}^{\beta}(\psi_p(D_{0+}^{\alpha}u(t)))\Big]_{t=1}=0, \quad 0<\beta<\alpha, \end{cases} \tag{6.3}$$

where $v(t) := D_{0+}^{\alpha}u(t)$, ψ_p is the p-Laplacian operator, $p>1$, $\psi_p(v)=|v|^{p-2}v$, $\psi_p^{-1}=\psi_q$ and $1/p+1/q=1$.

Let

$$f\,(t,u,v) = 2+3\sqrt{u}+\frac{t^2}{2}.$$

It is easy to verify that problem (6.3) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)\psi_q\left(\int_0^1 G(s,\tau)f\,(\tau,u(\tau),v(\tau))\,d\tau\right)ds, \tag{6.4}$$

$$v(t) = -\psi_q\left(\int_0^1 G(t,s)f\,(s,u(s),v(s))\,ds\right), \tag{6.5}$$

where the corresponding Green's function is

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1. \end{cases}$$

It is easy to check that $G(t,s)\ge 0$, for each $(t,s)\in[0,1]\times[0,1]$ (see [8]). Choose $\alpha=1.8,\ \beta=1.2,\ p=2.25$ and $q=1.8$. Then, by calculation, we obtain

$$a_1 = \max_{t\in[0,1]}\int_0^1 |G(t,s)|ds$$

$$= \max_{t\in[0,1]} \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}ds}{\Gamma(\alpha)} - \int_0^t \frac{(t-s)^{\alpha-1}ds}{\Gamma(\alpha)}$$
$$= \max_{t\in[0,1]} \left[\frac{t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$= \frac{\beta}{(\alpha-\beta)\Gamma(\alpha+1)}$$
$$\approx 1.1930.$$

By (6.5), using a way similar to the proof of Lemma **??**, we have

$$|\psi_p(v(t))| = \left|-\int_0^1 G(t,s) f\left(s, u(s), v(s)\right) ds\right| \le a_1 M.$$

Hence,

$$|v(t)| \le \psi_q(a_1 M) = (a_1 M)^{q-1}.$$

By (6.4), we obtain

$$|u(t)| = \left|-\int_0^1 G(t,s)[-v(t)]ds\right| \le a_1(a_1M)^{q-1} = a_1^q M^{q-1}.$$

Moreover, we choose a suitable $M > 0$ such that

$$|f\left(t, u, v\right)| \le 2 + 3\sqrt{a_1^q M^{q-1}} + \frac{1}{2} \le M.$$

Thus, choose $M = 13$. Then,

$$\gamma = \max\{a_1^q M^{q-1}, (a_1 M)^{q-1}\} \approx 10.6932.$$

On the other hand,

$$f(t, 0, 0) = 2 + \frac{t^2}{2} \ne 0.$$

This implies that all the conditions of Theorem 3.1 are satisfied. Further, it is easy to check that f satisfies the hypotheses of Theorem 4.1. Therefore, problem (6.2) has at least a solution, and the iterative method converges. Let

$$\Lambda_0^{(1)}(s) = \int_0^1 G(s,\tau) f\left(\tau, 0, 0\right) d\tau$$
$$= \int_0^1 \frac{s^{\alpha-1}(1-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha)} \left(2 + \frac{\tau^2}{2}\right) d\tau - \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(2 + \frac{\tau^2}{2}\right) d\tau,$$

$$\Lambda_0^{(2)}(s) = \int_0^1 G(s,\tau) f\left(\tau, \gamma, \gamma\right) d\tau$$
$$= \int_0^1 \frac{s^{\alpha-1}(1-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha)} \left(2 + 3\sqrt{\gamma} + \frac{\tau^2}{2}\right) d\tau - \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(2 + 3\sqrt{\gamma} + \frac{\tau^2}{2}\right)$$
$$d\tau$$

and

$$\Lambda_k(s) = \int_0^1 G(s,\tau) f\left(\tau, u_k(\tau), v_k(\tau)\right) d\tau$$

$$= \int_0^1 \frac{s^{\alpha-1}(1-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha)} \left(2 + 3\sqrt{u_k} + \frac{\tau^2}{2}\right) d\tau$$
$$- \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(2 + 3\sqrt{u_k} + \frac{\tau^2}{2}\right) d\tau.$$

