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The Regularity of Stochastic Convolution Driven by Tempered Fractional Brownian Motion and Its Application to Mean-field Stochastic Differential Equations*

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Abstract In this paper, some properties of a stochastic convolution driven by tempered fractional Brownian motion are obtained. Based on this result, we get the existence and uniqueness of stochastic mean-field equation driven by tempered fractional Brownian motion. Furthermore, combining with the Banach fixed point theorem and the properties of Mittag-Leffler functions, we study the existence and uniqueness of mild solution for a kind of time fractional mean-field stochastic differential equation driven by tempered fractional Brownian motion.

Keywords Mean-field stochastic differential equations, Tempered fractional Brownian motion, Caputo fractional derivative, Banach fixed point theorem.

MSC(2010) 60H15, 60H05, 60G22.

1. Introduction

In 2015, Sabzikar, Meerschaert and Chen [16] proposed a tempered fractional derivative that multiplies the power law kernel by an exponential tempering factor. Temper means that it produces a more tractable mathematical object, and can be made arbitrarily light. It has better properties than the fractional derivative that can be approximated by adjusting the parameters by resulting operator over a finite interval. The basic definitions and properties of tempered fractional Brownian motion (TFBM for its abbreviated form) was proposed in [12], which modifies the power law kernel in the moving average representation of a fractional Brownian motion adding an exponential tempering. Then, the theories of stochastic integrals for TFBM were developed in [13]. Along the way, they also developed some basic results on tempered fractional calculus. First, we give the definition of TFBM, as defined in [12]. Detailed description will be introduced in Section 2.

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Definition 1.1. For any $\sigma < 1/2$ and $\lambda > 0$, letting $\{B(t)\}_{t\in\mathbb{R}}$ be a real-valued Brownian motion on the real line, the stochastic integral

$$B^{\sigma,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\sigma} - e^{-\lambda(-x)_+}(-x)_+^{-\sigma}\right] B(dx)$$

is called a TFBM, where $(x)_+ = xI_{(x>0)}, 0^0 = 0$, and λ is called tempered parameter.

A solution to stochastic evolution equations can usually be as represented a stochastic convolution generated by infinitesimal generator driven by noises. It is requisite to give the regularity of stochastic convolution in function spaces where the solution exists. In this paper, we are concerned about the regularity of stochastic convolution driven by TFBM (Lemma 3.2 and Lemma 3.3). As an application, the existence and uniqueness of the following mean-field stochastic differential equations (SDEs for its abbreviated form) driven by TFBM is studied.

$$\begin{cases} dx(t) = Ax(t)dt + f\left(t, x(t), \mathbb{P}_{x(t)}\right)dt + \gamma\left(t, \mathbb{P}_{x(t)}\right)dB_Q^{\sigma,\lambda}(t), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where the operator A generates an exponentially stable semigroup, f, γ are continuous functions with additional properties which would be specified below. Assume that there exists a complete orthogonal basis $\{e_k\}_{k\in\mathbb{N}}$ in a Hilbert space $\mathbb{H}$, and that $B_Q^{\sigma,\lambda} = \left\{B_Q^{\sigma,\lambda}(t)\right\}_{t\geq 0}$ is cylindrical $\mathbb{H}$ -valued TFBM defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ with a finite trace nuclear covariance operator $Q \geq 0$. Denote $\mathrm{Tr}(Q) = \Sigma_{k=1}^{\infty}\lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k, k \in \mathbb{N}$. The cylindrical $\mathbb{H}$ -valued TFBM is defined as

$$B_Q^{\sigma,\lambda}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k^{\sigma,\lambda}(t) e_k, \quad t \geq 0,$$

where $\{B_k^{\sigma,\lambda}\}$ $(k \in \mathbb{N})$ are independent TFBMs.

Mean-field SDEs, also named as McKean-Vlasov equations, whose coefficients depend upon the marginal distribution of the solution, were discussed the first time by Kac, Uhlenbeck and Hibb [7] in their analysis of the Boltzmann equation for the particle density in dilute monotonic gases and a toy model for the Vlasov kinetic equation of plasma. Mean field SDEs have been used to study high-dimensional systems corresponding to a large number of particles, i.e., N-players stochastic differential games [9, 10]. The existence and uniqueness of almost automorphic solutions in distribution of mean field SDEs driven by fractional Brownian motion were established in [2]. The existence of stationary solutions adapted to dissipative finite-dimensional SDEs driven by TFBM was studied in [5, 6]. Recently, Wang, Liu and Caraballo [19] have considered the exponential behavior and upper noise excitation index of solutions to evolution equations with unbounded delay and tempered fractional Brownian motions. As far as we are concerned, it is meaningful to discuss the global existence and uniqueness of mean-field stochastic differential equation driven by TFBM. In addition, it has been found that fractional calculus can be useful in the most diverse areas of science, mainly due to the nonlocal character of the fractional differentiation. There have been many meaningful results in recent years. Carvalho and Planas [1] got the mild solutions to the time fractional Navier-Stokes equations in $\mathbb{R}^n$. Shen and Huang [18] discussed time-space fractional stochastic Ginzburg-Landau equation driven by Gaussian white noise. Zou explored a Galerkin finite element method for time-fractional stochastic heat equation in [20]. Li and Yang [11] got the stability analysis for the numerical simulation of hybrid stochastic differential equations. Recently, Kumar et al., [3, 4, 17] have explored the dynamics of fractional differential equation, and they also give computational algorithms for fractional partial differential equations in [8, 14].

Therefore, in this paper, we also give the regularity of Mittag-Leffler function driven by TFBM (Lemma 4.1 and Lemma 4.2). As an application, the global existence and uniqueness of the following mean-field SDEs driven by TFBM is given

$$\begin{cases} D_t^{\alpha} x(t) = Ax(t)dt + f\left(t, x(t), \mathbb{P}_{x(t)}\right)dt + \gamma\left(t, \mathbb{P}_{x(t)}\right)dB_Q^{\sigma,\lambda}(t), \\ x(0) = x_0. \end{cases} \tag{1.2}$$

The structure of the paper is given as follows: In Section 2, we prove some basic results on tempered fractional Brownian motion, and give the regularity of stochastic convolution of TFBM. In Section 3, we give the existence and uniqueness of mean-field SDEs driven by TFBM. In Section 4, we discuss the properties of time fractional Mittag-Leffler functions driven by TFBM, and give the regularity and existence and uniqueness of time fractional mean-field SDEs. We also give the existence and uniqueness of mean-field damped wave equation as an example in Section 5.

2. Preliminaries

In this section, we present some basic properties related to tempered fractional calculus and stochastic integrals with respect to TFBM.

Definition 2.1 ([16])**.** For any $f \in L^p(\mathbb{R})$ (where $1 \le p < \infty$), the positive and negative tempered fractional integral on $\mathbb{R}$ are defined by

$$\mathbb{I}_+^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} f(u)(t-u)^{\alpha-1} e^{-\lambda(t-u)} du \tag{2.1}$$

and

$$\mathbb{I}_-^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{+\infty} f(u)(u-t)^{\alpha-1} e^{-\lambda(u-t)} du \tag{2.2}$$

respectively. For any $\alpha > 0$ and $\lambda > 0$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$ is the Gamma function.

