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Exact Loop Wave Solutions and Cusp Wave Solutions of the Fujimoto-Watanabe Equation*

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Abstract In this paper, the theory of dynamical systems is employed to investigate loop waves and cusp waves of the Fujimoto-Watanabe equation. These waves contain solitary loop waves, periodic loop waves, peakons and periodic cusp waves. Under fixed parameter conditions, their exact explicit parametric expressions are given.

Keywords Fujimoto-Watanabe equation, Solitary loop wave, Periodic loop wave, Peakon, Periodic cusp wave.

MSC(2010) 35B10, 35C07, 35C08.

1. Introduction

The nonlinear equation

$$u_t - u^3 u_{xxx} - 3u^2(u_x u_{xx} + \alpha u_x) = 0 \tag{1.1}$$

was derived by Fujimoto and Watanabe [9], and is now known as the Fujimoto-Watanabe equation. Sakovich [18] showed that equation (1.1) can be connected with the famous KdV equation. Du [5] obtained some implicit expressions of traveling wave solutions of equation (1.1) by using an irrational equation method. Liu [14] gave the classifications of traveling wave solutions of equation (1.1) through the method of complete discrimination system. In [19], Shi and Wen obtained the implicit expressions of solitary wave solutions, periodic wave solutions, kink-like wave solutions and antikink-like wave solutions of equation (1.1). In [20–22], Shi and Wen continued to study three extension Fujimoto-Watanabe equations, and got some traveling wave solutions. These results enrich the research work of Fujimoto-Watanabe equation. The traveling wave of research on nonlinear differential equation has been greatly emphasized, some pieces of literature [1–3, 7, 10, 17] have shown some traveling wave solutions and properties of nonlinear equation. These results improve the content of traveling wave.

Here, our aim of this paper is to use the bifurcation method of planar systems and simulation method of differential equations [4, 6, 8, 11–13, 15, 16, 23–28] to investigate the loop wave solutions and cusp wave solutions of equation (1.1). The exact

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representation of loop wave solutions and cusp wave solutions of equation (1.1) are obtained. The planar graphs of the loop wave solutions and cusp wave solutions are shown under some parameters. These results are new.

For the given constant c, substituting $u(x,t)=\phi(\xi)$ with $\xi = x - ct$ in equation (1.1), it follows that

$$-c\phi' - \phi^3\phi''' - 3\phi^2(\phi'\phi'' + \alpha\phi') = 0. \tag{1.2}$$

Integrating equation (1.2) once with respect to ξ, we have the following traveling wave equation

$$\phi^3\phi'' = -\alpha\phi^3 - c\phi + g = 0, \tag{1.3}$$

where g is integral constant. Letting $\phi' = y$, then equation (1.3) becomes a planar system

$$\begin{cases} \dfrac{d\phi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{-\alpha\phi^3 - c\phi + g}{\phi^3}. \end{cases} \tag{1.4}$$

Clearly, on the straight line $\phi = 0$, system (1.4) is discontinuous. Such is called singular traveling wave system. Obviously, system (1.4) has the first integral

$$H(\phi, y) = \frac{y^2}{2} + \frac{2\alpha\phi^3 - 2c\phi + g}{2\phi^2} = h. \tag{1.5}$$

Let

$$f(\phi) = -2\alpha\phi^3 + 2h\phi^2 + 2c\phi - g. \tag{1.6}$$

According to the lemma of [19], we let the $(\phi^*, 0)$ is a saddle of system (1.4). We have main results as follows.

Proposition 1.1. *(1) If $\alpha < 0$, $c > 0$, and $g < 0$, or $\alpha < 0$, $c < 0$, and $g < 0$, then equation (1.1) has a solitary loop wave, and its parametric type is as follows:*

$$\begin{cases} \phi = \phi_1^* + (\phi^* - \phi_1^*)\tanh^2\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right], \\ \xi = \sqrt{\frac{2}{\alpha(\phi_1^*-\phi^*)}}\left\{\phi^*\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w - (\phi^* - \phi_1^*)\tanh\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right]\right\}, \end{cases} \tag{1.7}$$

where w is a parameter variable, $h = H(\phi^*, 0)$, ϕ_1^* and ϕ^* are a simple real zero and a double real zero of $f(\phi)$ and $\phi_1^* \le \phi < \phi^*$.

