Journal of Nonlinear Modeling and Analysis
Volume 4, Number 3, September 2022, 605–614
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.605

On Two-point Boundary Value Problems for Second-order Difference Equation*

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Abstract In this paper, we aim to investigate the difference equation

$$\Delta^2 y(t-1) + |y(t)| = 0, \qquad t \in [1, T]_{\mathbb{Z}}$$

with different boundary conditions $y(0) = 0$ or $\Delta y(0) = 0$ and $y(T+1) = B$ or $\Delta y(T) = B$, where $T \geq 1$ is an integer and $B \in \mathbb{R}$. We will show that how the values of T and B influence the existence and uniqueness of the solutions to the about problem. In details, for the different problems, the TB-plane explicitly divided into different parts according to the number of solutions to the above problems. These parts of TB-plane for the value of T and B guarantee the uniqueness, the existence and the nonexistence of solutions respectively.

Keywords Second-order difference equation, Different boundary conditions, Boundary value problems.

MSC(2010) 39A27, 39A05.

1. Introduction

Let c, d be two integers with $c < d$. We use $[c, d]_{\mathbb{Z}}$ to denote the set $\{c, c+1, \cdots, d\}$. Consider the following nonlinear second-order difference equation

$$\Delta^2 y(t-1) + |y(t)| = 0, \quad t \in [1, T]_{\mathbb{Z}} \tag{1.1}$$

with the boundary conditions (BCs)

$$y(0) = 0,\ y(T+1) = B, \tag{1.2}$$

or

$$y(0) = 0,\ \Delta y(T) = B, \tag{1.3}$$

or

$$\Delta y(0) = 0,\ y(T+1) = B, \tag{1.4}$$

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*The authors were supported by National Natural Science Foundation of China (No. 11961060) and Natural Science Foundation of Gansu Province (No. 18JR3RA084).

or

$$\Delta y(0) = 0,\ \Delta y(T) = B, \tag{1.5}$$

where $T \in [1, \infty)_{\mathbb{Z}}$ and $B \in \mathbb{R}$, Δ is the forward difference operator satisfying $\Delta y(t) = y(t+1) - y(t)$ and $\Delta^2 y(t) = \Delta(\Delta y(t))$.

In the past few years, boundary value problems for difference equations have been widely studied in different disciplines such as computer science, economics, mechanical engineering, control systems (see [1,2,9]). Great efforts have been made to study the existence, multiplicity and uniqueness of solutions of boundary value problems with various boundary conditions (1.2)-(1.5) (see [3–8, 10–12] and the references therein). In [4], Bailey, Shampine and Waltman considered the nonlinear problem

$$\begin{cases} y'' + |y(t)| = 0,\ t \in [0, b], \\ y(0) = 0,\ y(b) = B, \end{cases} \tag{1.6}$$

and obtained the results: problem (1.6) has a unique solution for every number B, if $b < \pi$. However, if $b \geq \pi$, then there is one solution, no solution (existence fails), or more than one solutions (uniqueness fails), depending upon the number B.

Motivated by the above facts, it is natural to raise a question: if the interval length is larger than a certain value, whether or not to influence the existence and uniqueness of the solution for the difference equation (1.1) with BCs (1.2), (1.3), (1.4), (1.5) respectively? In this paper, we will show how the values of T and B influence the existence and uniqueness of the solutions to the problem (1.1) with BCs (1.2), (1.3), (1.4), (1.5) respectively.

2. Preliminaries

Let $y(t)$ be a solution of (1.1). If $y(t) \geq 0$, then it is a solution of the equation

$$\Delta^2 y(t-1) + y(t) = 0, \qquad t \in [1, T]_{\mathbb{Z}}. \tag{2.1}$$

If $y(t) \leq 0$, it is a solution of the equation

$$\Delta^2 y(t-1) - y(t) = 0, \qquad t \in [1, T]_{\mathbb{Z}}. \tag{2.2}$$

Let us consider equation (2.1) and equation (2.2) with the initial conditions $y(0) = 0$ and $\Delta y(0) = 0$ respectively.