Definition 2.2 ([12])**.** For any $-1/2 < \sigma < 0, \lambda > 0$, and for any $a, b \in [0, T]$ with $b > a$, we define

$$\int_a^b f(t) dB^{\sigma,\lambda}(t) := \Gamma(k+1) \int_a^b \left(\mathbb{I}_-^{k,\lambda} f(t) - \lambda \mathbb{I}_-^{k+1,\lambda} f(t)\right) dB(t), \tag{2.3}$$

for any $f \in \mathcal{A}_1 := \left\{ f \in L^2(a,b) : \int_a^b \left|\mathbb{I}_-^{k,\lambda} f(t) - \lambda \mathbb{I}_-^{k+1,\lambda} f(t)\right|^2 dt < \infty \right\}$. Here, $k = -\sigma$ and $\mathcal{A}_1$ is a linear space with inner product $\langle f, g \rangle_{\mathcal{A}_1} := \langle F, G \rangle_{L^2(a,b)}$, where

$$F(t) = \Gamma(k+1)\left(\mathbb{I}_-^{k,\lambda} f(t) - \lambda \mathbb{I}_-^{k+1,\lambda} f(t)\right),$$
$$G(t) = \Gamma(k+1)\left(\mathbb{I}_-^{k,\lambda} g(t) - \lambda \mathbb{I}_-^{k+1,\lambda} g(t)\right).$$

Next, we have the following lemma to give the isometric property of stochastic integral with respect to TFBM. Throughout the paper, we assume that $(\mathbb{H}, \|\cdot\|)$ is a real separable Hilbert space. We denote by $\mathcal{L}(\mathbb{H}, \mathbb{H})$ the space of all bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$. Note that $\mathcal{L}(\mathbb{H}, \mathbb{H})$ is a Banach space, and we denote the norm by $\|\cdot\|_{\mathcal{L}(\mathbb{H},H)}$. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\mathcal{L}^2(\mathbb{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$ -valued random variables Y such that

$$\mathbb{E}\|Y\|^2 = \int_\Omega \|Y\|^2 \mathrm{d}\mathbb{P} < \infty.$$

It is $\mathcal{L}^2$ -bounded, if $\sup_{t\in\mathbb{R}} \|Y(t)\|_2 < \infty$.

A stochastic process $Y : \mathbb{R} \to \mathcal{L}^2(\mathbb{P}, \mathbb{H})$ is said to be $\mathcal{L}^2$-continuous or square-mean continuous, if for any $s \in \mathbb{R}$

$$\lim_{t\to s} \mathbb{E}\|Y(t) - Y(s)\|^2 = 0.$$

A function $g : \mathbb{R} \times \mathcal{L}^2(\mathbb{P}, \mathbb{H}) \to \mathcal{L}^2(\mathbb{P}, \mathbb{H}), (t, Y) \mapsto g(t, Y)$ is said to be square-mean continuous process in $t \in \mathbb{R}$ for each $Y \in \mathcal{L}^2(\mathbb{P}, \mathbb{H})$, if g is continuous in the following sense

$$\mathbb{E}\left\|g(t, Y) - g\left(t', Y'\right)\right\|^2 \to 0, \quad \text{as } \left(t', Y'\right) \to (t, Y).$$

The operator A is an infinitesimal generator on a Hilbert space $\mathbb{H}$, $(e_n, \alpha_n), n = 1, 2, \cdots$ are the eigenvectors and eigenvalues of A. Denote the operator A is well defined in the space of functions

$$\mathbb{H} = \left\{ f = \Sigma f_n e_n \middle| \|f\|_{\mathbb{H}} := \left(\Sigma f_n^2 \alpha_n^2\right)^{\frac{1}{2}} < \infty \right\}.$$

For $\psi, \phi \in \mathcal{L}(\mathbb{H})$, we define $(\psi, \phi)_Q = \operatorname{Tr}(\psi Q \phi^*)$, where ϕ^* is the adjoint of the operator ϕ. Then, for any bounded operator $\phi \in \mathcal{L}(\mathbb{H}, \mathbb{H})$,

$$\|\phi\|_Q^2 = \operatorname{Tr}(\phi Q \phi^*) = \sum_{k=1}^{\infty} \left\| \sqrt{\lambda_k} \phi e_k \right\|^2,$$

where $\{e_k\}$ is a complete orthogonal basis in $\mathbb{H}$. If $\|\phi\|_Q^2 < \infty$. Then, ϕ is called a Q -Hilbert-Schmidt operator. Then, we have

Lemma 2.1. *If $\phi : [0, T] \mapsto \mathcal{L}(\mathbb{H}, \mathbb{H})$ satisfies $\int_0^T \|\phi(s)\|_Q^2 ds < \infty$, then for any $t \in [0, T]$,*

$$\mathbb{E}\left\| \int_0^t \phi(s) dB_Q^{\sigma,\lambda}(s) \right\|^2 \le \Gamma^2(1-\sigma)\lambda^{2\sigma} \int_0^t \|\phi(s)\|_Q^2 ds, \tag{2.4}$$

where $-1/2 < \sigma < 0$, $\lambda > 0$.

Proof. According to the Definition 2.1, Definition 2.2 and the isometric property of Itô stochastic integration, we have

$$\begin{aligned} & \mathbb{E}\left\| \int_0^t \phi(s) dB_Q^{\sigma,\lambda}(s) \right\|^2 \\ = \ & \mathbb{E}\left\| \sum_{k=1}^{\infty} \int_0^t \sqrt{\lambda_k} \phi(s) e_k dB_Q^{\sigma,\lambda}(s) \right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left\|\sum_{k=1}^{\infty}\Gamma(1-\sigma)\int_0^t \left(\mathbb{I}_{-}^{-\sigma,\lambda}[\sqrt{\lambda_k}\phi(s)e_k] - \lambda\mathbb{I}_{-}^{1-\sigma,\lambda}[\sqrt{\lambda_k}\phi(s)e_k]\right) dB(s)\right\|^2 \\
&= \Gamma^2(1-\sigma)\int_0^t \left\|\mathbb{I}_{-}^{-\sigma,\lambda}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k] - \lambda\mathbb{I}_{-}^{1-\sigma,\lambda}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k]\right\|^2 ds.
\end{aligned}
$$

Then, by using the definition of tempered calculus and the Young's inequality, we can get

$$
\begin{aligned}
&\int_0^t \left\|\mathbb{I}_{-}^{-\sigma,\lambda}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k]\right\|^2 ds \\
\leq\ & \frac{1}{\Gamma^2(-\sigma)}\int_0^t \left\|\int_s^{\infty}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(u)e_k](u-s)^{-\sigma-1}e^{-\lambda(u-s)}du\right\|^2 ds \\
\leq\ & \frac{1}{\Gamma^2(-\sigma)}\int_0^t \left\|\int_{-\infty}^{\infty}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(u)e_k](u-s)_+^{-\sigma-1}e^{-\lambda(u-s)_+}du\right\|^2 ds \\
\leq\ & \frac{1}{\Gamma^2(-\sigma)}\int_0^t \left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k\right\|^2 ds \left(\int_0^t s_+^{-\sigma-1}e^{-\lambda s_+}ds\right)^2 \\
\leq\ & \frac{\left(\int_0^{\infty} s^{-\sigma-1}e^{-\lambda s}ds\right)^2}{\Gamma^2(-\sigma)}\int_0^t \left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k\right\|^2 ds \\
\leq\ & \lambda^{2\sigma}\int_0^t \|\phi(s)\|_Q^2 ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^t \left\|\mathbb{I}_{-}^{1-\sigma,\lambda}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k]\right\|^2 ds \\
\leq\ & \frac{1}{\Gamma^2(1-\sigma)}\int_0^t \left\|\int_s^{\infty}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(u)e_k](u-s)^{-\sigma}e^{-\lambda(u-s)}du\right\|^2 ds \\
\leq\ & \frac{1}{\Gamma^2(1-\sigma)}\int_0^t \left\|\int_{-\infty}^{\infty}[\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(u)e_k](u-s)_+^{-\sigma}e^{-\lambda(u-s)_+}du\right\|^2 ds \\
\leq\ & \frac{1}{\Gamma^2(1-\sigma)}\int_0^t \left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k\right\|^2 ds \left(\int_0^t s_+^{-\sigma}e^{-\lambda s_+}\right)^2 ds \\
\leq\ & \frac{\left(\int_0^{\infty} s^{-\sigma}e^{-\lambda s}ds\right)^2}{\Gamma^2(1-\sigma)}\int_0^t \left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}\phi(s)e_k\right\|^2 ds \\
\leq\ & \lambda^{2\sigma-2}\int_0^t \|\phi(s)\|_Q^2 ds.
\end{aligned}
$$

Then, by combining the two estimates, the proof can be finished. □

For the sake of convenience, from now on, we denote $H = \frac{1}{2} - \sigma$, and $C(\lambda, H) = \Gamma^2(1-\sigma)\lambda^{2\sigma}$.

