Stability Analysis of an Eco-epidemiological Model with Time Delay and Holling Type-III Functional Response*

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Abstract In this paper, an eco-epidemiological model with diseases in the predator and Holling type-III functional response is analyzed. A time delay due to the gestation of the predator is considered in this model. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria and the existence of Hopf bifurcations at the disease-free equilibrium and the endemic-coexistence equilibrium are established respectively. By using Lyapunov functionals and LaSalle's invariance principle, sufficient conditions are obtained for the global stability of the predator-extinction equilibrium, the disease-free equilibrium and the endemic-coexistence equilibrium respectively. Finally, numerical simulations are performed to illustrate the theoretical results.

Keywords Eco-epidemiological model, Time delay, Holling type-III functional response, Stability, Hopf bifurcation.

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1. Introduction

Dynamics of predator-prey model is one of important subjects in mathematical ecology, and some important results have been studied and derived by many researchers ([2, 8, 9, 13, 14, 16, 17, 22]). It is well-known that the effect of disease on ecological system is an important topic. Since the pioneering work of Anderson and May [1], many works have been devoted to the study of the effects of a disease on a predator-prey model recently ([3, 6, 7, 10, 12, 15, 18–21]). Most of these works dealt with predator-prey models with disease are in the prey. Recently, several authors have proposed different eco-epidemiological predator-prey models by assuming that the predator population suffers a transmissible disease ([3,6,7,10,15,18,21]). In [10], by assuming that a transmissible disease spreads among the predator population,

Lu et al., considered the following eco-epidemiological model

$$\dot{x}(t) = rx(t) - ax^2(t) - a_1 x(t)S(t),$$
$$\dot{S}(t) = a_2 x(t-\tau)S(t-\tau) - r_1 S(t) - \alpha S(t)I(t) + \sigma I(t), \qquad (1.1)$$
$$\dot{I}(t) = \alpha S(t)I(t) - (r_2 + \sigma)I(t),$$

where $x(t)$, $S(t)$ and $I(t)$ represent the densities of the prey population, the susceptible predator and the infected predator at time t respectively. The parameters a, a_1, a_2, r, r_1, r_2, α and σ are positive constants. In model (1.1), the following assumptions have been made:

(A1) In the absence of predation, the prey population x grows logistically with the intrinsic growth rate r and carrying capacity r/a.

(A2) The total predator population $N(t)$ is divided into two distinct classes: susceptible (sound) predator $S(t)$ and infected predator $I(t)$. Therefore, at any time t, the total density of predator population is $N(t) = S(t) + I(t)$.

(A3) The disease spreads among the predator species only by contact, and the disease cannot be transmitted vertically. The disease incidence is assumed to be the bilinear incidence αSI, where $\alpha > 0$ is called the disease transmission coefficient. The parameter σ represents the recovery rate, i.e., the rate at which the infected predators recover from the disease and become susceptible again.

(A4) Only the susceptible predators have the ability to capture the prey with capturing rate a_1, and the infected predator are unable to catch the prey because of a high infection. The ratio a_2/a_1 is the conversion rate of nutrients into the reproduction of the susceptible predators by consuming prey. The parameters r_1 and r_2 are the natural death rate of the susceptible predator and the infected predator respectively. Here, $r_1 \leq r_2$.

(A5) The reproduction of the predator population after consuming prey ia not instantaneous, and a duration of τ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the predator population.

In [10], the stability of each of feasible equilibria and the existence of Hopf bifurcations at the disease-free equilibrium and the positive equilibrium are established respectively. By means of Lyaponov functionals and LaSalle's invariance principle [4,5], sufficient conditions for the global stability of each of the nonnegative equilibria of model (1.1) have been figured out.

In model (1.1), it is assumed that the per capita rate of predation depends on the prey numbers only. However, Holling found that each predator increased its consumption rate when exposed to a higher prey density, and also predator density increased with growing prey density. In 1965, Holling [8] used the following function

$$p(x) = \frac{x^2}{1 + mx^2}$$

as one of the predator response functions. It is now referred to as the Holling type-III response function. Holling type-III functional response reveals that the risk of

being preyed upon is small at low prey density, but rises to a certain point as prey density increases, which is in accordance with some phenomena of natural world. Also, we know that many factors contribute to a type III functional response such as prey refuge, predator learning and the presence of alternative.

To this end, we study the following epidemiological model

$$
\begin{aligned}
\dot{x}(t) &= rx(t) - ax^2(t) - \frac{a_1 x^2(t)S(t)}{1 + mx^2(t)}, \\
\dot{S}(t) &= \frac{a_2 x^2(t-\tau)S(t-\tau)}{1 + mx^2(t-\tau)} - r_1 S(t) - \alpha S(t)I(t) + \sigma I(t), \\
\dot{I}(t) &= \alpha S(t)I(t) - (r_2 + \sigma)I(t),
\end{aligned}
\tag{1.2}
$$

where all the parameters are defined as in $(A1)$-$(A5)$.

The initial conditions for model (1.2) take the form

$$
\begin{aligned}
& x(\theta) = \varphi(\theta) \geq 0, \quad S(\theta) = \phi_1(\theta) \geq 0, \quad I(\theta) = \phi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0), \\
& \varphi(0) > 0, \quad \phi_1(0) > 0, \quad \phi_2(0) > 0, \quad (\varphi(\theta), \phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], R_{+0}^3),
\end{aligned}
\tag{1.3}
$$

where

$$
R_{+0}^3 = \{(y_1, y_2, y_3) : y_i \geq 0, i = 1, 2, 3\}.
\tag{1.4}
$$

By the fundamental theory of functional differential equations [11], it is well-known that model (1.2) has a unique solution $(x(t), S(t), I(t))$ satisfying initial conditions (1.3). It is easy to show that all solutions of model (1.2) with initial conditions (1.3) are defined on $[0, +\infty]$, and remain positive for all $t \geq 0$.