Take the initial approximation $u_0^{(1)}(t)$ or $u_0^{(2)}(t)$, and that is,

$$u_0^{(1)}(t) = \int_0^1 G(t,s)\psi_q(\Lambda_0^{(1)}(s))ds$$
$$= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\psi_q(\Lambda_0^{(1)}(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\psi_q(\Lambda_0^{(1)}(s))ds$$

or

$$u_0^{(2)}(t) = \int_0^1 G(t,s)\psi_q(\Lambda_0^{(2)}(s))ds$$
$$= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\psi_q(\Lambda_0^{(2)}(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\psi_q(\Lambda_0^{(2)}(s))ds.$$

Then, the following iteration formulation is obtained:

$$u_{k+1}(t) = Tu_k(t) = \int_0^1 G(t,s)\psi_q(\Lambda_k(s))ds$$
$$= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\psi_q(\Lambda_k(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\psi_q(\Lambda_k(s))ds \ (k = 1, 2, \cdots).$$

The simulations experiment shows that for $N = 100$, the iterative method achieves the accuracy 10^{-16}, after $k = 34$ iterations. Some approximations of the solution with different initial approximations are depicted in Figure 5. Comparison of the exact solution with two convergent solutions is depicted in Figure 6. The absolute errors of the iterative method for problem (6.3) are given in Table 3. From Figure 5, it is easy to see that the approximations u_{33}, and u_{34} almost coincide. It implies that iterative method converges very fast. This shows the effectiveness of our proposed method.

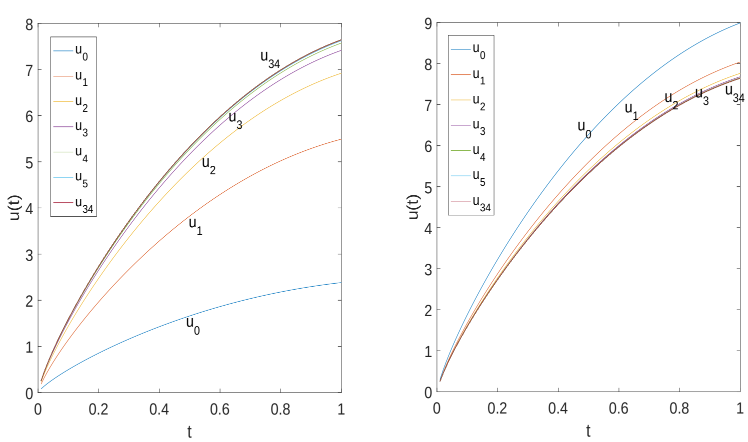

Figure 5. Some approximations of the solution with $u_0^{(1)}$ and $u_0^{(2)}$

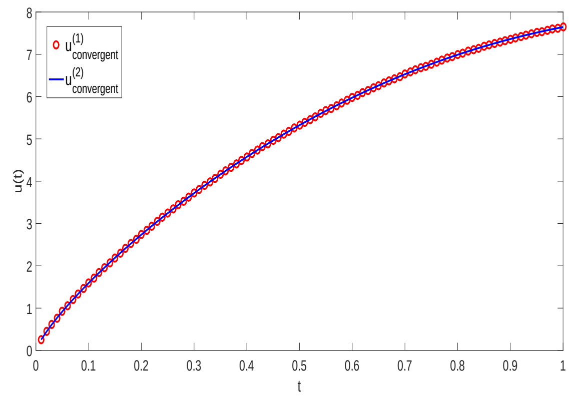

Figure 6. Comparison of two convergent solutions with different initial approximations

Table 3. The iterative errors in Example 6.3

k	$e(k)$ for $u_0^{(1)}$	$e(k)$ for $u_0^{(2)}$	k	$e(k)$ for $u_0^{(1)}$	$e(k)$ for $u_0^{(2)}$
1	3.107536e+00	9.527701e-01	18	1.049568e-08	1.653937e-09
2	1.429353e+00	2.759970e-01	19	3.220012e-09	5.074181e-10
3	4.954284e-01	8.307753e-02	20	9.878818e-10	1.556675e-10
4	1.577633e-01	2.533888e-02	21	3.030749e-10	4.776002e-11
5	4.895634e-02	7.759859e-03	22	9.297985e-11	1.465139e-11
6	1.507217e-02	2.379366e-03	23	2.852651e-11	4.496847e-12
7	4.629017e-03	7.298518e-04	24	8.751222e-12	1.381117e-12
8	1.420622e-03	2.239025e-04	25	2.684963e-12	4.227729e-13
9	4.358825e-04	6.869087e-05	26	8.242296e-13	1.314504e-13
10	1.337303e-04	2.107384e-05	27	2.549072e-13	4.263256e-14
11	4.102804e-05	6.465319e-06	28	7.904788e-14	1.598721e-14
12	1.258719e-05	1.983520e-06	29	2.575717e-14	5.329071e-15
13	3.861679e-06	6.085321e-07	30	1.243450e-14	5.329071e-15
14	1.184740e-06	1.866940e-07	31	3.552714e-15	1.776357e-15
15	3.634712e-07	5.727659e-08	32	5.329071e-15	2.664535e-15
16	1.115108e-07	1.757212e-08	33	1.776357e-15	0.000000e+00
17	3.421085e-08	5.391020e-09	34	0.000000e+00	0.000000e+00