Lemma 2.2. *For any $-1/2 < \sigma < 0, \lambda > 0$, assume that a function $F : \mathbb{H} \to \mathbb{R}^1$ and its first and second Frechét derivatives F_x, F_{xx}, are uniformly continuous on bounded subsets of $[0,T] \times \mathbb{H}$. f and g are predictable processes on $[0,T], \mathbb{P}$ -a.s., and $X(0)$ is a $\mathcal{F}_0$- measurable random variable. We have the following rules. If*

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t g(s)dB^{\sigma,\lambda}(s),$$

then

$$\begin{aligned} F(X(t)) =& F(X(0)) + \int_0^t \langle F_x(X(s)), f(s)\rangle ds + \int_0^t \langle F_x(X(s))g(s)\rangle dB^{\sigma,\lambda}(s) \\ &+ \int_0^t \frac{1}{2} \operatorname{Tr}\Big\{F_{xx}(X(s)) \left[\mathbb{I}_-^{k,\lambda}, g(s) - \lambda \mathbb{I}_-^{k+1,\lambda} g(s)\right] \left[\mathbb{I}_-^{k,\lambda} g(s) - \lambda \mathbb{I}_-^{k+1,\lambda} g(s)\right]^* \Big\} ds. \end{aligned}$$

Proof. According to Definition 2.2,

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t \mathbb{I}_-^{k,\lambda} g(s) - \lambda \mathbb{I}_-^{k+1,\lambda} g(s)dB(s).$$

Then, by using the Itô lemma for stochastic integral for Brownian motion, the proof can be finished. □

Let $(\mathbb{X}, d)$ be a separable, complete metric space and let $\operatorname{Pr}(\mathbb{H})$ be the space of Borel probability measures on $\mathbb{H}$, which is a Hilbert space.

For $\mu, \nu \in \operatorname{Pr}(\mathbb{H})$, we define

$$d_{BL}(\mu,\nu) = \sup_{\|h\|_{BL} \le 1} \left| \int_{\mathbb{H}} hd(\mu - \nu) \right|,$$

where h are Lipschitz continuous functions on $\mathbb{H}$ with the norms

$$\|h\|_{BL} = \max\left\{\|h\|_\infty, \|h\|_L\right\}, \|h\|_\infty := \sup_{x \in \mathbb{X}} |h(x)| < \infty,$$

$$\|h\|_L = \sup\left\{\frac{|h(x) - h(y)|}{\|x - y\|}, x, y \in \mathbb{H}, x \neq y\right\}.$$

It is known that d_{BL} is a complete metric on $\operatorname{Pr}(\mathbb{H})$, which generates the weak topology. For a random variable $x : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{H}$, we will denote by $\mu(x) := \mathbb{P} \circ x^{-1}$ its distribution. If $\mu, \nu \in \operatorname{Pr}(\mathbb{H})$, $\mathcal{W}(\mu,\nu)$ denotes the 2-Wasserstein metric

$$\mathcal{W}(\mu,\nu) := \inf\left\{\int \|x - y\|^2 \pi(dx, dy); \pi \in \operatorname{Pr}(\mathbb{H} \times \mathbb{H}) \text{ with marginals } \mu \text{ and } \nu\right\}.$$

By the definition, we have

$$\|\mathbb{E}[x_1] - \mathbb{E}[x_2]\| \le \mathcal{W}\left(\mathbb{P}_{x(t)}, \mathbb{P}_{x'(t)}\right) \le \left[\mathbb{E}\|x - x'\|^2\right]^{1/2}.$$

Here, $\mathbb{P}_{x(t)}$ denote the distribution of $x(t)$.

3. Existence and uniqueness of the mild solution of (1.1)

For simplicity, as Da Prato's work in [15], we assume that A is a dissipative semigroup with $\alpha_k < 0$, where α_k means the eigenvalues of A. Then, the operator generates a $\mathcal{C}_0$-semigroup $T(t)$ satisfying

$$\|T(t)\| \leq Ke^{-\omega t} \tag{3.1}$$

with $\omega, K > 0$. Then, we can give the definition of the mild solutions of (1.1).

Definition 3.1. A $\mathbb{H}$-valued stochastic process $x(t)$ is called a mild solution of (1.1) with initial value x_0, if the following equation is satisfied

$$x(t) = x_0 + \int_0^t T(t-s)f(s, x(s), \mathbb{P}_{x(s)})ds + \int_0^t T(t-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s).$$

Lemma 3.1. *Assume that $-1/2 < \sigma < 0$, $\lambda > 0$, $H = 1/2 - \sigma$, and γ is uniformly bounded on $[0,T]$, then*

$$\mathbb{E}\left\|\int_0^t T(t-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\right\|^2 < \infty. \tag{3.2}$$

Proof. According to the property of $T(t)$ and Lemma 2.1, we have

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t T(t-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\right\|^2 \\
&\leq \mathbb{E}\left\|\sum_{k=1}^{\infty}\int_0^t T(t-s)\gamma(s, \mathbb{P}_{x(s)})\sqrt{\lambda_k}e_k dB_k^{\sigma,\lambda}(s)\right\|^2 \\
&\leq C(\lambda, H)\int_0^t \left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}T(t-s)\gamma(s, \mathbb{P}_{x(s)})e_k\right\|^2 ds \\
&\leq C(\lambda, H)\sum_{k=1}^{\infty}\int_0^t \lambda_k\|T(t-s)\|^2\|\gamma(s, \mathbb{P}_{x(s)})\|^2 ds \\
&\leq C(\lambda, H)\|\gamma(s, \mathbb{P}_{x(s)})\|^2 < \infty.
\end{aligned}$$

Thus, it completes the proof. □

Lemma 3.2. *Let $-1/2 < \sigma < 0$, $\lambda > 0$, $H = 1/2 - \sigma$, and γ is uniformly bounded on $[0,T]$, $\sum_{k=0}^{\infty}\frac{\lambda_k}{\alpha_k^{1-\kappa}} < \infty$, where $B_Q^{\sigma,\lambda}(t) = \sum_{k=0}^{\infty}\sqrt{\lambda_k}\beta_k(t)e_k$, $Ae_k = \alpha_k e_k$. Then, the stochastic convolution $z(t) = \int_0^t T(t-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)$, $t \in [0,T]$ has a continuous version.*

Proof. For $s, t \in [0,t]$, $s < t$, we have

$$\begin{aligned}
&\mathbb{E}\|z(t_1, x) - z(t_2, x)\|^2 \\
&= \mathbb{E}\left\|\int_0^{t_1} T(t_1-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s) - \int_0^{t_2} T(t_2-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= 2\mathbb{E}\left\|\int_0^{t_1}(T(t_1-s)-T(t_2-s))\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\right\|^2\\
&\quad +2\mathbb{E}\left\|\int_{t_1}^{t_2}T(t_2-s)\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\right\|^2\\
&= I_1+I_2.
\end{aligned}
$$