(a) We rewrite (2.1) as

$$y(t+1) - y(t) + y(t-1) = 0, \tag{2.3}$$

and the general solution of the difference equation of (2.1) is

$$y(t) = c_1 \cos \frac{\pi}{3} t + c_2 \sin \frac{\pi}{3} t, \tag{2.4}$$

where c_1, c_2 are arbitrary constants. Therefore, the solution of (2.1) which satisfies $y(0) = 0$ is as follows:

$$y(t) = c \sin \frac{\pi}{3} t, \tag{2.5}$$

where c is an arbitrary nonnegative constant. Meanwhile, the solution of (2.1) with $\Delta y(0) = 0$ is

$$y(t) = c\sin(\frac{\pi}{3}t + \frac{\pi}{3}). \tag{2.6}$$

(b) Similarly, the solution of (2.2) satisfying $y(0) = 0$ is

$$y(t) = -c\Big(\Big(\frac{3}{2} + \frac{\sqrt{5}}{2}\Big)^t - \Big(\frac{3}{2} - \frac{\sqrt{5}}{2}\Big)^t \Big), \tag{2.7}$$

and the solution of (2.2) satisfying $\Delta y(0) = 0$ is

$$y(t) = -c\left(\frac{(\sqrt{5}-1)^2}{4}(\frac{3}{2} + \frac{\sqrt{5}}{2})^t + (\frac{3}{2} - \frac{\sqrt{5}}{2})^t \right). \tag{2.8}$$

Definition 2.1 ([9]). We say that $y(t)$ on $[0, T+1]_{\mathbb{Z}}$ has a zero-point at t_0, providing that $y(t_0) = 0$.

Lemma 2.1. *Assume that $y(t)$ is a solution of* (1.1) *and $y(0) = 0$.*
(i) If $\Delta y(0) < 0$, then $y(t)$ has no further zero-point, for $t > 0$;
(ii) If $\Delta y(0) > 0$, then $t = 3$ is the second zero-point.

Proof. (i) In view of $y(0) = 0$ and $\Delta y(0) < 0$, we know that $y(1) < 0$. Hence, $y(t)$ satisfies (2.5) on $t \in [0, 1]_{\mathbb{Z}}$. Furthermore, since $y(t)$ is decreasing now, we know that $y(t)$ has no further zero-point, for $t > 0$. Similarly, (ii) holds. □

Lemma 2.2. *Assume that $y(t)$ is a solution of* (1.1) *and $\Delta y(0) = 0$.*
(i) If $y(0) < 0$, then $y(t)$ has no zero-point, for $t > 0$;
(ii) If $y(0) > 0$, then $t = 2$ is the zero-point.

Proof. (i) In view of $y(0) < 0$ and $\Delta y(0) = 0$, we know that $y(0) = y(1) < 0$. Hence, $y(t)$ is the solution of (2.2) on the interval $[0, 1]_{\mathbb{Z}}$. We know that $y(t)$ has no further zero-point, for $t > 0$, since $y(t)$ is nonincreasing now. Similarly, (ii) holds. □

3. Main results

In this section, we will show how the value of T and the sign of B in the domain influence the existence and uniqueness of the solutions.

Theorem 3.1. *For* (1.1), (1.2), *the following conclusions hold.*
(F_1) *If $T = 1$, then* (1.1), (1.2) *has a unique solution for every number B;*
(F_2) *If $T \geq 2$, then* (1.1), (1.2) *has no solution for $B > 0$, and* (1.1), (1.2) *has at least one solution for $B \leq 0$.*

Proof. Let us analyze (1.1) and (1.2).
(F_1) If $T = 1$, for any nonnegative constant c, we define a function $y(t)$ on $t \in [0, 2]_{\mathbb{Z}}$ as follows:

$$y(t)=\begin{cases} c\sin\frac{\pi}{3}t, & B>0,\\ -c\Big(\big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big)^t-\big(\frac{3}{2}-\frac{\sqrt{5}}{2}\big)^t\Big), & B<0,\\ 0, & B=0.\end{cases}$$

Such $y(t)$ is the solution of (1.1), since it is a solution of (2.1) or (2.2), and the interval is small enough to preclude a second zero-point of $y(t)$. Further, in order to guarantee that $y(t)$ satisfies the boundary condition $y(2)=B(B\neq 0)$, we choose c as follows:

$$c=\begin{cases} \frac{2\sqrt{3}}{3}B, & B>0,\\ -\frac{\sqrt{5}}{15}B, & B<0.\end{cases}$$

Therefore, if $T=1$, then (1.1), (1.2) exists a unique solution.