Example 6.4. Consider the following coupled system of nonlinear FDEs (see [16])

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, w(t), D_{0+}^{q}w(t)) = 0, \quad t \in (0,1)\,, \\ D_{0+}^{\beta}w(t) + z(t, u(t), D_{0+}^{p}u(t)) = 0, \quad t \in (0,1)\,, \\ u(0) = u(1) = w(0) = w(1) = 0, \end{cases} \tag{6.6}$$

where $1 < \alpha,\ \beta < 2$ and $0 < p,\ q < 1$.

First, let

$$f\left(t, w, D_{0+}^{q}w\right) = \left|t - \frac{1}{2}\right| \sqrt{100w + 10(D_{0+}^{q}w)^{\rho} + 1},$$

and

$$z\left(t, u, D_{0+}^{p}u\right) = \frac{1}{2}\left|t - \frac{1}{3}\right| \sqrt{100u + 10(D_{0+}^{p}u)^{\delta} + 1}.$$

According to Definition 2.1 and Lemma 2.1, we have

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, w(s), D_{0+}^{q}w(s))ds.$$

Thus, we have

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, w(s), D_{0+}^{q}w(s))ds - c_1t^{\alpha-1} - c_2t^{\alpha-2}.$$

By means of the boundary condition $u(0) = u(1) = 0$, we get $c_2 = 0$, and

$$c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f(s, w(s), D_{0+}^{q}w(s))ds.$$

Hence,

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, w(s), D_{0+}^{q}w(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}f(s, w(s), D_{0+}^{q}w(s))ds \\ =&\int_0^1 G_1(t,s)f\left(s, w(s), D_{0+}^{q}w(s)\right)ds, \end{aligned}$$

where

$$G_1(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 \le t \le s \le 1, \end{cases}$$

which is the corresponding Green's function.

Analogously, we can prove that

$$w(t) = \int_0^1 G_2(t,s)z\left(s, u(s), D_{0+}^{p}u(s)\right)ds,$$

where

$$G_2(t,s)=\begin{cases}\frac{[t(1-s)]^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0\le s\le t\le 1,\\ \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & 0\le t\le s\le 1.\end{cases}$$

This means that problem (6.6) is equivalent to the integral equations

$$\begin{cases}u(t)=\int_0^1 G_1(t,s)f\left(s,w(s),D_{0+}^q w(s)\right)ds,\\ w(t)=\int_0^1 G_2(t,s)z\left(s,u(s),D_{0+}^p u(s)\right)ds.\end{cases}\tag{6.7}$$

Noticing (6.7), we have

$$\begin{cases}D_{0+}^p u(t)=\int_0^1 D_t^p G_1(t,s)f\left(s,w(s),D_{0+}^q w(s)\right)ds,\\ D_{0+}^q w(t)=\int_0^1 D_t^q G_2(t,s)z\left(s,u(s),D_{0+}^p u(s)\right)ds,\end{cases}$$

where

$$D_t^p G_1(t,s)=\begin{cases}\frac{t^{\alpha-p-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}, & 0\le s\le t\le 1,\\ \frac{t^{\alpha-p-1}(1-s)^{\alpha-1}}{\Gamma(\alpha-p)}, & 0\le t\le s\le 1,\end{cases}$$

and

$$D_t^q G_2(t,s)=\begin{cases}\frac{t^{\beta-q-1}(1-s)^{\beta-1}-(t-s)^{\beta-q-1}}{\Gamma(\beta-q)}, & 0\le s\le t\le 1,\\ \frac{t^{\beta-q-1}(1-s)^{\beta-1}}{\Gamma(\beta-q)}, & 0\le t\le s\le 1.\end{cases}$$