Then, for $T(t)$ is a C_0-semigroup, we calculate that

$$
\begin{aligned}
I_1 &= \mathbb{E}\|\int_0^{t_1}(T(t_1-s)-T(t_2-s))\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\|^2\\
&\le C(\lambda,H)\int_0^{t_1}\left\|\sum_{k=1}^{\infty}(T(t_1-s)-T(t_2-s))\gamma(s,\mathbb{P}_{x(s)})e_k\right\|^2 ds\\
&\le C(\lambda,H)\int_0^{t_1}\sum_{k=1}^{\infty}\lambda_k\left\|e^{A(t_1-s)}-e^{A(t_2-s)}\gamma(s,\mathbb{P}_{x(s)})e_k\right\|^2 ds\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\lambda_k\int_0^{t_1}\|e^{\alpha_k(t_1-s)}-e^{\alpha_k(t_2-s)}\gamma(s,\mathbb{P}_{x(s)})\|^2 ds\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\|\gamma(s,\mathbb{P}_{x(s)})\|^2\frac{\lambda_k}{\alpha_k}\Big[2\left(1-e^{-(t_1-t_2)\alpha_k}\right)-\left(1-e^{-2(t_1-t_2)\alpha_k}\right)\\
&\quad -\left(1-e^{-2(t_1-t_2)\alpha_k}\right)-\left(e^{-t_1\alpha_k}-e^{-t_2\alpha_k}\right)^2\Big]\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\|\gamma(s,\mathbb{P}_{x(s)})\|^2\frac{\lambda_k}{\alpha_k}\left(1-e^{-(t_1-t_2)\alpha_k}\right)\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\|\gamma(s,\mathbb{P}_{x(s)})\|^2\frac{\lambda_k}{\alpha_k^{1-\kappa}}|t_1-t_2|^{\kappa}\\
&\le C(\lambda,H)|t_1-t_2|^{\kappa}.
\end{aligned}
$$

Note that for arbitrary $\kappa\in[0,1]$ and all $x\ge 0, y\ge 0, |e^{-x}-e^{-y}|\le|x-y|^{\kappa}$. Here, we need to prove that $\|\gamma\|$ is uniformly bounded. Similarly, according to the dissipative property of $T(t)$, we have

$$
\begin{aligned}
I_2 &= \mathbb{E}\left\|\int_{t_1}^{t_2}T(t_2-s)\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\right\|^2\\
&\le C(\lambda,H)\int_{t_1}^{t_2}\left\|\sum_{k=1}^{\infty}T(t_2-s)\gamma(s,\mathbb{P}_{x(s)})e_k\right\|_Q^2 ds\\
&\le C(\lambda,H)\int_{t_1}^{t_2}\left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}T(t_2-s)\gamma(s,\mathbb{P}_{x(s)})e_k\right\|^2 ds\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\int_{t_1}^{t_2}\lambda_k\|\gamma(s,\mathbb{P}_{x(s)})\|^2e^{(t_2-s)\alpha_k}ds\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\frac{\lambda_k}{\alpha_k^{1-\kappa}}|t_1-t_2|^{\kappa}.
\end{aligned}
$$

Thus, there exists $\kappa \in [0,1]$ such that

$$\mathbb{E}\|z(t_1,x)-z(t_2,x)\|^2 \leq C(\lambda,H)|t_1-t_2|^{\kappa}.$$

By the Kolmmogorov's test theorem [15], there exists a continuous version of z. The lemma is proven. □

Theorem 3.1. *Assume that f,γ are square-mean continuous processes in $t\in\mathbb{R}$ for each $x\in L^2(\mathbb{P},\mathbb{H})$. Moreover, assume f and γ satisfy Lipschitz conditions, that is, for all $x,y\in L^2(\mathbb{P},\mathbb{H}),\mu_1,\mu_2\in \mathrm{Pr}(\mathbb{H})$ and $t\in\mathbb{R}$, $-1/2<\sigma<0$, $\lambda>0$, $H=1/2-\sigma$, there exist constants L and L' such that*

$$\|f(t,x,\mu_1)-f(t,y,\mu_2)\|^2 \leq L\left(\|x-y\|^2+\mathcal{W}^2(\mu_1,\mu_2)\right), \tag{3.3}$$

$$\|\gamma(t,\mu_1)-\gamma(t,\mu_2)\| \leq L\mathcal{W}(\mu_1,\mu_2). \tag{3.4}$$

where L satisfies $\frac{2K^2L}{\omega^2}+\frac{2K^2C(\lambda,H)L}{\omega^2}<1$, and γ is uniformly bounded on $[0,T]$. Then, equation (1.1) has a unique continuous solution.

Proof. We need to prove that the operator

$$\mathcal{L}: x\to x(t)=x_0+\int_0^t T(t-s)f(s,x(s),\mathbb{P}_{x(s)})ds+\int_0^t T(t-s)\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)$$

is a compress operator defined on $C([0,T],\mathbb{H})$. Taking norms of the $\mathcal{L}(x)$, we have

$$\mathbb{E}\|x(t)\|^2\leq\|x_0\|^2+\mathbb{E}\left\|\int_0^t T(t-s)f(s,x(s),\mathbb{P}_{x(s)})ds\right\|^2+\mathbb{E}\left\|\int_0^t T(t-s)\gamma(s,\mathbb{P}_{x(s)})ds\right\|^2.$$

By Lemma 3.1, we get that the operator $\mathcal{L}$ maps $C([0,T],\mathbb{H})$ to $C([0,T],\mathbb{H})$. Then, we need to prove that it is compressed. Let $x(t)$, $y(t)\in L^2$, taking norms of the $\mathcal{L}(x-y)$, we have

$$\begin{aligned}\mathbb{E}\|x(t)-y(t)\|^2 \leq&\mathbb{E}\left\|\int_0^t T(t-s)[f(s,x(s),\mathbb{P}_{x(s)})-f(s,y(s),\mathbb{P}_{y(s)})]ds\right\|^2\\&+\mathbb{E}\left\|\int_0^t T(t-s)[\gamma(s,\mathbb{P}_{x(s)})-\gamma(s,\mathbb{P}_{y(s)})]ds\right\|^2\\\triangleq& I_1+I_2.\end{aligned}$$

Then, we calculate that

$$\begin{aligned}I_1 &\leq \mathbb{E}\left\|\int_{-\infty}^t T(t-s)\left[f\left(s,x(s),\mathbb{P}_{x(s)}\right)-f\left(s,y(s),\mathbb{P}_{y(s)}\right)\right]ds\right\|^2\\&\leq K^2\int_{-\infty}^t e^{-\omega(t-s)}ds\left(\int_{-\infty}^t e^{-\omega(t-s)}\mathbb{E}\|f\left(s,x(s),\mathbb{P}_{x(s)}\right)-f\left(s,y(s),\mathbb{P}_{y(s)}\right)\|^2ds\right)\\&\leq K^2L\left(\int_{-\infty}^t e^{-\omega(t-s)}ds\right)^2\sup_{t\in\mathbb{R}}\left(\mathbb{E}\|x(t)-y(t)\|^2+\mathcal{W}^2\left(\mathbb{P}_{x(t)},\mathbb{P}_{y(t)}\right)\right)\\&\leq\frac{2K^2L}{\omega^2}\sup_{t\in\mathbb{R}}\mathbb{E}\|x(t)-y(t)\|^2\end{aligned}$$

and

$$\begin{aligned} I_2 &\leq \mathbb{E}\left\|\int_{-\infty}^{t} T(t-s)\left[\gamma\left(s, \mathbb{P}_{x(s)}\right)-\gamma\left(s, \mathbb{P}_{y(s)}\right)\right] dB_Q^{\sigma,\lambda}(s)\right\|^2 \\ &\leq K^2 C(\lambda, H) L\left(\int_{-\infty}^{t} e^{-\omega(t-s)} ds\right)^2 \sup_{t\in\mathbb{R}}\left(\mathbb{E}\|x(t)-y(t)\|^2+\mathcal{W}^2\left(\mathbb{P}_{x(t)}, \mathbb{P}_{y(t)}\right)\right) \\ &\leq \frac{2K^2 C(\lambda, H) L}{\omega^2} \sup_{t\in\mathbb{R}} \mathbb{E}\|x(t)-y(t)\|^2. \end{aligned}$$