(2) If $\alpha > 0$, $c < 0$, and $g < 0$, or $\alpha > 0$, $c > 0$, and $g < 0$, then equation (1.1) has a solitary loop wave, and has parametric type as follows:

$$\begin{cases} \phi = \phi_1^* - (\phi_1^* - \phi^*)\tanh^2\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right], \\ \xi = \sqrt{\frac{2}{\alpha(\phi_1^*-\phi^*)}}\left\{\phi^*\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w + (\phi_1^* - \phi^*)\tanh\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right]\right\}, \end{cases} \tag{1.8}$$

where w is a parameter variable, $h = H(\phi^*, 0)$, ϕ_1^* and ϕ^* are a simple real zero and double real zero of $f(\phi)$, and $\phi^* < \phi \le \phi_1^*$.

Remark 1.1. Under $\alpha c < 0$, when $g \to 0$, then $\phi_1^* \to 0$, the solitary loop wave becomes peakon, and its parametric type becomes as follows:

$$\begin{cases} \phi = \phi^* \tanh^2\left(\sqrt{\frac{-\alpha\phi^*}{2}} w\right), \\ \xi = \sqrt{\frac{2}{-\alpha\phi^*}}\left[\phi^* \sqrt{\frac{-\alpha\phi^*}{2}} w - \phi^* \tanh\left(\sqrt{\frac{-\alpha\phi^*}{2}} w\right)\right]. \end{cases} \tag{1.9}$$

Proposition 1.2. *(1) If $\alpha < 0$, $c > 0$ and $g < 0$, or $\alpha < 0$, $c < 0$ and $g < 0$, then equation (1.1) has a periodic loop wave, and its parametric type is as follows:*

$$\begin{cases} \phi = \phi_1 + (\phi_2 - \phi_1)\mathrm{sn}^2\left[\sqrt{\frac{\alpha(\phi_1-\phi_3)}{2}} w, \sqrt{\frac{\phi_2-\phi_1}{\phi_3-\phi_1}}\right], \\ \xi = \sqrt{\frac{2}{\alpha(\phi_1-\phi_3)}}\left\{\phi_3 \sqrt{\frac{\alpha(\phi_1-\phi_3)}{2}} w - (\phi_3 - \phi_1)\mathrm{E}\left[\sqrt{\frac{\alpha(\phi_1-\phi_3)}{2}} w, \sqrt{\frac{\phi_2-\phi_1}{\phi_3-\phi_1}}\right]\right\}, \end{cases} \tag{1.10}$$

where w is a parameter variable, $h = H(\phi_0, 0)$, $0 < \phi_0 < \phi^*$, ϕ_1, ϕ_2 and ϕ_3 are three simple real zeros of $f(\phi)$, $\phi_0 = \phi_2$ and $\phi_1 \le \phi \le \phi_2 < \phi_3$.

(2) If $\alpha > 0$, $c < 0$ and $g < 0$, or $\alpha > 0$, $c > 0$ and $g < 0$, then equation (1.1) has a periodic loop wave, and its parametric type is as follows:

$$\begin{cases} \phi = \phi_3 - (\phi_3 - \phi_2)\mathrm{sn}^2\left[\sqrt{\frac{\alpha(\phi_3-\phi_1)}{2}} w, \sqrt{\frac{\phi_3-\phi_2}{\phi_3-\phi_1}}\right], \\ \xi = \sqrt{\frac{2}{\alpha(\phi_3-\phi_1)}}\left\{\phi_1 \sqrt{\frac{\alpha(\phi_3-\phi_1)}{2}} w + (\phi_3 - \phi_1)\mathrm{E}\left[\sqrt{\frac{\alpha(\phi_3-\phi_1)}{2}} w, \sqrt{\frac{\phi_3-\phi_2}{\phi_3-\phi_1}}\right]\right\}, \end{cases} \tag{1.11}$$

where w is a parameter variable, $h = H(\phi_0, 0)$, $\phi^* < \phi_0 < 0$, ϕ_1, ϕ_2 and ϕ_3 are three simple real zeros of $f(\phi)$, $\phi_0 = \phi_2$ and $\phi_1 < \phi_2 \le \phi \le \phi_3$.