It is easy to check that $G_1(t,s)\ge 0,\ G_2(t,s)\ge 0,\ D_t^pG_1(t,s)\ge 0$ and $D_t^qG_2(t,s)\ge 0$ for $(t,s)\in[0,1]\times[0,1]$. Choose $\alpha=1.5,\ \beta=1.75,\ p=0.5,\ q=0.25,\ \rho=2$ and $\delta=2$. Then, by calculation, we obtain

$$\begin{aligned}a_1&=\max_{t\in[0,1]}\int_0^1|G_1(t,s)|ds\\ &=\max_{t\in[0,1]}\int_0^1\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}ds-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &=\max_{t\in[0,1]}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\\ &=\frac{(\alpha-1)^{(\alpha-1)}}{\alpha^\alpha\Gamma(\alpha+1)}\approx 0.2895\end{aligned}$$

and

$$\begin{aligned}a_2&=\max_{t\in[0,1]}\int_0^1\left|D_t^pG_1(t,s)\right|ds\\ &=\max_{t\in[0,1]}\left[\int_0^1\frac{t^{\alpha-p-1}(1-s)^{\alpha-1}}{\Gamma(\alpha-p)}ds-\int_0^t\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]\\ &=\max_{t\in[0,1]}\left[\frac{t^{\alpha-p-1}}{\alpha\Gamma(\alpha-p)}-\frac{t^{\alpha-p}}{\Gamma(\alpha-p+1)}\right]=\frac{2}{3}.\end{aligned}$$

Analogously,

$$\begin{aligned}\overline{a}_1 &= \max_{t\in[0,1]}\int_0^1 |G_2(t,s)|ds\\ &= \max_{t\in[0,1]}\int_0^1 \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}ds - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}ds\\ &= \frac{(\beta-1)^{(\beta-1)}}{\beta^\beta\Gamma(\beta+1)} \approx 0.1882\end{aligned}$$

and

$$\begin{aligned}\overline{a}_2 &= \max_{t\in[0,1]}\int_0^1 |D_t^q G_2(t,s)|\,ds\\ &= \max_{t\in[0,1]}\left[\int_0^1 \frac{t^{\beta-q-1}(1-s)^{\beta-1}}{\Gamma(\beta-q)}ds - \int_0^t \frac{(t-s)^{\beta-q-1}}{\Gamma(\beta-q)}ds\right]\\ &= \max_{t\in[0,1]}\left[\frac{t^{\beta-q-1}}{\beta\Gamma(\beta-q)} - \frac{t^{\beta-q}}{\Gamma(\beta-q+1)}\right]\\ &\approx 0.2298.\end{aligned}$$

In view of Lemma (**??**), we have $|u| \le a_1 M$, $|D_{0^+}^p u| \le a_2 M$, $|w| \le \overline{a}_1 M$ and $|D_{0^+}^q w| \le \overline{a}_2 M$. Now, we choose a suitable $M > 0$ such that

$$|f| \le \frac{1}{2}\sqrt{100\overline{a}_1 M + 10(\overline{a}_2 M)^2 + 1} \le M,$$

$$|z| \le \frac{1}{3}\sqrt{100 a_1 M + 10(a_2 M)^2 + 1} \le M.$$

By the two inequalities above, we choose $M = 7$.

On the other hand,

$$f(t,0,0) = \left|t - \frac{1}{2}\right| \ne 0 \quad \text{and} \quad z\,(t,0,0) = \frac{1}{2}\left|t - \frac{1}{3}\right| \ne 0.$$

This implies that all the conditions of Theorem 3.1 are satisfied. Further, it is easy to check that f and z satisfy the hypotheses of Theorem 4.1. Therefore, problem (6.2) has at least a solution, and the iterative method converges.

Take the initial approximation (u_0, w_0), and that is,

$$\begin{aligned}u_0(t) &= \int_0^1 G_1(t,s) f(s,0,0) ds\\ &= \int_0^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}\left|s - \frac{1}{2}\right| ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|s - \frac{1}{2}\right| ds\end{aligned}$$

and

$$\begin{aligned}w_0(t) &= \int_0^1 G_2(t,s) z(s,0,0) ds\\ &= \int_0^1 \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}\frac{1}{2}\left|s - \frac{1}{3}\right| ds - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{1}{2}\left|s - \frac{1}{3}\right| ds.\end{aligned}$$