Thus, it follows that, for each $t \in \mathbb{R}$,

$$\mathbb{E}\left\|(\mathcal{L}x)(t)-(\mathcal{L}y)(t)\right\|^2 \leqslant \left[\frac{2K^2 L}{\omega^2}+\frac{2K^2 C(\lambda, H) L}{\omega^2}\right] \sup_{s\in\mathbb{R}} \mathbb{E}\left\|x(s)-y(s)\right\|^2.$$

Under the condition $\left[\frac{2K^2 L}{\omega^2}+\frac{2K^2 C(\lambda,H) L}{\omega^2}\right] < 1$, it follows that $\mathcal{L}$ is a contraction mapping on $C([0,T], \mathbb{H}))$. Hence, using the Banach fixed point theorem, we get the existence and uniqueness of the equation. □

4. Existence and uniqueness of the mild solution of (1.2)

To establish the existence and uniqueness of

$$\begin{cases} D_t^{\alpha} x(t)=A x(t) dt+f\left(t, x(t), \mathbb{P}_{x(t)}\right) dt+\gamma\left(t, \mathbb{P}_{x(t)}\right) dB_Q^{\sigma,\lambda}(t), \\ x(0)=x_0, \end{cases} \tag{4.1}$$

which is a time-fractional stochastic mean-field equation driven by TFBM. The definition of A, f and γ are introduced in Section 2. Here, the D_t^{α} denote the Caputo fractional derivative with $\alpha \in (0,1)$. First, we give some definition of fractional calculus.

Definition 4.1 ([1])**.** For $\alpha > 0$, the Riemann-Liouville fractional integral operator of order α for function $f \in L^1([0,T], \mathbb{R})$ is defined by

$$I_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad t \in [0,T]. \tag{4.2}$$

Definition 4.2 ([1])**.** For $\alpha \in (0,1)$, the Caputo fractional derivative of order α for function $f \in C([0,T]; \mathbb{R})$ is defined by

$$D_t^{\alpha} f(t):=\frac{d}{dt}[I_t^{1-\alpha}(f(t)-f(0))]=\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds. \tag{4.3}$$

Definition 4.3 ([1])**.** The Mainardi's function is given by

$$M_{\alpha}(z)=\sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k+1-\alpha)}, \ 0<\alpha<1, \ z \in \mathbb{C}. \tag{4.4}$$

Moreover, $M_\alpha(z) \geq 0$ for all $t \geq 0$, and satisfies the following equality

$$\int_0^\infty t^r M_\alpha(t)dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r+1)},\ r > -1,\ 0 < \alpha < 1. \tag{4.5}$$

Definition 4.4 ([1]). The Mittag-Leffler functions is defined by

$$E_{\alpha,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\eta)},\ \alpha,\eta \in \mathbb{C},\ Re(z) > 0. \tag{4.6}$$

Definition 4.5 ([1]). The Mittag-Leffler families operators based on the analytic semigroup $S(t)$ generated by the space fractional operator $(-\Delta)^\gamma$ is defined by

$$T_{\alpha,\gamma}(t) = \int_0^\infty M_\alpha(s)S(st^\alpha)ds = \int_0^\infty M_\alpha(s)e^{-st^\alpha A^\gamma}ds \tag{4.7}$$

and

$$S_{\alpha,\gamma}(t) = \int_0^\infty \alpha s M_\alpha(s)S(st^\alpha)ds = \int_0^\infty \alpha s M_\alpha(s)e^{-st^\alpha A^\gamma}ds. \tag{4.8}$$

Now, we shall introduce some notations of functional spaces given as following

$$\mathbb{H}^\kappa = \left\{ u = \sum u_n e_n \in L^2[\mathbb{R}] \,\middle|\, \|u\|_{\mathbb{H}^\kappa} = (\kappa u_n^2 a_n^\kappa)^{\frac{1}{2}} < \infty \right\}$$

with the norm $\|u\|_{\mathbb{H}^\kappa} = (\kappa u_n^2 a_n^\kappa)^{\frac{1}{2}}$. Then, $(e_n(x), a_n^\gamma)$ are the eigenvectors and eigenvalues of A^γ with Dirichlet boundary conditions, and the operator $A^{\frac{\kappa}{2}}$ is well defined in the Banach space $\mathbb{H}^\kappa$.

Definition 4.6. A $\mathbb{H}^\kappa$-valued stochastic process $x(t)$ is called a mild solution of (1.2) with initial value u_0 and n_0, if the following equation is satisfied,

$$\begin{aligned} x(t) = T_{\alpha,2}x_0 + \int_0^t (t-s)^{\alpha-1} S_{\alpha,2}(t-s) f(s, x(s), \mathbb{P}_{x(s)})ds \\ + \int_0^t (t-s)^{\alpha-1} S_{\alpha,2}(t-s)\gamma(s, \mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s). \end{aligned}$$

Let $z(t) = \int_0^t S_{\alpha,2}(t-s)\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)$. Next, we will give the regularity of $z(t)$.

Lemma 4.1. *When $-1/2 < \sigma < 0$, $a > 0$, $H = 1/2 - \sigma$, and γ is uniformly bounded on $[0,T]$. Then,*

$$\mathbb{E}\left\| \int_0^t S_{\alpha,2}(t-s)\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s) \right\|_{\mathbb{H}^\kappa} < \infty. \tag{4.9}$$

Proof. It follows from the fact that $\|e_k\|_\infty < 1$. Then, by the disspative property

(3.1),

$$
\begin{aligned}
&\mathbb{E}\left\|z(t)\right\|^2_{\mathbb{H}^\kappa}\\
&=\mathbb{E}\sum_{k=1}^{\infty}\left\langle A^{\frac{\kappa}{2}}\int_0^t(t-\tau)^{\alpha-1}\int_0^\infty \alpha sM_\alpha(s)e^{-s(t-\tau)^\alpha}ds\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(\tau),e_k\right\rangle^2\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\int_0^t\left[\int_D A^{\frac{\kappa}{2}}(t-\tau)^{\alpha-1}\int_0^\infty \alpha sM_\alpha(s)e^{-s(t-\tau)^\alpha A}e_k\gamma(s,\mathbb{P}_{x(s)})dsdx\right]^2d\tau\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}\int_0^t\left[\int_0^\infty a_k^{\frac{\kappa}{2}-\theta}\alpha s^{1-\theta}M_\alpha(s)(t-\tau)^{\alpha-1-\alpha\theta}\gamma(s,\mathbb{P}_{x(s)})ds\right]^2d\tau\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}k^{2(\kappa-2\theta)}\|\gamma(s,\mathbb{P}_{x(s)})\|^2\int_0^t(t-\tau)^{2\alpha-2-2\alpha\theta}d\tau\\
&\le C(\lambda,H)\sum_{k=1}^{\infty}k^{2(\kappa-2\theta)}T^{2\alpha-1-2\alpha\theta}\|\gamma(s,\mathbb{P}_{x(s)})\|^2<\infty.
\end{aligned}
$$

