Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 4, Number 4, December 2022, 615–627 DOI:10.12150/jnma.2022.615

Rational Solutions to the KdV Equation in Terms of Particular Polynomials

Pierre Gaillard[1,†]

Abstract Here, we construct rational solutions to the KdV equation by particular polynomials. We get the solutions in terms of determinants of the order n for any positive integer n, and we call these solutions, solutions of the order n. Therefore, we obtain a very efficient method to get rational solutions to the KdV equation, and we can construct explicit solutions very easily. In the following, we present some solutions until order 10.

Keywords Polynomial, Bilinear differential operator, Rational solution.

MSC(2010) 35C99, 35Q35, 35Q53.

1. Introduction

We consider the KdV equation in the following normalization

$$4u_t = 6uu_x + u_{xxx}. \tag{1.1}$$

As usual, the subscripts x and t denote partial derivatives. This equation appeared in the footnote on Page 360 of Boussinesq's massive 680-page memoir [4] written in 1872. Korteweg and Vries [10] studied equation (1.1) in a paper published in 1895, and from then on, this equation had carried their names. This equation has described the propagation of waves with weak dispersion in various nonlinear media. Gardner et al., [7] proposed a method of resolution in 1967. In 1971, Zakharov and Faddeev [19] proved this equation as a complete integrable system. In the same year, Hirota [8] constructed the solutions by using the bilinear method, and lot of works came into existence in the following years. We can mention, for example, Its and Matveev [9] in 1975, Lax [12] in the same year, Airault et al., [3] in 1977, Adler and Moser [2] in 1978, Ablowitz and Cornille [1] in 1979, Freeman and Nimmo [5] in 1984, Matveev [17] in 1992, Ma [16] in 2004, Kovalyov [11] in 2005 and Ma [14] in 2015.

In the following, we consider certain polynomials and construct rational solutions using determinants of the order n. The proof of the result is based on the verification of the corresponding Hirota bilinear expression for the KdV equation.

Therefore, we get a very efficient method to construct rational solutions to the KdV equation. We give explicit solutions in the simplest cases for orders $n = 1$ until 10.

† the corresponding author.
Email address: pierre.gaillard@u-bourgogne.fr (P. Gaillard)
[1] Institut de Mathématiques, Université de Bourgogne-Franche Comté, Dijon, 21000, France

2. Rational solutions to the KdV equation

We consider the polynomial p_n defined by

$$\begin{cases} p_{3k}(x,t) = \sum_{l=0}^{k} \frac{x^{3l}}{(3l)!}\frac{(t)^{k-l}}{(k-l)!}, \quad k \geq 0, \\ p_{3k+1}(x,t) = \sum_{l=0}^{k} \frac{x^{3l+1}}{(3l+1)!}\frac{(t)^{k-l}}{(k-l)!}, \quad k \geq 0, \\ p_{3k+2}(x,t) = \sum_{l=0}^{k} \frac{x^{3l+2}}{(3l+2)!}\frac{(t)^{k-l}}{(k-l)!}, \quad k \geq 0, \\ p_n(x,t) = 0, \quad n < 0. \end{cases} \tag{2.1}$$

We denote $A_n(x,t)$ the following determinant

$$A_n(x,t) = \det(p_{n+1-2i+j}(x,t))_{\{1\leq i\leq n,\, 1\leq j\leq n\}}. \tag{2.2}$$

With these notations, we have the following result.

Theorem 2.1. *The function $v_n(x,t)$ defined by*

$$v_n(x,t) = 2\partial_x^2(\ln(A_n(x,t))) \tag{2.3}$$

is a rational solution to the (KdV) equation (1.1)

$$4u_t = 6uu_x + u_{xxx}. \tag{2.4}$$

Proof. We know that $v_n(x,t) = 2\partial_x^2(\ln f(x,t))$ is a solution to the KdV equation, if f satisfies the following equation

$$(D_x^4 - 4D_xD_t)f \cdot f = 0, \tag{2.5}$$

where D is the bilinear differential operator.