Then, the following iteration formulation is obtained:

$$\begin{aligned}u_{k+1}(t) =&Tu_k(t)\\ =&\int_0^1 G_1(t,s)f(s,w_k,D^q_{0^+}w_k)ds\\ =&\int_0^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}\left|s-\frac{1}{2}\right|\sqrt{100w_k+10(D^q_{0^+}w_k)^\rho+1}ds\\ &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|s-\frac{1}{2}\right|\sqrt{100w_k+10(D^q_{0^+}w_k)^\rho+1}ds,\quad (k=1,2,\cdots),\end{aligned}$$

$$\begin{aligned}D^p_{0^+}u_{k+1}(t) =& \int_0^1 D^p_t G_1(t,s)f(s,w_k,D^q_{0^+}w_k)ds\\ =&\int_0^1 \frac{t^{\alpha-p-1}(1-s)^{\alpha-1}}{\Gamma(\alpha-p)}\left|s-\frac{1}{2}\right|\sqrt{100w_k+10(D^q_{0^+}w_k)^\rho+1}ds\\ &-\int_0^t \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left|s-\frac{1}{2}\right|\sqrt{100w_k+10(D^q_{0^+}w_k)^\rho+1}ds,\\ &(k=1,2,\cdots),\end{aligned}$$

$$\begin{aligned}w_{k+1}(t) =&Tw_k(t)\\ =&\int_0^1 G_2(t,s)z(s,u_k,D^p_{0^+}u_k)ds\\ =&\frac{1}{2}\int_0^1 \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}\left|s-\frac{1}{3}\right|\sqrt{100u_k+10(D^p_{0^+}u_k)^\delta+1}ds\\ &-\frac{1}{2}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left|s-\frac{1}{3}\right|\sqrt{100u_k+10(D^p_{0^+}u_k)^\delta+1}ds,\quad (k=1,2,\cdots)\end{aligned}$$

and

$$\begin{aligned}D^q_{0^+}w_{k+1}(t) =& \int_0^1 D^q_t G_2(t,s)z(s,u_k,D^p_{0^+}u_k)ds\\ =&\frac{1}{2}\int_0^1 \frac{t^{\beta-q-1}(1-s)^{\beta-1}}{\Gamma(\beta-q)}\left|s-\frac{1}{3}\right|\sqrt{100u_k+10(D^p_{0^+}u_k)^\delta+1}ds\\ &-\frac{1}{2}\int_0^t \frac{(t-s)^{\beta-q-1}}{\Gamma(\beta-q)}\left|s-\frac{1}{3}\right|\sqrt{100u_k+10(D^p_{0^+}u_k)^\delta+1}ds,\\ &(k=1,2,\cdots).\end{aligned}$$

The simulations experiment shows that for $N = 100$, the iterative method achieves the accuracy 10^{-16}, after $k = 21$ iterations. Some approximations of the solution are depicted in Figure 7. Convergent solution and convergence of the iterative process are depicted in Figure 8. The iteration errors for problem (6.6) are indicated in Table 4. From Figure 7, it is easy to see that the approximations (u_{20}, w_{20}) and (u_{21}, w_{21}) almost coincide. It implies that iterative method converges very fast. This shows the applicability and the effectiveness of the proposed approach and iterative method.

Table 4. The iteration errors in Example 6.6

k	$e_u(k)$	$e_v(k)$	k	$e_u(k)$	$e_v(k)$
0	8.438501e-02	3.560760e-02	11	2.511370e-09	8.104813e-10
1	2.447377e-02	8.243518e-03	12	4.987868e-10	1.609709e-10
2	5.168321e-03	1.678239e-03	13	9.906481e-11	3.197066e-11
3	1.035939e-03	3.345667e-04	14	1.967543e-11	6.349768e-12
4	2.059927e-04	6.648132e-05	15	3.907707e-12	1.261088e-12
5	4.091634e-05	1.320449e-05	16	7.761569e-13	2.504663e-13
6	8.126349e-06	2.622551e-06	17	1.541267e-13	4.980738e-14
7	1.613970e-06	5.208667e-07	18	3.066991e-14	9.964252e-15
8	3.205519e-07	1.034499e-07	19	6.133982e-15	2.081668e-15
9	6.366521e-08	2.054634e-08	20	1.276756e-15	5.273559e-16
10	1.264464e-08	4.080738e-09	21	4.440892e-16	2.220446e-16