(F_2)(a) If $T=2$, then $y(t)=c\sin\frac{\pi}{3}t(c\geq 0),\ t\in[0,3]_{\mathbb{Z}}$ provides an infinite number of nonnegative solutions in the case $B=0$. By Lemma 2.1, there are no solutions, if $B>0$. If $T=2,\ B<0$, then

$$y(t)=\frac{\sqrt{5}}{40}B\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^t-\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^t\right)$$

is a negative solution of (1.1), (1.2).

(b) For $T>2$.

(i) If $B>0$, then (1.1), (1.2) has no solution. Since the solution of (2.1) and $y(0)=0$ satisfying $\Delta y(0)>0$ have another zero point $t=3$ and $\Delta^2y(t-1)+|y(t)|=0$ at $t=3$, it follows that $y(4)=-y(2)<0$. Hence, all of those solutions of (1.1) must remain negative, for $t>3$.

(ii) If $B=0$, then $y(t)\equiv 0$ is the unique solution of (1.1), (1.2), the uniqueness following from the fact that no nontrivial solution which has a zero at $t=0$ and has another zero, for $t>3$.

(iii) If $B<0$, we will show that there are exactly two solutions satisfying (1.1), (1.2).

One negative solution is apparent, namely,

$$y_1(t)=B\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^{T+1}-\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^{T+1}\right)^{-1}\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^t-\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^t\right).$$

In addition, we define

$$y_2(t)=\begin{cases} c_1\sin\frac{\pi}{3}t, & t\in[0,3]_{\mathbb{Z}},\\ c_2\big((\frac{3}{2}+\frac{\sqrt{5}}{2})^t-(\frac{3+\sqrt{5}}{3-\sqrt{5}})^3(\frac{3}{2}-\frac{\sqrt{5}}{2})^t\big), & t\in[3,T+1]_{\mathbb{Z}}.\end{cases}$$

We only need to choose c_2 satisfying $y(T+1)=B$ and choose c_1 such that $\Delta^2y(t-1)+|y(t)|=0$ at $t=3$. The first task is accomplished by choosing

$$c_2=B\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^{T+1}-\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^3\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^{T+1}\right)^{-1},$$

and the second task accomplished by choosing

$$c_1 = -\frac{2\sqrt{3}B\Big(\big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big)^4 - \frac{(3+\sqrt{5})^2}{4}\Big)}{3\Bigg(\Big(\frac{3}{2}+\frac{\sqrt{5}}{2}\Big)^{T+1} - \Big(\frac{3+\sqrt{5}}{3-\sqrt{5}}\Big)^3\Big(\frac{3}{2}-\frac{\sqrt{5}}{2}\Big)^{T+1}\Bigg)}.$$

Thus, the sign-changing solution of (1.1), (1.2) is given by

$$y_2(t) = \begin{cases} -\dfrac{2\sqrt{3}B\Big(\big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big)^4 - \frac{(3+\sqrt{5})^2}{4}\Big)}{3\Big(\big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big)^{T+1} - \big(\frac{3+\sqrt{5}}{3-\sqrt{5}}\big)^3\big(\frac{3}{2}-\frac{\sqrt{5}}{2}\big)^{T+1}\Big)} \sin\frac{\pi}{3}t, & t\in[0,3]_{\mathbb{Z}}, \\ B\dfrac{\big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big)^{t} - \big(\frac{3+\sqrt{5}}{3-\sqrt{5}}\big)^3\big(\frac{3}{2}-\frac{\sqrt{5}}{2}\big)^{t}}{\big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big)^{T+1} - \big(\frac{3+\sqrt{5}}{3-\sqrt{5}}\big)^3\big(\frac{3}{2}-\frac{\sqrt{5}}{2}\big)^{T+1}}, & t\in[3,T+1]_{\mathbb{Z}}. \end{cases}$$

□

By Theorem 3.1, the graph of the number of solutions of (1.1), (1.2) depends upon the length of the interval, and the sign of B is given in Figure 1.