This paper is organized as follows. In Section 2, by using the theory on characteristic equation of delay differential equations developed by [11], we discuss the local stability of each of feasible equilibria of model (1.2). We establish the existence of Hopf bifurcations at the disease-free equilibrium and the endemic-coexistence equilibrium. In Section 3, by means of Lyaponov functionals and LaSalle's invariance principle, we obtain sufficient conditions for the global stability of the endemic-coexistence equilibrium, the disease-free equilibrium and the predator-extinction equilibrium of model (1.2) respectively. In Section 4, some numerical simulations are presented to illustrate the main results. The paper ends with a brief discussion in Section 5.

2. Local stability and Hopf bifurcations

In this section, we discuss the local stability of each of feasible equilibria of model (1.2) by analyzing the corresponding characteristic equations respectively.

It is easy to show that model (1.2) always has a trivial equilibrium $E_0(0, 0, 0)$ and a predator-extinction equilibrium $E_1(r/a, 0, 0)$. Model (1.2) has a disease-free equilibrium $E_2(x_2, S_2, 0)$ provided that

$$
\mathcal{R}_0 = \frac{a_2 r^2}{r_1(a^2 + mr^2)} > 1,
\tag{2.1}
$$

where

$$x_2 = \sqrt{\frac{r_1}{a_2 - mr_1}}, \quad S_2 = \frac{a_2(r - ax_2)}{a_1 x_2(a_2 - mr_1)}. \tag{2.2}$$

Further, if the following holds:

$$\mathcal{R}_1 = \frac{\alpha S_2}{r_2 + \sigma} > 1, \tag{2.3}$$

then model (1.2) has a positive (endemic-coexistence) equilibrium $E_+(x_+, S_+, I_+)$, where

$$S_+ = \frac{r_2 + \sigma}{\alpha}, \quad I_+ = \frac{r_2 + \sigma}{\alpha r_2}\left(\frac{a_2 x_+^2}{1 + m x_+^2} - r_1\right), \tag{2.4}$$

in which x_+ is a positive real root of the following cubic equation

$$a\alpha m x^3 - \alpha m r x^2 + (a\alpha + a_1 r_2 + a_1 \sigma)x - \alpha r = 0. \tag{2.5}$$

2.1. Local stability of boundary equilibria and Hopf bifurcation

It is not difficult to show that $E_0(0, 0, 0)$ is always unstable.

The characteristic equation of model (1.2) at the equilibrium $E_1(r/a, 0, 0)$ is of the form

$$(\lambda + r)(\lambda + r_2 + \sigma)\left(\lambda + r_1 - \frac{a_2 r^2}{a^2 + mr^2}e^{-\lambda\tau}\right) = 0. \tag{2.6}$$

Note that equation (2.1) always has two negative real roots: $\lambda_1 = -r$, $\lambda_2 = -(r_2 + \sigma)$. All other roots of (2.1) are determined by the following equation:

$$\lambda + r_1 - \frac{a_2 r^2}{a^2 + mr^2}e^{-\lambda\tau} = 0. \tag{2.7}$$

Let

$$g(\lambda) = \lambda + r_1 - \frac{a_2 r^2}{a^2 + mr^2}e^{-\lambda\tau}.$$

For λ is real, a direct calculation shows that

$$g(0) = r_1 - \frac{a_2 r^2}{a^2 + mr^2} = r_1(1 - \mathcal{R}_0), \quad \lim_{\lambda \to +\infty} g(\lambda) = +\infty.$$

Hence, if $\mathcal{R}_0 > 1$, then $g(\lambda) = 0$ has at least one positive real root. Accordingly, the equilibrium E_1 is unstable. If $\mathcal{R}_0 < 1$, we show that E_1 is locally asymptotically stable. Otherwise, there is a root λ satisfying $Re\lambda \geq 0$. It follows from (2.2) that

$$Re\lambda = \frac{a_2 r^2 e^{-\tau Re\lambda}}{a^2 + mr^2}\cos(\tau Im\lambda) - r_1 \leq \frac{a_2 r^2}{a^2 + mr^2} - r_1 = r_1(\mathcal{R}_0 - 1) < 0.$$

A contradiction occurs. Hence, if $\mathcal{R}_0 < 1$, the equilibrium E_1 is locally asymptotically stable.

The characteristic equation of model (1.2) at the equilibrium $E_2(x_2, S_2, 0)$ takes the form

$$(\lambda + r_2 + \sigma - \alpha S_2)[\lambda^2 + h_1\lambda + h_0 + (l_1\lambda + l_0)e^{-\lambda\tau}] = 0, \tag{2.8}$$

where

$$h_1 = r_1 + 2ax_2 - r + \frac{2a_1x_2S_2}{(1+mx_2^2)^2}, \; h_0 = r_1\left(2ax_2 - r + \frac{2a_1x_2S_2}{(1+mx_2^2)^2}\right),$$

$$l_1 = -r_1, \qquad\qquad\qquad l_0 = -r_1(2ax_2 - r).$$

Clearly, equation (2.3) always has a root

$$\lambda_1 = \alpha S_2 - (r_2 + \sigma).$$

If $\mathcal{R}_1 > 1$, then $\lambda_1 > 0$. Thus, the equilibrium E_2 is unstable in this case.