Remark 1.2. Under $\alpha < 0$ and $c > 0$, when $g \to 0$, then $\phi_1 \to 0$, the periodic loop wave becomes periodic cusp wave, and its parametric type becomes as follows:

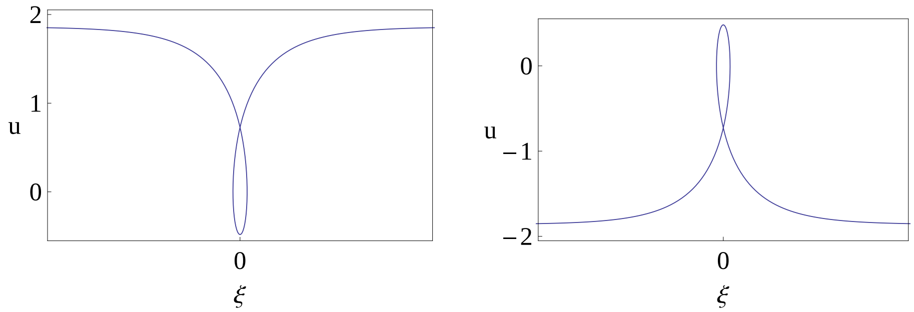
$$\begin{cases} \phi = \phi_2 \mathrm{sn}^2\left(\sqrt{\frac{-\alpha\phi_3}{2}} w, \sqrt{\frac{\phi_2}{\phi_3}}\right), \\ \xi = \sqrt{\frac{2}{-\alpha\phi_3}}\left[\phi_3 \sqrt{\frac{-\alpha\phi_3}{2}} w - \phi_3 \mathrm{E}\left(\sqrt{\frac{-\alpha\phi_3}{2}} w, \sqrt{\frac{\phi_2}{\phi_3}}\right)\right]. \end{cases} \tag{1.12}$$

Remark 1.3. Under $\alpha > 0$ and $c < 0$, when $g \to 0$, then $\phi_3 \to 0$, the periodic loop wave becomes periodic cusp wave, and its parametric type becomes as follows:

$$\begin{cases} \phi = \phi_2 \mathrm{sn}^2\left(\sqrt{\frac{-\alpha\phi_1}{2}} w, \sqrt{\frac{\phi_2}{\phi_1}}\right), \\ \xi = \sqrt{\frac{2}{-\alpha\phi_1}}\left[\phi_1 \sqrt{\frac{-\alpha\phi_1}{2}} w - \phi_1 \mathrm{E}\left(\sqrt{\frac{-\alpha\phi_1}{2}} w, \sqrt{\frac{\phi_2}{\phi_1}}\right)\right]. \end{cases} \tag{1.13}$$

Example 1.1. Letting $\alpha = -6$, $c = 10$ and $g = -20$, then $\phi_1^* \doteq -0.481842$, $\phi^* \doteq 1.85983$. Substituting these data into (1.7), on $\xi - u$ plane, we draw solitary loop wave graph as Figure 1(a).

Letting $\alpha = 6$, $c = -10$ and $g = -20$, then $\phi^* \doteq -1.85983$, $\phi_1^* \doteq 0.481842$. Substituting these data into (1.8), on $\xi - u$ plane, we draw solitary loop wave graph as Figure 1(b).