Choosing $\theta>\frac{1+2\kappa}{4}$ to ensure the $\mathbb{H}^\kappa$ norm of $z(t)$ is well defined. □

Lemma 4.2. *Let $-1/2<\sigma<0$, $a>0$, $\alpha\in(\frac{3}{4},1)$, and γ is uniformly bounded, then the stochastic convolution $z(t)$, $t\in[0,T]$ has a continuous version.*

Proof. For $s,t\in[0,t]$, $s<t$, we have

$$
\begin{aligned}
&\mathbb{E}\left|z(t_1)-z(t_2)\right|^2\\
&\le 2C(\lambda,H)\sum_{k=1}^{\infty}\int_s^t\left|(t_1-\tau)^{\alpha-1}\int_0^\infty \alpha sM_\alpha(s)e^{-s(t_1-\tau)^\alpha A}e_k(x)ds\right|^2d\tau\\
&\quad+2C(\lambda,H)\sum_{k=1}^{\infty}\int_0^s\left|\left[(t_1-\tau)^{\alpha-1}\int_0^\infty \alpha sM_\alpha(s)e^{-s(t_1-\tau)^\alpha A}ds\right.\right.\\
&\quad\left.\left.-(t_2-\tau)^{\alpha-1}\int_0^\infty \alpha sM_\alpha(s)e^{-s(t_2-\tau)^\alpha A}ds\right]e_k(x)\right|^2d\tau\\
&\triangleq 2C(\lambda,H)I_1+2C(\lambda,H)I_2.
\end{aligned}
$$

Minkowski's inequality yields that

$$
\begin{aligned}
I_1&\le\sum_{k=1}^{\infty}\left[\int_0^\infty \alpha sM_\alpha(s)\left(\int_{t_2}^{t_1}(t_1-\tau)^{2(\alpha-1)}e^{-2s(t_1-\tau)^\alpha a_k^2}d\tau\right)^{\frac{1}{2}}ds\right]^2\\
&\le\sum_{k=1}^{\infty}\left[\int_0^\infty \alpha sM_\alpha(s)\left(\int_{t_2}^{t_1}(t_1-\tau)^{2(\alpha-1)-\alpha\theta}s^{-\theta}a_k^{-\theta}d\tau\right)^{\frac{1}{2}}ds\right]^2\\
&\le\sum_{k=1}^{\infty}\left[\int_0^\infty \alpha sM_\alpha(s)\left(\int_{t_2}^{t_1}(t_1-\tau)^{4(\alpha-1)}s^{-\theta}a_k^{-\theta}d\tau\right.\right.\\
&\quad\left.\left.+\int_{t_1}^{t_2}(t_1-\tau)^{-2\alpha\theta}s^{-\theta}a_k^{-\theta}d\tau\right)^{\frac{1}{2}}ds\right]^2
\end{aligned}
$$

$$\leq \sum_{k=1}^{\infty}\left[|t_1-t_2|^{\frac{4\alpha-3}{2}}a_k^{-\frac{\theta}{2}}\int_0^\infty \alpha s^{1-\frac{\theta}{2}}M_\alpha(s)ds\right.$$
$$\left.+|t_1-t_2|^{\frac{1-2\alpha\theta}{2}}a_k^{-\frac{\theta}{2}}\int_0^\infty \alpha s^{1-\frac{\theta}{2}}M_\alpha(s)ds\right]^2$$
$$\leq C\sum_{\infty}^{k=1}k^{-2\theta}\Big(|t_1-t_2|^{4\alpha-3}+|t_1-t_2|^{1-2\alpha\theta}\Big).$$

Choosing $\theta<\frac{1}{2\alpha},\alpha>\frac{3}{4}$ ensures that $I_1<\infty$, we obtain

$$I_2 \leq \sum_{k=1}^{\infty}\left[\int_0^\infty \alpha s M_\alpha(s)\left(\int_0^{t_2}|(t_1-\tau)^{\alpha-1}e^{-s(t_1-\tau)^\alpha a_k}-(t_1-\tau)^{\alpha-1}e^{-s(t_2-\tau)^\alpha a_k}\right.\right.$$
$$\left.\left.+(t_1-\tau)^{\alpha-1}e^{-s(t_2-\tau)^\alpha a_k}-(t_2-\tau)^{\alpha-1}e^{-s(t_2-\tau)^\alpha a_k^2}\Big|^2d\tau\right)^{\frac{1}{2}}ds\right]^2$$
$$\leq \sum_{k=1}^{\infty}\left[\int_0^\infty \alpha s M_\alpha(s)\left(\int_0^{t_2}|(t_1-\tau)^{2(\alpha-1)}\left(e^{-s(t_1-\tau)^\alpha a_k}-e^{-s(t_2-\tau)^\alpha a_k}\right)^2d\tau\right.\right.$$
$$\left.\left.+\int_0^{t_2}e^{-2s(t_2-\tau)^\alpha a_k}\left((t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}\right)^2d\tau\right)^{\frac{1}{2}}ds\right]^2$$
$$\leq C\sum_{k=1}^{\infty}\left[\int_0^\infty \alpha s M_\alpha(s)\left(\int_0^{t_2}a_k^2s^2|t_1-t_2|^2(t_1-\tau)^{2(\alpha-1)}(t-\tau)^{2(\alpha-1)}\right.\right.$$
$$\left.\left.e^{-2s(t-\tau)^\alpha a_k}d\tau+\int_0^{t_2}s^{-\theta}(t_2-\tau)^{-\alpha\theta}a_k^{-\theta}|t_1-t_2|^2(t-\tau)^{2(\alpha-2)}d\tau\right)^{\frac{1}{2}}ds\right]^2$$
$$\leq C\sum_{k=1}^{\infty}\left[a_k^{1-\frac{\theta_1}{2}}\int_0^\infty \alpha s^{2-\frac{\theta_1}{2}}M_\alpha(s)ds\left(\int_0^{t_2}|t_1-t_2|^2(t_1-\tau)^{2\alpha-2}(t-\tau)^{2\alpha-2-\alpha\theta_1}d\tau\right)^{\frac{1}{2}}\right.$$
$$\left.+\int_0^\infty \alpha s^{2-\frac{\theta}{2}}M_\alpha(s)ds\left(|t_1-t_2|^2a_k^{-\theta}\int_0^{t_2}(t_2-\tau)^{-\alpha\theta}(t-\tau)^{2(\alpha-2)}d\tau\right)^{\frac{1}{2}}\right]^2$$
$$\leq C\sum_{k=1}^{\infty}a_k^{2-\theta}\left[(t_1-t_2)^2\left((t_1-t_2)^{4\alpha-3}+t_1^{4\alpha-3}+(t-t_2)^{4\alpha-3-\alpha\theta_1}+t^{4\alpha-3-\alpha\theta_1}\right)\right]$$
$$+C\sum_{k=1}^{\infty}a_k^{-\theta}\left[|t_1-t_2|^2\left(t_2^{1-2\alpha\theta}+(t-t_2)^{4\alpha-7}+t_2^{4\alpha-7}\right)\right]$$
$$\leq C\sum_{k=1}^{\infty}\left(k^{4-2\theta_1}+k^{-2\theta}\right)\left((t_1-t_2)^{4\alpha-1}+|t_1-t_2|^2\right),$$

where $\frac{1}{2}<\theta<\frac{1}{2\alpha},\theta_1>2+\frac{1}{2},\alpha>\frac{3}{4}$. Then, by the Kolmmogorov's test theorem [15], the lemma is proven. □