If $\mathcal{R}_1 < 1$, then $\lambda_1 < 0$. All other roots of equation (2.3) are determined by the following equation:

$$\lambda^2 + h_1\lambda + h_0 + (l_1\lambda + l_0)e^{-\lambda\tau} = 0. \tag{2.9}$$

When $\tau = 0$, equation (2.4) reduces to

$$\lambda^2 + (h_1 + l_1)\lambda + (h_0 + l_0) = 0. \tag{2.10}$$

It is easy to show that

$$h_1 + l_1 = 2ax_2 - r + \frac{2a_1x_2S_2}{(1+mx_2^2)^2},$$

$$h_0 + l_0 = \frac{2a_1r_1x_2S_2}{(1+mx_2^2)^2} > 0.$$

Hence, if $\mathcal{R}_1 < 1$ and $h_1 + l_1 > 0$, then the equilibrium E_2 is locally asymptotically stable, when $\tau = 0$.

If $i\omega(\omega > 0)$ is a solution of (2.4), we obtain that

$$\omega^4 + (h_1^2 - 2h_0 - l_1^2)\omega^2 + h_0^2 - l_0^2 = 0. \tag{2.11}$$

It is easy to show that

$$h_1^2 - 2h_0 - l_1^2 = (h_1 + l_1)^2 > 0,$$

$$h_0^2 - l_0^2 = 2r_1(h_0 + l_0)\left(2ax_2 - r + \frac{a_1x_2S_2}{(1+mx_2^2)^2}\right).$$

Let

$$\gamma_0 = 2ax_2 - r + \frac{a_1x_2S_2}{(1+mx_2^2)^2}.$$

Hence, if $\gamma_0 > 0$, then equation (2.6) has no positive real roots. Note that $\gamma_0 > 0$ implies $h_1 + l_1 > 0$. Therefore, if $\mathcal{R}_1 < 1$ and $\gamma_0 > 0$, the equilibrium E_2 is locally asymptotically stable, for all $\tau \geq 0$.

If $\gamma_0 < 0$, equation (2.6) has a unique positive root ω_0. Accordingly, equation (2.4) has a pair of imaginary roots of the form $\pm i\omega_0$. Denote

$$\tau_k = \frac{1}{\omega_0}\arccos\frac{l_0(\omega_0^2 - h_0) - h_1l_1\omega_0^2}{l_0^2 + h_1^2\omega_0^2} + \frac{2k\pi}{\omega_0}, \quad k = 0, 1, 2, \cdots.$$

By Theorem 3.4.1, in the work of Kuang [11], we see that if $\mathcal{R}_1 < 1$ and $\gamma_0 < 0$ hold, then E_2 remains stable for $\tau < \tau_0$.

Let $\lambda(\tau) = \upsilon(\tau) + i\omega(\tau)$ be a root of equation (2.4) satisfying $\upsilon(\tau_0) = 0, \omega(\tau_0) = \omega_0$. Differentiating equation (2.4) with respect to τ, we obtain that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + h_1}{-\lambda(\lambda^2 + h_1\lambda + h_0)} + \frac{l_1}{\lambda(l_1\lambda + l_0)} - \frac{\tau}{\lambda}.$$

A direct calculation shows that

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} = sign\left\{\frac{d\lambda}{d\tau}\right\}_{\lambda=i\omega_0}^{-1}$$
$$= sign\left\{\frac{2\omega_0^2 + h_1^2 - 2h_0}{(h_0 - \omega_0^2)^2 + h_1^2\omega_0^2} - \frac{l_1^2}{l_0^2 + l_1^2\omega_0^2}\right\}$$
$$= sign\left\{\frac{2\omega_0^2 + h_1^2 - 2h_0 - l_1^2}{l_0^2 + l_1^2\omega_0^2}\right\} > 0.$$

This shows that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [5] are then satisfied by yielding a periodic solution.

In conclusion, we have the following results.

Theorem 2.1. *For model (1.2), we have*

(i) *if $\mathcal{R}_0 < 1$, then the equilibrium $E_1(r/a, 0, 0)$ is locally asymptotically stable; if $\mathcal{R}_0 > 1$, then E_1 is unstable.*

(ii) *let $\mathcal{R}_0 > 1$ hold, and if $\gamma_0 > 0$, then the equilibrium $E_2(x_2, S_2, 0)$ is locally asymptotically stable for all $\tau \geq 0$; if $\mathcal{R}_1 < 1$ and $\gamma_0 < 0$, then there exists a positive number τ_0 such that E_2 is locally asymptotically stable, if $0 < \tau < \tau_0$ and unstable, if $\tau > \tau_0$. Further, model (1.2) undergoes a Hopf bifurcation at E_2, when $\tau = \tau_0$; if $\mathcal{R}_1 > 1$, then the equilibrium E_2 is unstable.*

2.2. Local stability of positive equilibrium and Hopf bifurcation

Now, we study the local stability of the positive equilibrium $E_+(x_+, S_+, I_+)$ of model (1.2). The characteristic equation of model (1.2) at the equilibrium E_+ takes the form

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_2\lambda^2 + q_1\lambda)e^{-\lambda\tau} = 0, \tag{2.12}$$

where

$$p_2 = r_1 + \alpha I_+ + 2ax_+ - r + \frac{2a_1 x_+ S_+}{(1 + mx_+^2)^2},$$
$$p_1 = r_2\alpha I_+ + (r_1 + \alpha I_+)\left(2ax_+ - r + \frac{2a_1 x_+ S_+}{(1 + mx_+^2)^2}\right),$$
$$p_0 = r_2\alpha I_+ \left(2ax_+ - r + \frac{2a_1 x_+ S_+}{(1 + mx_+^2)^2}\right),$$
$$q_2 = -(r_1 + \alpha I_+) + \sigma\frac{I_+}{S_+},$$
$$q_1 = (2ax_+ - r)\left(-(r_1 + \alpha I_+) + \sigma\frac{I_+}{S_+}\right).$$