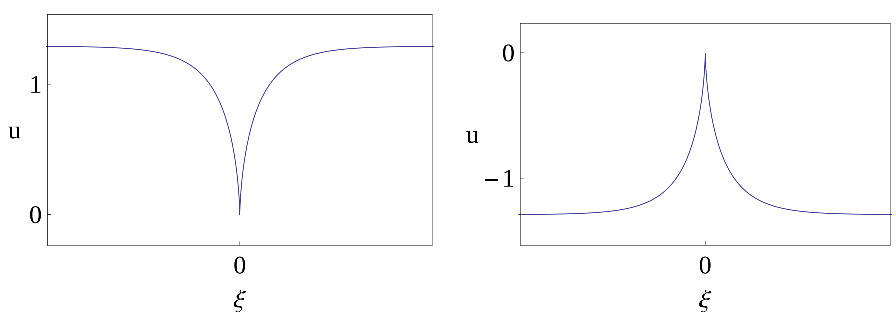


a. When $\alpha = -6$, $c = 10$ and $g = -20$ b. When $\alpha = 6$, $c = -10$ and $g = -20$

Figure 1. The solitary loop wave graphs of equation (1.1)

Example 1.2. Letting $\alpha = -6$, $c = 10$ and $g = 0$, then $\phi_1^* = 0$, $\phi^* \doteq 1.29099$. Substituting these data into (1.9), on $\xi - u$ plane, we draw peakon graph as Figure 2(a).

Letting $\alpha = 6$, $c = -10$ and $g = 0$, then $\phi^* \doteq -1.29099$, $\phi_1^* = 0$. Substituting these data into (1.9), on $\xi - u$ plane, we draw peakon graph as Figure 2(b).



a. When $\alpha = -6$, $c = 10$ and $g = 0$ b. When $\alpha = 6$, $c = -10$ and $g = 0$

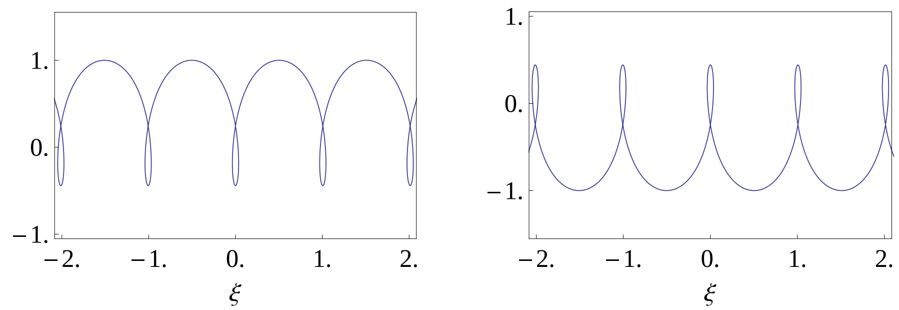
Figure 2. The peakon graphs of equation (1.1)

Example 1.3. Letting $\alpha = -6$, $c = 10$ and $g = -20$, then $\phi^* \doteq 1.85983$. Taking $\phi_0 = 1$, we have $h = -26$, $\phi_1 \doteq -0.441518$, $\phi_2 = 1$, and $\phi_3 \doteq 3.77485$. Substituting these data into (1.10), on $\xi - u$ plane, we draw periodic loop wave graph as Figure 3(a).

Letting $\alpha = 6$, $c = -10$ and $g = -20$, then $\phi^* \doteq -1.85983$. Taking $\phi_0 = -1$, we have $h = -26$, $\phi_1 \doteq -3.77485$, $\phi_2 = -1$, and $\phi_3 \doteq 0.441518$. Substituting these data into (1.11), on $\xi - u$ plane, we draw periodic loop wave graph as Figure 3(b).

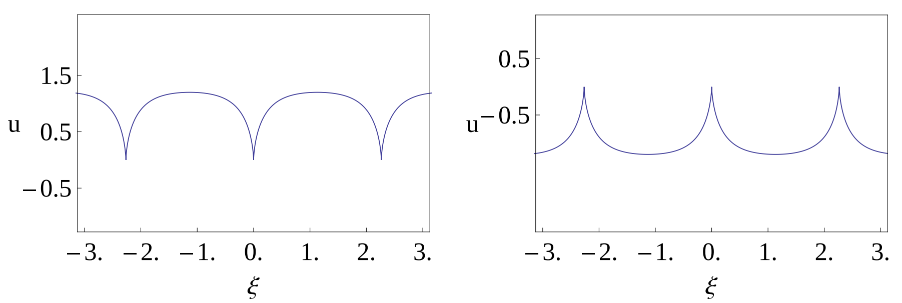
Example 1.4. Letting $\alpha = -6$, $c = 10$ and $g = 0$, then $\phi^* \doteq 1.29099$. Taking $\phi_0 = 1.2$, we have $h \doteq -15.5333$, $\phi_1 = 0$, $\phi_2 = \phi_0 = 1.2$, and $\phi_3 \doteq 1.38889$. Substituting these data into (1.12), on $\xi - u$ plane, we draw periodic cusp wave graph as Figure 4(a).