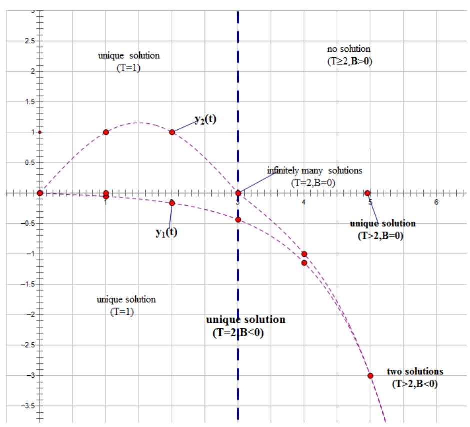

Figure 1. (1.1), (1.2)

Theorem 3.2. *For* (1.1), (1.3)*, the following conclusions hold.*

(H_1) *If $B > 0$, then* (1.1), (1.3) *has no solution for every number T;*

(H_2) *If $B \le 0$, then* (1.1), (1.3) *has at least one solution for every number T.*

Proof. Let us analyze (1.1), (1.3).

(H_1) Suppose on the contrary that $y(t)$ is the solution of (1.1), (1.3). By Lemma 2.2, the form of $y(t)$ can only be (2.3), there are $\Delta y(1) = 0$ and $\Delta y(2) < 0$. In addition, in view of $\Delta^2 y(t-1) + |y(t)| = 0$ at $t = 3$, we know that $y(4) = -y(2) < 0$. Hence, the solution of (1.1) remain $\Delta y(T) < 0$, for $T \ge 2$, which contradicts the fact that $\Delta y(T) > 0$.

(H_2)(a) If $T = 1$, then $y(t) = c\sin\frac{\pi}{3}t (c \ge 0)$ on $t \in [0,2]_{\mathbb{Z}}$ are infinite many nonnegative solutions in the case $B = 0$. (1.1), (1.3) has a negative solution

$$y(t) = \frac{\sqrt{5}}{10}B\Bigg(\Big(\frac{3}{2}+\frac{\sqrt{5}}{2}\Big)^t - \Big(\frac{3}{2}-\frac{\sqrt{5}}{2}\Big)^t\Bigg),\ t\in[0,2]_{\mathbb{Z}},$$

for $B<0$.

(b) For $T=2$.

(i) If $B=0$, then $y(t)\equiv 0$ is the unique solution of (1.1), (1.3). Since $y(2)=y(3)=0$ and $y(3)-2y(2)+y(1)+|y(2)|=0$, we see that $y(t)\equiv 0$ on $[0,3]_{\mathbb{Z}}$.

(ii) If $B<0$, then (1.1), (1.3) has a positive solution

$$y^{(1)}(t)=-\frac{2\sqrt{3}}{3}B\sin\frac{\pi}{3}t,\quad t\in[0,3]_{\mathbb{Z}}$$

and a negative solution

$$y^{(2)}(t)=\frac{\sqrt{5}}{25}B\Bigg(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^t-\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^t\Bigg),\ t\in[0,3]_{\mathbb{Z}}.$$

(c) For $T>2$.

(i) If $B=0$, then $y(t)\equiv 0$ is the unique solution of (1.1), (1.3). By Lemma 2.1, the uniqueness following from the fact that no nontrivial solution which $y(T+1)=y(T)$, for $T>2$.

(ii) If $B<0$, then (1.1), (1.3) has a negative solution

$$y_1(t)=B\frac{(\frac{3}{2}+\frac{\sqrt{5}}{2})^t-(\frac{3}{2}-\frac{\sqrt{5}}{2})^t}{(\frac{1}{2}+\frac{\sqrt{5}}{2})(\frac{3}{2}+\frac{\sqrt{5}}{2})^T+(\frac{\sqrt{5}}{2}-\frac{1}{2})(\frac{3}{2}-\frac{\sqrt{5}}{2})^T},\ t\in[0,T+1]_{\mathbb{Z}}$$

and a sign-changing solution

$$y_2(t)=\begin{cases}-\dfrac{2\sqrt{3}B\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^4-\frac{(3+\sqrt{5})^2}{4}\right)\sin\frac{\pi}{3}t}{3\left(\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^T+\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^3\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^T\right)}, & t\in[0,3]_{\mathbb{Z}},\\[2ex] B\dfrac{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^t-\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^3\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^t}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^T+\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^3\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^T}, & t\in[3,T+1]_{\mathbb{Z}}.\end{cases}$$

□

By Theorem 3.2, the graph of the number of solutions of (1.1), (1.3) depends upon the length of the interval, and the sign of B is given in Figure 2.