When $\tau = 0$, equation (2.7) reduces to

$$\lambda^3 + (p_2 + q_2)\lambda^2 + (p_1 + q_1)\lambda + p_0 = 0. \tag{2.13}$$

By direct calculation, we obtain

$$p_2 + q_2 = 2ax_+ - r + \frac{2a_1x_+S_+}{(1 + mx_+^2)^2} + \sigma\frac{I_+}{S_+},$$

$$(p_2 + q_2)(p_1 + q_1) - p_0$$
$$= r_2\alpha\sigma\frac{I_+^2}{S_+} + \left(\sigma\frac{I_+}{S_+}(2ax_+ - r + \frac{2a_1x_+S_+}{(1 + mx_+^2)^2}) + \frac{2\alpha a_1x_+^2S_+}{(1 + mx_+^2)^2}\right)(p_2 + q_2).$$

Hence, if $\gamma_0 > 0$, then
$p_2 + q_2 > 0$, and $(p_2 + q_2)(p_1 + q_1) > p_0$,
Hence, by Routh-Hurwitz theorem, the positive equilibrium E_+ is locally asymptotically stable, when $\tau = 0$.

If $\lambda = i\omega(\omega > 0)$ is a solution of (2.7), separating the real and imaginary parts, we obtain that

$$\begin{aligned} q_2\omega^2\cos\omega\tau - q_1\omega\sin\omega\tau &= -p_2\omega^2 + p_0, \\ q_2\omega^2\sin\omega\tau + q_1\omega\cos\omega\tau &= \omega^3 - p_1\omega. \end{aligned} \tag{2.14}$$

Squaring and adding the two equations of (2.9), it follows that

$$\omega^6 + h_2\omega^4 + h_1\omega^2 + h_0 = 0, \tag{2.15}$$

where

$$h_2 = p_2^2 - q_2^2 - 2p_1, \ h_1 = p_1^2 - q_1^2 - 2p_0p_2, \ h_0 = p_0^2.$$

Letting $z = \omega^2$, equation (2.10) can be written as

$$h(z) = z^3 + h_2z^2 + h_1z + h_0 = 0. \tag{2.16}$$

Denote $\triangle = h_2^2 - 3h_1$. It is easy to show that if $\triangle \leq 0$, function $h(z)$ is strictly monotonically increasing. If $\triangle > 0$ and $z_* = (\sqrt{\triangle} - h_2)/3 < 0$ or $\triangle > 0$, $z_* = (\sqrt{\triangle} - h_2)/3 > 0$ but $h(z_*) > 0$, then $h(z)$ has always no positive roots. Hence, under these conditions, equation (2.7) has no purely imaginary roots, for any $\tau > 0$ and accordingly the equilibrium E_+ is locally asymptotically stable, for all $\tau \geq 0$.

In the following, we assume that
$(H_1)\ \triangle > 0,\ z_* = (\sqrt{\triangle} - h_2)/3 > 0,\ h(z_*) \leq 0.$
In this case, by Lemma 2.2 in [17], we see that equation (2.11) has at least one positive root. Without loss of generality, we assume that (2.11) has two positive roots, namely, z_1 and z_2 respectively. Accordingly, equation (2.10) has two positive roots $\omega_k = \sqrt{z_k}(k = 1, 2)$.

For k=1, 2, from (2.10), one can obtain the corresponding $\tau_k^j > 0$ such that (2.7) has a pair of purely imaginary roots $\pm i\omega_k$ given by

$$\tau_k^j = \frac{1}{\omega_k}\arccos\left[-\frac{(p_2q_2 - q_1)\omega_k^2 + p_1q_1 - p_0q_2}{q_2^2\omega_k^2 + q_1^2}\right] + \frac{2\pi j}{\omega_k}, \ j = 0, 1, 2, \cdots. \tag{2.17}$$

Let $\lambda(\tau) = \upsilon(\tau) + i\omega(\tau)$ be a root of equation (2.7) satisfying $\upsilon(\tau_k^j) = 0$, $\omega(\tau_k^j) = \omega_k$

Differentiating the two sides of (2.7) with respect τ, it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{2q_2\lambda + q_1}{\lambda(q_2\lambda^2 + q_1\lambda)} - \frac{\tau}{\lambda}.$$

By direct calculation, one obtains that

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega_k} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_k}$$

$$= sign\left\{Re\left[\frac{3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{2q_2\lambda + q_1}{\lambda(q_2\lambda^2 + q_1\lambda)}\right]_{\lambda=i\omega_k}\right\}$$

$$= sign\left\{-\frac{(p_1 - 3\omega_k^2)(\omega_k^2 - p_1) + 2p_2(p_0 - p_2\omega_k^2)}{(\omega_k^3 - p_1\omega_k)^2 + (p_0 - p_2\omega_k^2)^2} - \frac{q_1^2 + 2q_2^2\omega_k^2}{(q_2\omega_k^2)^2 + q_1^2\omega_k^2}\right\}.$$

It follows from (2.9) that

$$(\omega_k^3 - p_1\omega_k)^2 + (p_0 - p_2\omega_k^2)^2 = (q_2\omega_k^2)^2 + q_1^2\omega_k^2.$$

Hence, we obtain that

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega_k} = sign\left\{\frac{3\omega_k^4 + 2(p_2^2 - 2p_1 - q_2^2)\omega_k^2 + p_1^2 - 2p_0p_2 - q_1^2}{(q_2\omega_k^2)^2 + q_1^2\omega_k^2}\right\}$$

$$= sign\left[\frac{h'(z_k)}{(q_2\omega_k^2)^2 + q_1^2\omega_k^2}\right].$$

Accordingly, if $(H1)$ holds, then we have that

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\tau=\tau_0} > 0.$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\tau = \tau_0$.