Letting $\alpha = 6$, $c = -10$ and $g = 0$, then $\phi^* \doteq -1.29099$. Taking $\phi_0 = -1.2$, we have $h \doteq -15.5333$, $\phi_1 \doteq -1.38889$, $\phi_2 = \phi_0 = -1.2$, and $\phi_3 = 0$. Substituting these data into (1.13), on $\xi - u$ plane, we draw periodic cusp wave graph as Figure 4(b).



a. When $\alpha = -6$, $c = 10$ and $g = -20$ b. When $\alpha = 6$, $c = -10$ and $g = -20$

Figure 3. The periodic loop wave graphs of equation (1.1)



a. When $\alpha = -6$, $c = 10$ and $g = 0$ b. When $\alpha = 6$, $c = -10$ and $g = 0$

Figure 4. The periodic cusp wave graphs of equation (1.1)

2. The proof of main results

Using the bifurcation method of planar systems, as [19], in different regions parameter plane, we draw the bifurcation phase portraits of system (1.4), which are shown in Figures 5 and 6.

In order to derive the expressions of solutions above. We assume that $(\phi_0, 0)$ is the initial point of an orbit of system (1.5), from equation (1.5), it has the expression $H(\phi, y) = h_0$, where $h_0 = H(\phi_0, 0)$.

From $H(\phi, y) = h_0$, the following equation determines the orbit of passing through $(\phi_0, 0)$.

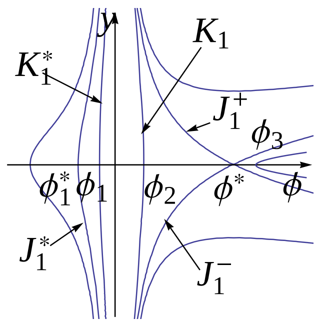
$$\phi^2 y^2 = f(\phi). \tag{2.1}$$

By applying the transformation

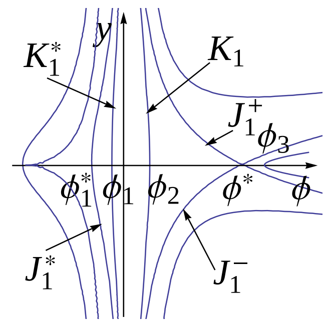
$$d\xi = \phi dw \tag{2.2}$$

to the first expression of system (1.4), we obtain

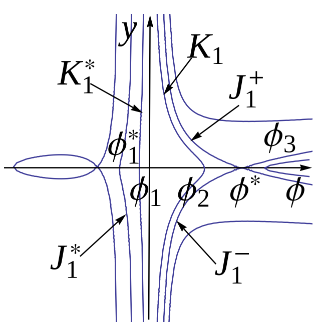
$$\frac{d\phi}{dw} = \phi y. \tag{2.3}$$



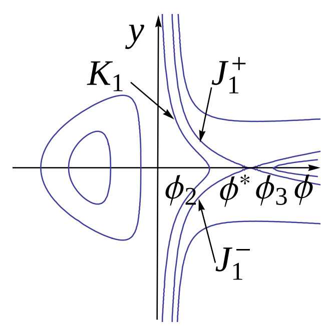
a. When $\alpha < 0$, $c > 0$ and $g < -2c/3 \cdot \sqrt{-c/(3\alpha)}$, or $\alpha < 0$, $c < 0$ and $g < 0$



b. When $\alpha < 0$, $c > 0$ and $g = -2c/3 \cdot \sqrt{-c/(3\alpha)}$



c. When $\alpha < 0$, $c > 0$ and $-2c/3 \cdot \sqrt{-c/(3\alpha)} < g < 0$



d. When $\alpha < 0$, $c > 0$ and $g = 0$

Figure 5. The phase portraits of system (1.4)

2.1. The proof of Proposition 1.1

Proof. Under any one of the following conditions, letting ϕ_0 be original, we have $H(\phi_0, 0) = H(\phi^*, 0) = h_0$. $H(\phi, y) = h_0$ determines two heteroclinic orbits and an open curve. Corresponding to these orbits, equation (1.1) has a solitary loop wave.