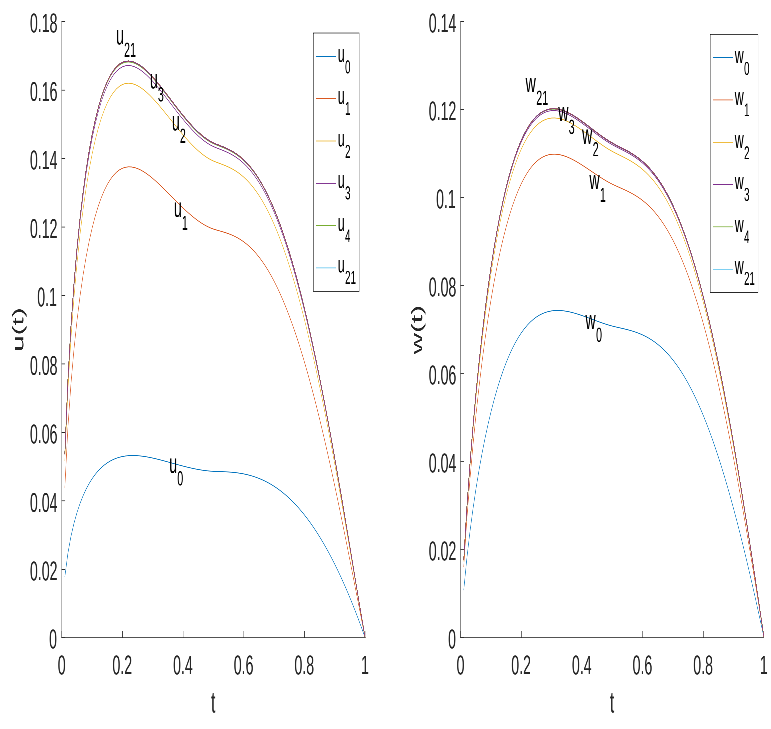

Figure 7. Some approximations of solutions $u(t)$ and $w(t)$

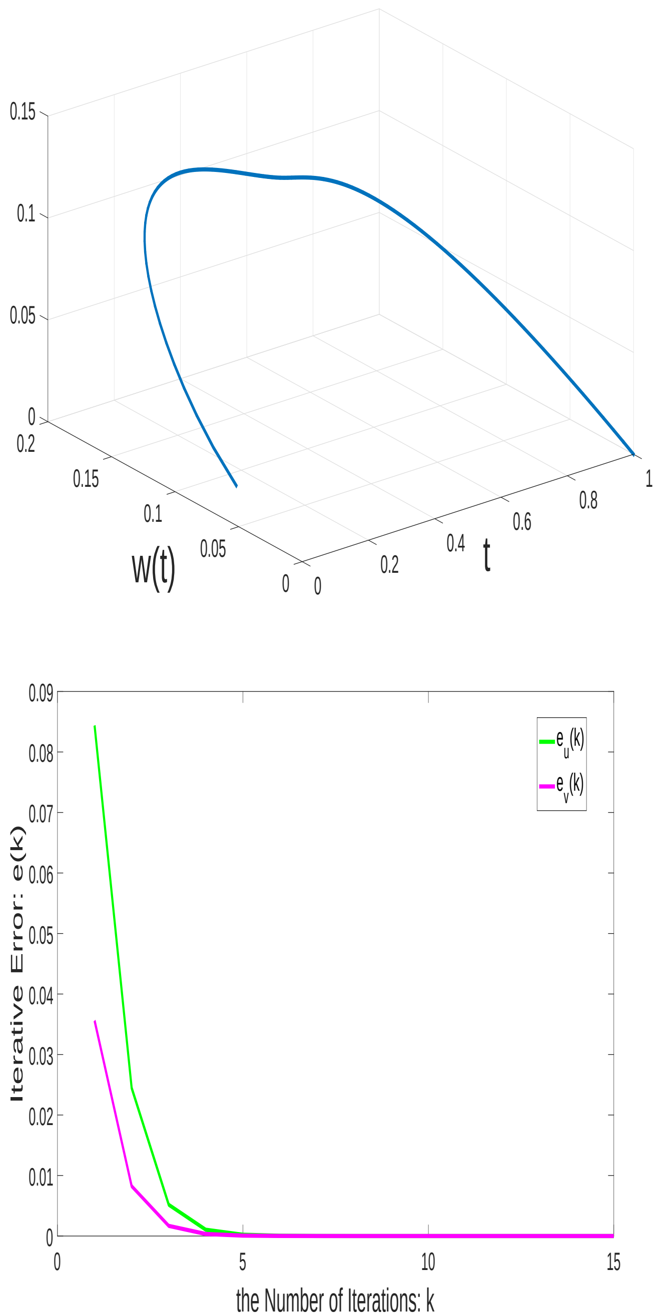

Figure 8. Convergent solution (u, w) and convergence of the iterative process

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