From what has been discussed previously, we obtain the following results.

Theorem 2.2. *For model (1.2), assume that $\mathcal{R}_1 > 1$ and $\gamma_0 > 0$ hold. Then, the following results hold true:*

(i) *If $\triangle \leq 0$ or $\triangle > 0$ and $z_* = (\sqrt{\triangle} - h_2)/3 < 0$ or $\triangle > 0$, $z_* = (\sqrt{\triangle} - h_2)/3 > 0$ and $h(z_*) > 0$, then the equilibrium E_+ is locally asymptotically stable for all $\tau \geq 0$.*

(ii) *Let (H_1) hold, then $h(z)$ has at least one positive root z_k, and all roots of (2.7) have negative real parts for $\tau \in [0, \tau_k^0)$, and the equilibrium E_+ of model (1.2) is locally asymptotically stable for $\tau \in [0, \tau_k^0)$.*

(iii) *If all conditions as stated in (ii) hold true and $h'(z_k) \neq 0$, then model (1.2) undergoes a Hopf bifurcation at E_+, when $\tau = \tau_k^j (j = 0, 1, 2, \cdots)$.*

3. Global stability

In this section, we are concerned with the global stability of the positive equilibrium $E_+(x_+, S_+, I_+)$, the disease-free equilibrium $E_2(x_2, S_2, 0)$ and the predator-extinction equilibrium $E_1(r/a, 0, 0)$ of model (1.2) respectively. The strategy of proofs is to use global Lyapunov functional and LaSalle's invariance principle.

Theorem 3.1. *Assume that $\mathcal{R}_1 > 1$, then the positive equilibrium E_+ of model (1.2) is globally attractive, provided (H_2) $\underline{x} > r/(2a)$.*
Here, $\underline{x}$ is the persistency constant for x satisfying $\liminf_{t \to +\infty} x(t) \geq \underline{x}$.

Proof. Let $(x(t), S(t), I(t))$ be any positive solution of model (1.2) with initial conditions (1.3). Denote $f(x) = x^2/(1 + mx^2)$. Define

$$
V_+(t) = c \left(x(t) - x_+ - \int_{x_+}^{x(t)} \frac{f(x_+)}{f(u)} du \right) + S(t) - S_+ - S_+ \ln \frac{S(t)}{S_+}
$$
$$
+ k \left(I(t) - I_+ - I_+ \ln \frac{I(t)}{I_+} \right),
$$

where $c = a_2/a_1$, $k = 1 - \sigma/(\alpha S_+)$. Calculating the derivative of $V_+(t)$ along positive solutions of model (1.2), it follows that

$$
\dot{V}_+(t) = c \left(1 - \frac{f(x_+)}{f(x(t))} \right) (rx(t) - ax^2(t)) - a_2 f(x(t)) S(t) + a_2 f(x(t - \tau)) S(t - \tau)
$$
$$
- \frac{a_2 S_+}{S(t)} f(x(t - \tau)) S(t - \tau) - \sigma I(t) \left(\frac{S(t)}{S_+} + \frac{S_+}{S(t)} - 2 \right) + a_2 f(x_+) S_+.
\tag{3.1}
$$

Define

$$
V(t) = V_+(t) + a_2 \int_{t-\tau}^{t} \left[f(x(u)) S(u) - f(x_+) S_+ - f(x_+) S_+ \ln \frac{f(x(u)) S(u)}{f(x_+) S_+} \right].
\tag{3.2}
$$

A direct calculation shows that

$$
\dot{V}(t) = c \frac{f(x(t)) - f(x_+)}{f(x(t))} (x(t) - x_+)[r - ax(t) - ax_+]
$$
$$
- a_2 f(x_+) S_+ \left[\frac{f(x(t-\tau)) S(t-\tau)}{f(x_+) S(t)} - 1 - \ln \frac{f(x(t-\tau)) S(t-\tau)}{f(x_+) S(t)} \right]
$$
$$
- a_2 f(x_+) S_+ \left[\frac{f(x_+)}{f(x(t))} - 1 - \ln \frac{f(x_+)}{f(x(t))} \right] - \sigma I(t) \left[\frac{S(t)}{S_+} + \frac{S_+}{S(t)} - 2 \right]
$$
$$
= \frac{-(x(t) + x_+)(x(t) - x_+)^2}{x^2(t)(1 + mx_+^2)} [a(x(t) + x_+) - r]
$$
$$
- a_2 f(x_+) S_+ \left[\frac{f(x(t-\tau)) S(t-\tau)}{f(x_+) S(t)} - 1 - \ln \frac{f(x(t-\tau)) S(t-\tau)}{f(x_+) S(t)} \right]
$$
$$
- a_2 f(x_+) S_+ \left[\frac{f(x_+)}{f(x(t))} - 1 - \ln \frac{f(x_+)}{f(x(t))} \right] - \sigma I(t) \left[\frac{S(t)}{S_+} + \frac{S_+}{S(t)} - 2 \right].
\tag{3.3}
$$