(2.1a) $\alpha < 0$, $c > 0$, $g < 0$ and $\phi_0 = \phi^*$ (see Figure 5(a,b,c) J_1^-, J_1^+ and J_1^*).
(2.1b) $\alpha < 0$, $c < 0$, $g < 0$ and $\phi_0 = \phi^*$ (see Figure 5(a) J_1^-, J_1^+ and J_1^*).
(2.1c) $\alpha > 0$, $c < 0$, $g < 0$ and $\phi_0 = \phi^*$ (see Figure 6(a,b,c) J_2^-, J_2^+ and J_2^*).
(2.1d) $\alpha > 0$, $c > 0$, $g < 0$ and $\phi_0 = \phi^*$ (see Figure 6(a) J_2^-, J_2^+ and J_2^*).

In cases (2.1a, 2.1b), we have

$$f(\phi) = -2\alpha(\phi - \phi_1^*)(\phi - \phi^*)^2, \tag{2.4}$$

where $\phi_1^* \le \phi < \phi^*$. From equation (2.1), J_1^-, J_1^+ and J_1^* have the expression

$$y = \pm(\phi^* - \phi)\sqrt{\frac{-2\alpha(\phi - \phi_1^*)}{\phi^2}}. \tag{2.5}$$

Substitute equation (2.5) into equation (2.3), and then integrate it along J_1^-, J_1^+ and J_1^* to obtain

$$\phi = \phi_1^* + (\phi^* - \phi_1^*)\tanh^2\left[\sqrt{\frac{\alpha(\phi_1^* - \phi^*)}{2}}w\right]. \tag{2.6}$$

Substitute equation (2.6) into equation (2.2), and then integrate it to obtain

$$\xi = \sqrt{\frac{2}{\alpha(\phi_1^* - \phi^*)}}\left\{\phi^*\sqrt{\frac{\alpha(\phi_1^* - \phi^*)}{2}}w - (\phi^* - \phi_1^*)\tanh\left[\sqrt{\frac{\alpha(\phi_1^* - \phi^*)}{2}}w\right]\right\}. \tag{2.7}$$

Thus, we obtain a solitary loop wave solution $u(x,t) = \phi(\xi)$ of parametric type as (1.7).

In cases (2.1c, 2.1d), we have

$$f(\phi) = -2\alpha(\phi - \phi^*)^2(\phi - \phi_1^*), \tag{2.8}$$

where $\phi^* \le \phi < \phi_1^*$. From equation (2.1), J_2^-, J_2^+ and J_2^* have the expression

$$y = \pm(\phi - \phi^*)\sqrt{\frac{-2\alpha(\phi - \phi_1^*)}{\phi^2}}. \tag{2.9}$$

Substitute equation (2.9) into equation (2.3), and then integrate it along J_2^-, J_2^+ and J_2^* to obtain

$$\phi = \phi_1^* - (\phi_1^* - \phi^*)\tanh^2\left[\sqrt{\frac{\alpha(\phi_1^* - \phi^*)}{2}}w\right]. \tag{2.10}$$