Theorem 4.1. *Assume that f,γ are square-mean continuous processes in $t\in\mathbb{R}$ for each $x\in L^2(\mathbb{P},\mathbb{H})$, and γ is uniformly bounded on $[0,T]$. Moreover, assume f and γ satisfy Lipschitz conditions. That is, there exists L such that all $x,y\in$*

$L^2(\mathbb{P},\mathbb{H}),\mu_1,\mu_2\in\Pr(\mathbb{H})$ and $t\in\mathbb{R}$, $-1/2<\sigma<0$, $a>0$, $H=1/2-\sigma$,

$$\|f(t,x,\mu_1)-f(t,y,\mu_2)\|^2\le L\left(\|x-y\|^2+\mathcal{W}^2(\mu_1,\mu_2)\right),$$

$$\|\gamma(t,\mu_1)-\gamma(t,\mu_2)\|\le L'\mathcal{W}(\mu_1,\mu_2).$$

Let $-1/2<\sigma<0$, $a>0$, $H=1/2-\sigma$, $\alpha\in(\frac{3}{4},1)$, $\kappa\in(\frac{1}{6},\frac{1}{2})$, if the T is small enough. Then, the equation has a unique solution belonging to $C([0,T],\mathbb{H}^\kappa)$.

Proof. First, we need to prove the operator

$$\begin{aligned}\mathcal{L}:x\to T_{\alpha,2}x_0&+\int_0^t(t-s)^{\alpha-1}S_{\alpha,2}(t-s)f(s,x(s),\mathbb{P}_{x(s)})ds\\&+\int_0^t(t-s)^{\alpha-1}S_{\alpha,2}(t-s)\gamma(s,\mathbb{P}_{x(s)})dB_Q^{\sigma,\lambda}(s)\end{aligned}$$

maps $\mathbb{H}^\kappa$ to $\mathbb{H}^\kappa$. First,

$$\begin{aligned}&\left\|A^{\frac{\kappa}{2}}\int_0^\infty\alpha sM_\alpha(s)e^{-s(t-\tau)^\alpha A}f(s,x(s),\mathbb{P}_{x(s)})d\tau\right\|^2\\&=\sum_{k=1}^\infty\left\langle a_k\int_0^\infty\alpha sM_\alpha(s)e^{-s(t-\tau)^\alpha a_k}f(s,x(s),\mathbb{P}_{x(s)}),e_k\right\rangle^2\\&\le C\sum_{k=1}^\infty\left\langle a_k^{\frac{\sigma}{2}-\theta}\int_0^\infty\alpha s^{1-\theta}M_\alpha(s)(t-\tau)^{-\alpha\theta}f(s,x(s),\mathbb{P}_{x(s)}),e_k\right\rangle^2\\&\le C\sum_{k=1}^\infty k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\langle f(s,x(s),\mathbb{P}_{x(s)}),e_k\rangle^2\\&\le C\sum_{k=1}^\infty k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\|f(s,x(s),\mathbb{P}_{x(s)})\|^2<\infty.\end{aligned}$$

For stochastic term $z(t)$, we have

$$\begin{aligned}&\mathbb{E}\,\|z(t,x)\|_{\mathbb{H}^\kappa}^2\\=&\mathbb{E}\sum_{k=1}^\infty\left\langle A^{\frac{\kappa}{2}}\int_0^t(t-\tau)^{\alpha-1}\int_0^\infty\alpha sM_\alpha(s)e^{-s(t-\tau)^\alpha A}\gamma(s,\mathbb{P}_{x(s)})dsdB_Q^{\sigma,\lambda}(\tau),e_k\right\rangle^2\\\le&C(\lambda,H)\sum_{k=1}^\infty\int_0^t\left[\int_D A^{\frac{\kappa}{2}}(t-\tau)^{\alpha-1}\int_0^\infty\alpha sM_\alpha(s)e^{-s(t-\tau)^\alpha A}e_k\gamma(s,\mathbb{P}_{x(s)})dsdx\right]^2d\tau\\\le&C(\lambda,H)\sum_{k=1}^\infty\int_0^t\left[\int_0^\infty a_k^{\frac{\kappa}{2}-\theta}\alpha s^{1-\theta}M_\alpha(s)(t-\tau)^{\alpha-1-\alpha\theta}\gamma(s,\mathbb{P}_{x(s)})ds\right]^2d\tau\\\le&C(\lambda,H)\sum_{k=1}^\infty k^{2(\kappa-2\theta)}\|\gamma(s,\mathbb{P}_{x(s)})\|^2\int_0^t(t-\tau)^{2\alpha-2-2\alpha\theta}d\tau<\infty.\end{aligned}$$

Then, we want to prove that $\mathcal{L}$ is a contraction mapping,

$$\int_0^t(t-\tau)^{\alpha-1}S_{\alpha,2}(t-\tau)\left[f(s,x(s),\mathbb{P}_{x(s)})-f(s,y(s),\mathbb{P}_{y(s)})\right]d\tau$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty} k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta} Re\langle f(s,x(s),\mathbb{P}_{x(s)})-f(s,y(s),\mathbb{P}_{y(s)}),e_k\rangle^2\\
&\leq C\sum_{k=1}^{\infty} k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\|f(s,x(s),\mathbb{P}_{x(s)})-f(s,y(s),\mathbb{P}_{y(s)})\|_{L^2}^2\\
&\leq C\sum_{k=1}^{\infty} k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta} L\left(\|x-y\|^2+\mathcal{W}^2\left(\mu_1,\mu_2\right)\right)\\
&\leq CL^2\sum_{k=1}^{\infty} k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\|x-y\|^2.
\end{aligned}
$$

As for the stochastic term,

$$
\begin{aligned}
&\mathbb{E}\left\|z(t,x)-z(t,y)\right\|_{\mathbb{H}^\kappa}^2\\
&=\mathbb{E}\sum_{k=1}^{\infty}\Big\langle A^{\frac{\kappa}{2}}\int_0^t(t-\tau)^{\alpha-1}\int_0^\infty \alpha s M_\alpha(s)e^{-s(t-\tau)^\alpha A}\\
&\quad\left[\gamma(s,\mathbb{P}_{x(s)})-\gamma(s,\mathbb{P}_{y(s)})\right]dsdB_Q^{\sigma,\lambda}(\tau),e_k\Big\rangle^2\\
&\leq C(\lambda,H)\sum_{k=1}^{\infty}\int_0^t\left[\int_D A(t-\tau)^{\alpha-1}\int_0^\infty \alpha s M_\alpha(s)e^{-s(t-\tau)^\alpha A^{\frac{\beta}{2}}}e_k\right.\\
&\quad\left.\left[\gamma(s,\mathbb{P}_{x(s)})-\gamma(s,\mathbb{P}_{y(s)})\right]dsdx\right]^2d\tau\\
&\leq C(\lambda,H)\sum_{k=1}^{\infty}\int_0^t\left[\int_0^\infty a_k^{\frac{\kappa}{2}-\theta}\alpha s^{1-\theta}M_\alpha(s)(t-\tau)^{\alpha-1-\alpha\theta}\right.\\
&\quad\left.\left[\gamma(s,\mathbb{P}_{x(s)})-\gamma(s,\mathbb{P}_{y(s)})\right]ds\right]^2d\tau\\
&\leq C(\lambda,H)\sum_{k=1}^{\infty}k^{2(\kappa-2\theta)}\left\|\gamma(s,\mathbb{P}_{x(s)})-\gamma(s,\mathbb{P}_{y(s)})\right\|^2\int_0^t(t-\tau)^{2\alpha-2-2\alpha\theta}d\tau\\
&\leq C(\lambda,H)\sum_{k=1}^{\infty}k^{2(\kappa-2\theta)}T^{2\alpha-1-2\alpha\theta}\|\gamma(s,\mathbb{P}_{x(s)})-\gamma(s,\mathbb{P}_{y(s)})\|^2\\
&\leq CL^2\sum_{k=1}^{\infty}k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\|x-y\|^2.
\end{aligned}
$$

Let T be small enough, so $\mathcal{L}$ has a unique fix point in $C\left([0,T];\mathbb{H}^\kappa\right)$, which is the unique solution of (4.1) on $[0,T]$. The proof of Theorem 4.1 is completed.