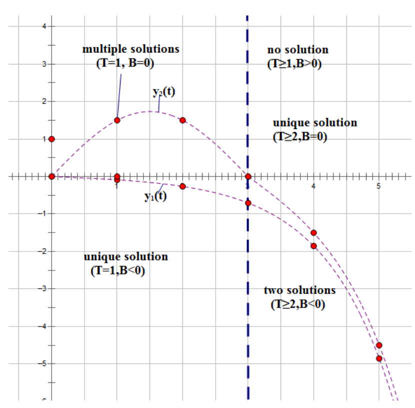

Figure 2. (1.1), (1.3)

Theorem 3.3. *For* (1.1), (1.4)*, the following conclusions hold.*

(I_1) *If* $B > 0$*, then* (1.1), (1.4) *has no solution;*

(I_2) *If* $B \leq 0$*, then* (1.1), (1.4) *has at least one solution.*

Proof. Let us analyze (1.1), (1.4).

(I_1) Suppose on the contrary that $y(t)$ is the solution of (1.1), (1.4). By Lemma 2.2, the form of $y(t)$ can only be (2.4), there are $y(2) = 0$. In addition, in view of $\Delta^2 y(t-1) + |y(t)| = 0$ at $t = 2$, we see that $y(3) = -y(1) < 0$. Therefore, the solution of (1.1) remain $y(t) < 0$, for $t > 2$, which contradicts the fact that $y(T+1) > 0$.

(I_2) (a) If $T = 1$, then $y(t) = c\sin(\frac{\pi}{3}t + \frac{\pi}{3})(c \geq 0)$ on $[0,2]_{\mathbb{Z}}$ provides an infinite number of nonnegative solutions, for $B = 0$. (1.1), (1.4) has a negative solution

$$y(t) = \frac{B}{5-\sqrt{5}}\left(\frac{(\sqrt{5}-1)^2}{4}(\frac{3}{2}+\frac{\sqrt{5}}{2})^t + (\frac{3}{2}-\frac{\sqrt{5}}{2})^t\right),$$

for $B < 0$.

(b) For $T > 1$.

(i) If $B = 0$, then $y(t) \equiv 0$ is the unique solution of (1.1), (1.4), the uniqueness following from the fact that no nontrivial solution with $y(T+1) = 0$, for $T > 1$.

(ii) If $B < 0$, (1.1), (1.4) has a negative solution is apparent, namely,

$$y_1(t) = B\frac{\frac{(\sqrt{5}-1)^2}{4}(\frac{3}{2}+\frac{\sqrt{5}}{2})^t + (\frac{3}{2}-\frac{\sqrt{5}}{2})^t}{\frac{(\sqrt{5}-1)^2}{4}(\frac{3}{2}+\frac{\sqrt{5}}{2})^{T+1} + (\frac{3}{2}-\frac{\sqrt{5}}{2})^{T+1}}.$$

In addition, we define

$$y_2(t) = \begin{cases} c_1\sin(\frac{\pi}{3}+\frac{\pi}{3}t), & t \in [0,2]_{\mathbb{Z}}, \\ c_2\big((\frac{3}{2}+\frac{\sqrt{5}}{2})^t - (\frac{3+\sqrt{5}}{3-\sqrt{5}})^2(\frac{3}{2}-\frac{\sqrt{5}}{2})^t\big), & t \in [2,T+1]_{\mathbb{Z}}. \end{cases}$$

We only need to choose c_2 satisfying $y(T+1) = B$ and choose c_1 such that $\Delta^2 y(t-1) + |y(t)| = 0$ at $t = 2$. Thus, the sign-changing solution of (1.1), (1.4) is given by

$$y_2(t) = \begin{cases} -\frac{2\sqrt{3}B\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^3 - \frac{3+\sqrt{5}}{2}\right)\sin(\frac{\pi}{3}+\frac{\pi}{3}t)}{3\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^{T+1} - \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^{T+1}\right)}, & t \in [0,2]_{\mathbb{Z}}, \\ B\frac{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^t - \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^t}{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^{T+1} - \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^{T+1}}, & t \in [2,T+1]_{\mathbb{Z}}. \end{cases}$$

□

By Theorem 3.3, the graph of the number of solutions of (1.1), (1.4) depends upon the length of the interval, and the sign of B is given in Figure 3.