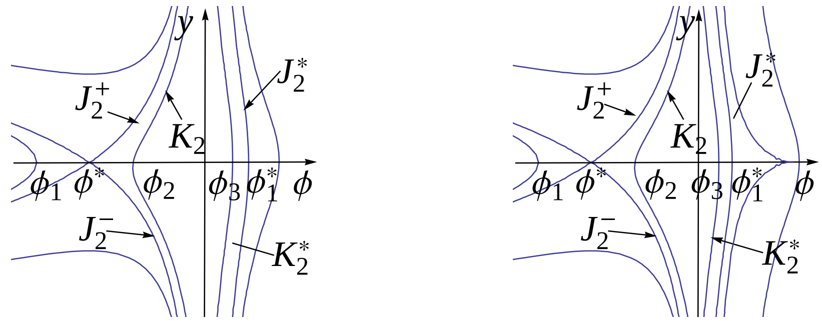
Substitute equation (2.10) into equation (2.2), and then integrate it to obtain

$$\xi = \sqrt{\frac{2}{\alpha(\phi_1^* - \phi^*)}}\left\{\phi^*\sqrt{\frac{\alpha(\phi_1^* - \phi^*)}{2}}w + (\phi_1^* - \phi^*)\tanh\left[\sqrt{\frac{\alpha(\phi_1^* - \phi^*)}{2}}w\right]\right\}. \tag{2.11}$$

Thus, we obtain a solitary loop wave solution $u(x,t) = \phi(\xi)$ of parametric type as (1.8).

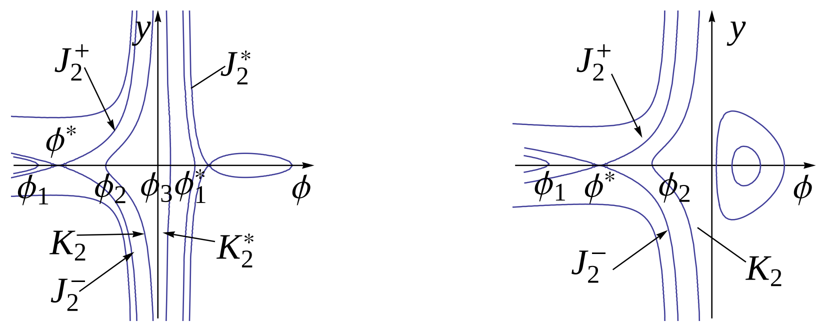
Specially, from (1.6), when $g = 0$, then $\phi = 0$ is a zero of $f(\phi)$ (see Figure 5(d) J_1^-, J_1^+ and Figure 6(d) J_2^-, J_2^+). Thus, under $\alpha c < 0$, when $g \to 0$, then $\phi_1^* \to 0$, the solitary loop wave becomes peakon, and parametric type becomes as (1.9).

Here, we complete the proof of Proposition 1.1. □



a. When $\alpha > 0$, $c < 0$ and $g < 2c/3 \cdot \sqrt{-c/(3\alpha)}$, or $\alpha > 0$, $c > 0$ and $g < 0$

b. When $\alpha > 0$, $c < 0$ and $g = 2c/3 \cdot \sqrt{-c/(3\alpha)}$



c. When $\alpha > 0$, $c < 0$ and $2c/3 \cdot \sqrt{-c/(3\alpha)} < g < 0$

d. When $\alpha > 0$, $c < 0$ and $g = 0$

Figure 6. The phase portraits of system (1.4)

2.2. The proof of Proposition 1.2

Proof. Under any one of the following conditions, we have $H(\phi_0, 0) = h_0$. $H(\phi, y) = h_0$ determines two open curves. Corresponding to these orbits, equation (1.1) has a periodic loop wave.

(2.2a) $\alpha < 0, c > 0, g < 0$ and $0 < \phi_0 < \phi^*$ (see Figure 5(a,b,c) K_1, and K_1^*).
(2.2b) $\alpha < 0, c < 0, g < 0$ and $0 < \phi_0 < \phi^*$ (see Figure 5(a) K_1, and K_1^*).
(2.2c) $\alpha > 0, c < 0, g < 0$ and $\phi^* < \phi_0 < 0$ (see Figure 6(a,b,c) K_2, and K_2^*).
(2.2d) $\alpha > 0, c > 0, g < 0$ and $\phi^* < \phi_0 < 0$ (see Figure 6(a) K_2, and K_2^*).

In any one of the above cases, we have

$$f(\phi) = -2\alpha(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3), \tag{2.12}$$

where ϕ_1, ϕ_2 and ϕ_3 are the real roots of the equation $f(\phi) = 0$.