□

Lemma 4.3. *Assume that x is the solution of (1.2) over $[0,T]$, and γ and f are uniformly $\mathbb{H}^\kappa$ − bounded on $[0,T]$. Let $-1/2<\sigma<0$, $a>0$, $H=1/2-\sigma$, $\alpha\in(\frac{3}{4},1)$, $\kappa\in(\frac{1}{6},\frac{1}{2})$. Then, it holds that*

$$
\|x(t)\|_{\mathbb{H}^\kappa}^2\leq C\left(T,\|x_0\|_{\mathbb{H}^\kappa},\max\|f\left(t,x(s),\mathbb{P}_{x(s)}\right)\|_{\mathbb{H}^\kappa}^2,\max\|\gamma(s,\mathbb{P}_{x(s)})\|_{\mathbb{H}^\kappa}^2\right).
$$

Proof. By the form of the solution, we have

$$
\mathbb{E}\|x(t)\|^2\ \leq 3\left\{\|S^{\alpha,2}(t)x_0\|_{\mathbb{H}^\kappa}^2+\mathbb{E}\left[\left(\int_0^t\|S^{\alpha,2}(t-s)f(x(s))\|ds\right)^2\right]\right.
$$

$$
\begin{aligned}
&+\mathbb{E}\left[\left\|\int_0^t S^{\alpha,2}(t-s)\gamma\left(t,\mathbb{P}_{x(s)}\right)dB^{\sigma,\lambda}(s)\right\|_{\mathbb{H}^\kappa}^2\right\}\Bigg]\\
&\leq 3\left\{M_T^2\mathbb{E}\|x_0\|_{\mathbb{H}^\kappa}^2+TM_T^2\int_0^t\mathbb{E}\|f\left(t,x(s),\mathbb{P}_{x(s)}\right)\|_{\mathbb{H}^\kappa}^2ds\right.\\
&\left.+C(\lambda,H)M_T^2\left[\int_0^t\left(\mathbb{E}\|\gamma\left(t,\mathbb{P}_{x(s)}\right)\|_{\mathbb{H}^\kappa}^2\right)ds\right]\right\}.
\end{aligned}
$$

Here, $M_T^2 = CL^2\sum_{k=1}^{\infty} k^{2(\sigma-2\theta)}T^{-2\alpha\theta}$, which is derived from Lemma 4.2. Choosing $\theta < \frac{1}{2\alpha}, \alpha > \frac{3}{4}$ ensures the norm is well defined. Moreover,

$$
\int_0^t \mathbb{E}\|f\left(s,x(s),\mathbb{P}_{x(s)}\right)\|_{\mathbb{H}^\kappa}^2 ds \leq C\int_0^t\left(1+\mathbb{E}\|x(s)\|^2\right)ds
$$

and

$$
\begin{aligned}
&\int_0^t\left(\mathbb{E}\|S^{\alpha,2}(t-s)\gamma\left(s,\mathbb{P}_{x(s)}\right)\|_{\mathbb{H}^\kappa}^2\right)ds\\
\leq&2\left(\int_0^t CL^2\sum_{k=1}^{\infty}k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\left(1+\mathbb{E}\|x(s)\|_{\mathbb{H}^\kappa}^2\right)ds\right)\\
\leq&2\left(CL^2\sum_{k=1}^{\infty}k^{2(\sigma-2\theta)}(t-\tau)^{-2\alpha\theta}\right)\int_0^t\left(1+\mathbb{E}\|x(s)\|_{\mathbb{H}^\kappa}^2\right)ds\\
\leq&C(T,L)\int_0^t\left(1+\mathbb{E}\|x(s)\|_{\mathbb{H}^\kappa}^2\right)ds.
\end{aligned}
$$

By Gronwall inequality, it is now clear that

$$
\|x(t)\|_{\mathbb{H}^\kappa}^2 \leq C\left(T,\|x_0\|_{\mathbb{H}^\kappa},\max\|f\left(t,x(s),\mathbb{P}_{x(s)}\right)\|_{\mathbb{H}^\kappa}^2,\max\|\gamma(s,\mathbb{P}_{x(s)})\|_{\mathbb{H}^\kappa}^2\right).
$$

The proof is completed. □

Therefore, we can extend the conclusion of the Theorem 4.1 to intervals $[0,T]$, $[T,2T]$... Thus, the global well-posedness of the equation is obtained.

Theorem 4.2. *Assume that f,γ are square-mean continuous processes in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P},\mathbb{H})$. Moreover, assume f and γ satisfy Lipschitz conditions, just as Theorem 4.1 and f,γ is uniformly bounded on $[0,T]$. Let $-1/2 < \sigma < 0$, $a > 0$, $H = 1/2-\sigma$, $\alpha \in (\frac{3}{4},1)$, $\kappa \in (\frac{1}{6},\frac{1}{2})$, then equation* (1.2) *has a global mild solution $\{x(t), t \in [0,T]\}$ in $C([0,T];\mathbb{H}^\kappa)$ for all $x_0 \in \mathbb{H}^\kappa$.*

5. Example

Consider the the mean-field damped wave equation driven by TFBM on the interval $[0,1]$ with Dirichlet boundary condition

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2}(t,\xi) = -2\alpha\dfrac{\partial u}{\partial t}(t,\xi)+\dfrac{\partial^2 u}{\partial \xi^2}(t,\xi)+f(t,u,\mathbb{P}_{u(t)})+g(t,\mathbb{P}_{u(t)})\dfrac{\partial B_Q^{\sigma,\lambda}}{\partial t}(t,\xi),\ t>0,\\
u(t,0)=u(t,1)=0,\ t>0.
\end{cases}
$$

Here, $\alpha > 0$ is a constant, and f,g are continuous functions with additional properties, which would be specified below. Let A be the Laplace operator, then

$A : D(A) = H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1)$. Denote $\mathbb{H} := H_0^1(0,1) \times L^2(0,1)$, and let

$$Y := \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then, the stochastic damped wave equation can be written as an abstract evolution equation

$$dY = (\mathcal{A}Y + \mathbb{F}(t, Y, \mathbb{P}_{Y(t)}))dt + \mathbb{G}(t, \mathbb{P}_{Y(t)})dB_Q^{\sigma,\lambda}(t)$$

on the Hilbert space $\mathbb{H}$, where

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & -2\alpha I \end{pmatrix}, \quad \mathbb{F}(t, Y) := \begin{pmatrix} 0 \\ f(t, u, \mathbb{P}_{u(t)}) \end{pmatrix}, \quad \mathbb{G}(t, Y) := \begin{pmatrix} 0 \\ g(t, \mathbb{P}_{u(t)}) \end{pmatrix}.$$

Note that the operator $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup satisfying the exponential dissipation property (3.1) for some positive constants K and ω. If the functions f, g satisfy the Lipschitz condition, and g is uniformly bounded, we can get that the damped wave equation has a unique equation in $C([0,1], L^2)$ by Theorem 3.1.

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