For $\underline{x} > r/(2a)$ holds, there is a constant $T > 0$ such that if $t \geq T$, $x(t) > r/(2a)$. In this case, we have that, for $t \geq T$,

$$
\frac{-(x(t) + x_+)(x(t) - x_+)^2}{f(x(t))(1 + mx_+^2)(1 + mx^2(t))} [a(x(t) + x_+) - r] \leq 0
$$

with equality, if and only if $x(t) = x_+$. Note that the function $g(x) = x - 1 - \ln x$ is always non-negative for any $x > 0$ and $g(x) = 0$, if and only if $x = 1$. Therefore,

we have that if $t \geq T$, $\dot{V}(t) \leq 0$ with equality, if and only if $x(t) = x_+$, $S(t) = S(t - \tau) = s_+$. Now, we look for the invariant subset $\mathcal{M}$ within the set.

$$M = \{(x, S, I) : x(t) = x_+, \ S(t) = S_+\}$$

We obtain from the first equation of model (1.2) that

$$0 = \dot{S}(t) = \frac{a_2 x_+^2 S_+}{1 + m x_+^2} - r_1 S_+ - \alpha S_+ I(t) + \sigma I(t),$$

which leads to $I(t) = I_+$. Hence, the only invariant set in M is $\mathcal{M} = \{(x_+, s_+, I_+)\}$. Therefore, the global asymptotic stability of E_+ follows from LaSalle's invariance principle for delay differential systems (see, for example, [4]). This completes the proof.

Theorem 3.2. *Assume that $\mathcal{R}_0 > 1$, $\mathcal{R}_1 < 1$, $\gamma_0 > 0$ and (H_2) hold, then the disease-free equilibrium $E_2(x_2, S_2, 0)$ of model (1.2) is globally asymptotically stable.*

Proof. By Theorem 2.1, we see that if $\mathcal{R}_0 > 1$, $\mathcal{R}_1 < 1$ and $\gamma_0 > 0$ hold, the equilibrium E_2 is locally asymptotically stable for all $\tau \geq 0$. Hence, we only prove that all positive solutions of model (1.2) with initial conditions (1.3) converge to E_2. Let $(x(t), S(t), I(t))$ be any positive solution of model (1.2) with initial conditions (1.3). Define

$$V_{21}(t) = c \left(x(t) - x_2 - \int_{x_2}^{x(t)} \frac{f(x_2)}{f(u)} du \right) + S(t) - S_2 - S_2 \ln \frac{S(t)}{S_2} + k_1 I(t),$$

where $k_1 = 1 - \sigma/(\alpha S_2)$. Calculating the derivative of $V_{21}(t)$ along positive solutions of model (1.2), we obtain that

$$
\begin{aligned}
\dot{V}_{21}(t) &= c \left(1 - \frac{f(x_2)}{f(u)} \right) \dot{x}(t) + \left(1 - \frac{S_2}{S(t)} \right) \dot{S}(t) + k \dot{I}(t) \\
&= c \left(1 - \frac{f(x_2)}{f(u)} \right) \left[rx(t) - ax^2(t) \right] - a_1 f(x(t)) S(t) + a_2 f(x(t - \tau)) S(t - \tau) \\
&\quad - \frac{a_2 S_2}{S(t)} f(x(t - \tau)) S(t - \tau) - \sigma I(t) \left(\frac{S(t)}{S_2} + \frac{S_2}{S(t)} - 2 \right) - k_1 (r_2 + \sigma - \alpha S_2) I(t) \\
&\quad + r_1 S_2.
\end{aligned}
$$

$$(3.4)$$

Define

$$V_2(t) = V_{21}(t) + a_2 \int_{t-\tau}^{t} \left[f(x(u) S(u)) - f(x_2) S_2 - f(x_2) S_2 \ln \frac{f(x(u)) S(u)}{f(x_2) S_2} \right]. \quad (3.5)$$

A direct calculation shows that

$$
\begin{aligned}
\dot{V}_2(t) &= \frac{-c(x(t) + x_2)(x(t) - x_2)^2}{x^2(t)(1 + m x_2^2)} (a(x(t) + x_2) - r) \\
&\quad - a_2 f(x_2) S_2 \left[\frac{f(x(t - \tau)) S(t - \tau)}{f(x_2) S(t)} - 1 - \ln \frac{f(x(t - \tau)) S(t - \tau)}{f(x_2) S(t)} \right] \\
&\quad - a_2 f(x_2) S_2 \left[\frac{f(x_2)}{f(x(t))} - 1 - \ln \frac{f(x_2)}{f(x(t))} \right] - \delta I(t) \left[\frac{S(t)}{S_2} + \frac{S_2}{S(t)} - 2 \right] \\
&\quad - k(r_2 + \sigma)(1 - \mathcal{R}_1) I(t).
\end{aligned}
$$

$$(3.6)$$

Hence, it follows from (3.6) that if $\mathcal{R}_1 < 1$ and (H_2) hold true, then $\dot{V}_2(t) \leq 0$ for all $t > T$ with equality, if and only if $S(t) = S(t - \tau) = S_2$, $I(t) = 0$ and $x(t) = x_2$. Using a similar argument as that in the proof of Theorem 3.1, we show that the only invariant set in $M = \{(x, S, I) : \dot{V}_2(t) = 0\}$ is $\mathcal{M} = \{(x_2, S_2, 0)\}$. Accordingly, the global asymptotic stability of E_2 follows from LaSalle's invariant principle for delay differential systems. This completes the proof.