From equation (2.1), K_1 and K_1^* have the expression

$$y = \pm\sqrt{\frac{-2\alpha(\phi - \phi_1)(\phi_2 - \phi)(\phi_3 - \phi)}{\phi^2}}. \tag{2.13}$$

In any one of cases (2.2a, 2.2b), we have $\phi_1 \le \phi \le \phi_2 < \phi_3$. Substitute equation (2.13) into equation (2.3), and then integrate it along K_1 and K_1^* to obtain

$$\phi = \phi_1 + (\phi_2 - \phi_1)\mathrm{sn}^2\left[\sqrt{\frac{\alpha(\phi_1-\phi_3)}{2}}w, \sqrt{\frac{\phi_2-\phi_1}{\phi_3-\phi_1}}\right]. \tag{2.14}$$

Substitute equation (2.14) into equation (2.2), and then integrate it to obtain

$$\xi = \sqrt{\frac{2}{\alpha(\phi_1-\phi_3)}}\left\{\phi_3\sqrt{\frac{\alpha(\phi_1-\phi_3)}{2}}w - (\phi_3-\phi_1)\mathrm{E}\left[\sqrt{\frac{\alpha(\phi_1-\phi_3)}{2}}w, \sqrt{\frac{\phi_2-\phi_1}{\phi_3-\phi_1}}\right]\right\}. \tag{2.15}$$

Thus, we obtain a periodic loop wave solution $u(x,t) = \phi(\xi)$ of parametric type as (1.10).

Specially, when $g = 0$, then $\phi_1 = 0$ (see Figure 5(d) K_1, K_1^*). Thus, under $\alpha < 0$ and $c > 0$, when $g \to 0$, then $\phi_1 \to 0$, the periodic loop wave becomes periodic cusp wave, and parametric type becomes as (1.12).

In cases (2.2c, 2.2d), we have $\phi_1 < \phi_2 \le \phi \le \phi_3$. From equation (2.1), K_2 and K_2^* have the expression:

$$y = \pm\sqrt{\frac{2\alpha(\phi-\phi_1)(\phi-\phi_2)(\phi_3-\phi)}{\phi^2}}. \tag{2.16}$$

Substitute equation (2.16) into equation (2.3), and then integrate it along K_2 and K_2^* to obtain

$$\phi = \phi_3 - (\phi_3 - \phi_2)\mathrm{sn}^2\left[\sqrt{\frac{\alpha(\phi_3-\phi_1)}{2}}w, \sqrt{\frac{\phi_3-\phi_2}{\phi_3-\phi_1}}\right]. \tag{2.17}$$

Substitute equation (2.17) into equation (2.2), and then integrate it to obtain

$$\xi = \sqrt{\frac{2}{\alpha(\phi_3-\phi_1)}}\left\{\phi_1\sqrt{\frac{\alpha(\phi_3-\phi_1)}{2}}w + (\phi_3-\phi_1)\mathrm{E}\left[\sqrt{\frac{\alpha(\phi_3-\phi_1)}{2}}w, \sqrt{\frac{\phi_3-\phi_2}{\phi_3-\phi_1}}\right]\right\}. \tag{2.18}$$

Thus, we obtain a periodic loop wave solution $u(x,t) = \phi(\xi)$ of parametric type as (1.11).

Specially, when $g = 0$, then $\phi_3 = 0$ (see Figure 6(d) K_2, K_2^*). Thus, under $\alpha > 0$ and $c < 0$, when $g \to 0$, then $\phi_3 \to 0$, the periodic loop wave becomes periodic cusp wave, and parametric type becomes as (1.13).

Here, we complete the proof of Proposition 1.2. □

3. Conclusion

In this paper, we obtained exact loop wave solutions of equation (1.1) by using the theory of dynamical system. We draw the bifurcation phase portraits of the singular traveling wave system (1.4). Through studying the shape of periodic waves, we showed that the loop of equation (1.1) have solitary loop waves and periodic loop waves (see Figures 1 and 3). The limit of periodic loop waves are periodic cusp waves, the limit of solitary loop waves are peakons (see Figures 2 and 4).

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