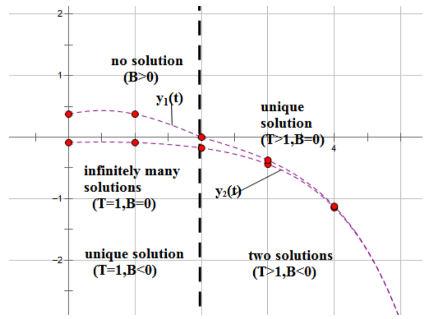

Figure 3. (1.1), (1.4)

Theorem 3.4. *For* (1.1), (1.5)*, the following conclusions hold.*

(L_1) *If* $B > 0$*, then* (1.1), (1.5) *has no solution;*

(L_2) *If* $B = 0$*, then* $y(t) \equiv 0$ *is the unique solution of* (1.1), (1.5)*;*

(L_3) *If* $B < 0$*, then* (1.1), (1.5) *has two solutions.*

Proof. Let us analyze (1.1), (1.5).

(L_1) Suppose on the contrary that $y(t)$ is the solution of (1.1), (1.5). By Lemma 2.2, the form of $y(t)$ can only be (2.4), there are $\Delta y(1) < 0$. In addition, in view of $\Delta^2 y(t-1) + |y(t)| = 0$ at $t = 2$, we see that $y(3) = -y(1) < 0$. Therefore, the solution of (1.1) remain $\Delta y(t) < 0$, for $t > 2$, which contradicts the fact that $\Delta y(T) > 0$.

(L_2) If $B = 0$, then $y(t) \equiv 0$ is the unique solution of (1.1), (1.5), By Lemma 2.2, the uniqueness following from the fact that no nontrivial solution which $y(T+1) = y(T)$, for $T \geq 1$.

(L_3) For $B < 0$.

(i) If $T = 1$, then (1.1), (1.5) has a positive solution

$$y^1(t) = -\frac{2\sqrt{3}}{3} B \sin(\frac{\pi}{3} t + \frac{\pi}{3}), t \in [0, 2]_{\mathbb{Z}}$$

and a negative solution

$$y^2(t) = \frac{2B}{5-\sqrt{5}} \left(\frac{(\sqrt{5}-1)^2}{4} (\frac{3}{2} + \frac{\sqrt{5}}{2})^t + (\frac{3}{2} - \frac{\sqrt{5}}{2})^t \right), \ t \in [0, 2]_{\mathbb{Z}}.$$

(ii) If $T > 1$, then (1.1), (1.5) has a negative solution

$$y(t) = \frac{2B}{\sqrt{5}-1} \left((\frac{3}{2} + \frac{\sqrt{5}}{2})^T - (\frac{3}{2} - \frac{\sqrt{5}}{2})^T \right)^{-1} \left(\frac{(\sqrt{5}-1)^2}{4} (\frac{3}{2} + \frac{\sqrt{5}}{2})^t + (\frac{3}{2} - \frac{\sqrt{5}}{2})^t \right)$$

and a sign-changing solution

$$y_2(t) = \begin{cases} -\frac{2\sqrt{3}B}{3} \frac{\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^3 - \frac{3+\sqrt{5}}{2}\right) \sin(\frac{\pi}{3} + \frac{\pi}{3} t)}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^T + \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2 \left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^T}, & t \in [0, 3]_{\mathbb{Z}}, \\ B \frac{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^t - \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2 \left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^t}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)^T + \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2 \left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)^T}, & t \in [3, T+1]_{\mathbb{Z}}. \end{cases}$$

□

By Theorem 3.4, the graph of the number of solutions of (1.1), (1.5) depends upon the length of the interval, and the sign of B is given in Figure 4.

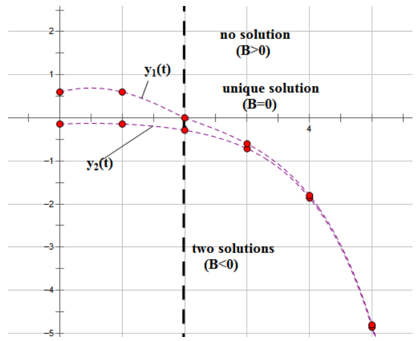

Figure 4. (1.1), (1.5)

Acknowledgements

The authors would like to express their sincere gratitude to the reviewers for their detailed comments on our manuscript and the editors for their generous assistance, which has contributed to its improvement in many ways.

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