Theorem 3.3. *Let $\mathcal{R}_0 < 1$, then the semi-trivial equilibrium $E_1(0, 0, r/a)$ of model (1.2) is globally asymptotically stable.*

Proof. By Theorem 2.1, we see that if $\mathcal{R}_0 < 1$, E_1 is locally asymptotically stable. Hence, we only prove that all positive solutions of model (1.2) with initial conditions (1.3) converge to E_1. Let $(x(t), S(t), I(t))$ be any positive solution of model (1.2) with initial conditions (1.3). Denote $x_1 = r/a$. Define

$$V_1(t) = c\left(x(t) - x_1 - \int_{x_1}^{x(t)} \frac{f(x_1)}{f(u)} du\right) + S(t) + I(t) + a_2 \int_{t-\tau}^{t} f(x(u))S(u)du.$$

Calculating the derivative of $V_1(t)$ along positive solutions of model (1.2), it follows that

$$\dot{V}_1(t) = c\left(1 - \frac{f(x_1)}{f(x(t))}\right)\dot{x}(t) + \dot{S}(t) + \dot{I}(t) + a_2 f(x(t))S(t) - a_2 f(x(t-\tau))S(t-\tau)$$

$$= c\left(1 - \frac{f(x_1)}{f(x(t))}\right)\left[rx(t) - ax^2(t)\right] - (r_1 - a_2 f(x_1))S(t) - r_2 I(t)$$

$$= -\frac{acx(t)(x(t) + x_1)}{x^2(t)(1 + mx_1^2)}(x(t) - x_1)^2 - r_1(1 - \mathcal{R}_0)S(t) - r_2 I(t).$$

(3.7)

If $\mathcal{R}_0 < 1$, it then follows from (3.7) that $\dot{V}_1(t) \leq 0$. By Theorem 5.3.1 in [5], solutions limit to Λ, the largest invariant subset of $\{\dot{V}_1(t) = 0\}$. Clearly, we see from (3.7) that $\dot{V}_1(t) = 0$, if and only if $x(t) = x_1$, $S(t) = 0$ and $I(t) = 0$. Accordingly, the global asymptotic stability of E_1 follows from LaSalle's invariant principle for delay differential systems. This completes the proof.

4. Numerical simulation

In this section, we give two examples to illustrate the main results in Section 2.

Example 1 In model (1.2), let $a = 0.15, a_1 = 1.5, a_2 = 1.2, m = 0.15, \alpha = 0.25, r = 2, r_1 = 1.5, r_2 = 0.5, \sigma = 0.7$. It is easy to show that $\mathcal{R}_0 = 6.4257 > 1$, and therefore model (1.2) has a disease-free equilibrium $E_2(1.2403, 1.2, 0)$. By calculation, we obtain that $\mathcal{R}_1 = 0.25 < 1$, $\gamma_0 = -0.1541 < 0$ and $\tau_0 = 5.8102$. By Theorem 2.1, we see that the equilibrium E_2 is locally asymptotically stable, if $0 < \tau < \tau_0$ and is unstable, if $\tau > \tau_0$. Further, model (1.2) undergoes a Hopf bifurcation at E_2, when $\tau = \tau_0$. An investigation of model (1.2) with the previous coefficients can be conducted via a numerical integration using the standard Matlab algorithm (Figures 1 and 2).

Example 2 In model (1.2), let $a = 0.25, a_1 = 1.5, a_2 = 3.2, m = 0.55, \alpha = 0.25, r = 2, r_1 = 0.5, r_2 = 0.5, \sigma = 0.2$. By calculation, we obtain $\mathcal{R}_1 = 1.1949 > 1$. Therefore, model (1.2) has a unique endemic-coexistence $E_+(0.5095, 2.8, 1.2710)$. Direct calculations show that $\Delta \approx 6.9893 > 0, z_* \approx 0.1580, f(z_*) \approx -0.0028 < 0$,

$\omega_1^0 \approx 0.4365$ and $\tau_1^0 \approx 5.7968$. By Theorem 2.2, we see that the equilibrium E_+ is locally asymptotically stable, if $0 < \tau < \tau_1^0$ and model (1.2) undergoes a Hopf bifurcation at E_+, when $\tau = \tau_1^0$. Numerical simulation illustrates the previous result (Figures 3 and 4).
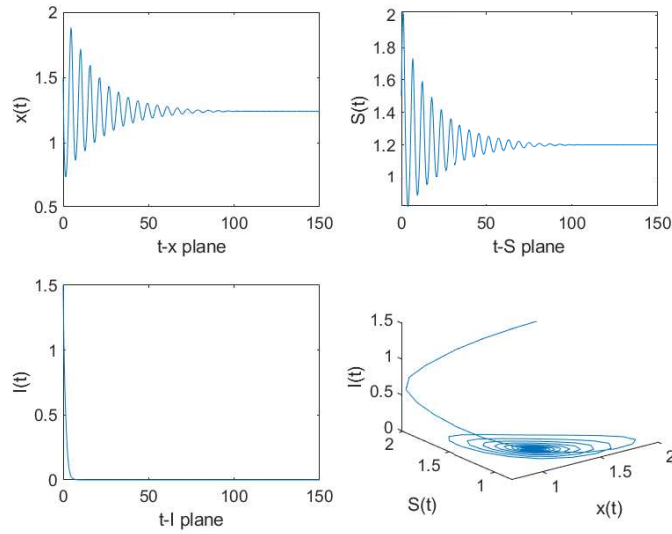

Figure 1. The temporal solution found by numerical integration of system (1.2) with $\tau = 1$ and $(\varphi(0), \phi_1(0), \phi_2(0)) = (1.5, 1.5, 1.5)$
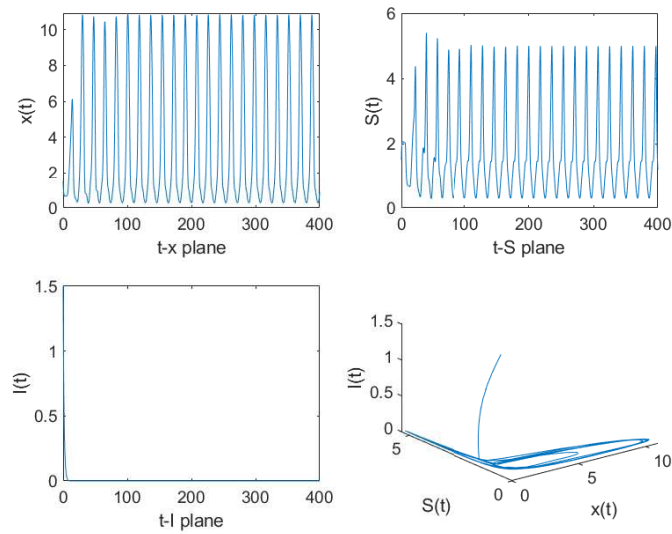

Figure 2. The temporal solution found by numerical integration of system (1.2) with $\tau = 6$ and $(\varphi(0), \phi_1(0), \phi_2(0)) = (1.5, 1.5, 1.5)$
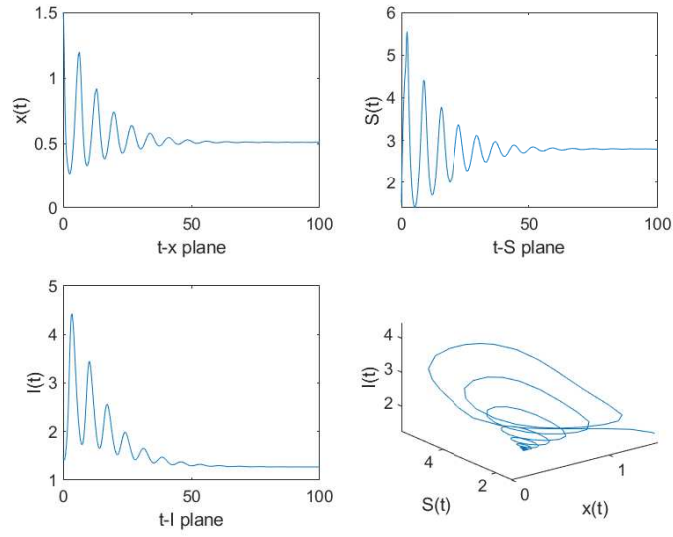
Figure 3. The temporal solution found by numerical integration of system (1.2) with $\tau = 1.6$ and $(\phi_1(0), \phi_2(0), \varphi(0)) = (1.5, 1.5, 1.5)$
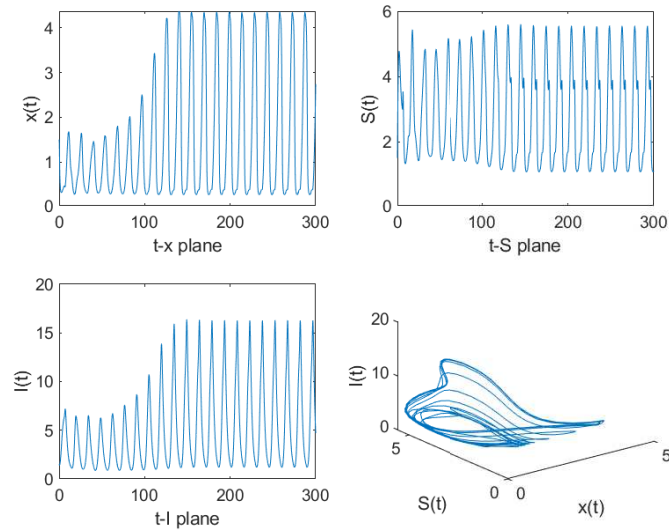

Figure 4. The temporal solution found by numerical integration of system (1.2) with $\tau = 6$ and $(\phi_1(0), \phi_2(0), \varphi(0)) = (1.5, 1.5, 1.5)$

5. Discussion

In this paper, we have investigated the global dynamics of an eco-epidemiological model with time delay and Holling type-III functional response. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria has been established. From the analysis in Section 2, we see that the ecological basic reproduction number $\mathcal{R}_0$ determines the local stability of the predator-extinction equilibrium $E_1(r/a, 0, 0)$. If the ecological basic reproduction number $\mathcal{R}_0 < 1$, the equilibrium E_1 is locally asymptotically stable. If $\mathcal{R}_0 > 1$ and the disease basic reproduction number $\mathcal{R}_1 < 1$, the model has a disease-free equilibrium and the endemic-coexistence equilibrium is not feasible. At the disease-free equilibrium, a threshold τ_0 for the time delay was identified such that below it, the equilibrium is locally asymptotically stable, but if the delay is greater than the threshold, sustained oscillation arise. If the disease basic reproduction number $\mathcal{R}_1 > 1$, the model has an endemic-coexistence equilibrium. A second threshold τ_k^0 for the time delay due to the gestation of the predators was also identified to characterize the existence of Hopf bifurcation at the endemic-coexistence equilibrium, when the delay crosses it. This implies that the disease spreading among the predators becomes periodically endemic.

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