We have to verify (2.5) for $f = \det(p_{n+1-2i+j}(x,t))_{\{1\leq i\leq n,\, 1\leq j\leq n\}}$. We denote by C_j the following column, $1 \leq j \leq n$

$$C_j = \begin{pmatrix} p_{n-1+j} \\ p_{n-3+j} \\ \vdots \\ p_{-n+1+j} \end{pmatrix}. \tag{2.6}$$

With these notations, $A_n(x,t)$ can be written as $A_n(x,t) = |C_1,\ldots,C_n|$.

We denote H as the expression $H = (D_x^4 - 4D_xD_t)f \cdot f$. We have to evaluate H. The polynomials p_k verify $\partial_x(p_k) = p_{k-1}$ and $\partial_t(p_k) = p_{k-3}$.

Therefore, H can be written as

$$\begin{aligned} H =& [|C_{-3}, C_2, C_3, C_4, C_5, \ldots, C_n| + 3|C_{-2}, C_1, C_3, C_4, C_5, \ldots, C_n| \\ &+ 2|C_{-1}, C_0, C_3, C_4, C_5, \ldots, C_n||C_{-1}, C_1, C_2, C_4, C_5, \ldots, C_n| \\ &+ |C_0, C_1, C_2, C_3, C_5, \ldots, C_n|] \times |C_1, C_2, C_3, C_4, C_5, \ldots, C_n| \\ &- 4[|C_{-2}, C_2, C_3, C_4, C_5, \ldots, C_n| + 2|C_{-1}, C_1, C_3, C_4, C_5, \ldots, C_n| \end{aligned}$$

$$
\begin{aligned}
&+|C_0,C_1,C_2,C_4,C_5,\dots,C_n|] \times |C_0,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3[|C_{-1},C_2,C_3,C_4,C_5,\dots,C_n| + |C_0,C_1,C_2,C_4,C_5,\dots,C_n|]^2 \\
&-4[|C_{-3},C_2,C_3,C_4,C_5,\dots,C_n| + |C_{-2},C_1,C_3,C_4,C_5,\dots,C_n| \\
&+|C_0,C_{-1},C_3,C_4,C_5,\dots,C_n| + |C_1,C_{-2},C_3,C_4,C_5,\dots,C_n| \\
&+|C_1,C_{-1},C_2,C_4,C_5,\dots,C_n| + |C_1,C_2,C_{-1},C_4,C_5,\dots,C_n| \\
&+|C_1,C_2,C_0,C_3,C_5,\dots,C_n|] \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+4[|C_{-2},C_2,C_3,C_4,C_5,\dots,C_n| + |C_1,C_{-1},C_3,C_4,C_5,\dots,C_n| \\
&+|C_1,C_2,C_0,C_4,C_5,\dots,C_n|] \times |C_0,C_2,C_3,C_4,C_5,\dots,C_n|. \qquad (2.7)
\end{aligned}
$$

H can be reduced to

$$
\begin{aligned}
H = &-3|C_{-3},C_2,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3|C_{-2},C_1,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+6|C_{-1},C_0,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3|C_{-1},C_1,C_2,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&-3|C_0,C_1,C_2,C_3,C_5,\dots,C_n|] \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&-12[|C_{-1},C_1,C_3,C_4,C_5,\dots,C_n| \times |C_0,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3|C_{-1},C_2,C_3,C_4,C_5,\dots,C_n|^2 + 3|C_0,C_1,C_3,C_4,C_5,\dots,C_n|^2 \\
&+6|C_{-1},C_2,C_3,C_4,C_5,\dots,C_n| \times |C_0,C_1,C_3,C_4,C_5,\dots,C_n|. \qquad (2.8)
\end{aligned}
$$

The following relation

$$
\begin{aligned}
&6[|C_{-1},C_0,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&-|C_{-1},C_1,C_3,C_4,C_5,\dots,C_n| \times |C_0,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+|C_{-1},C_2,C_3,C_4,C_5,\dots,C_n| \times |C_0,C_1,C_3,C_4,C_5,\dots,C_n|] = 0 \qquad (2.9)
\end{aligned}
$$

reduces the expression of H, which can be written as

$$
\begin{aligned}
H = &-3|C_{-3},C_2,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3|C_{-2},C_1,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3|C_{-1},C_1,C_2,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&-3|C_0,C_1,C_2,C_3,C_5,\dots,C_n|] \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n| \\
&-6[|C_{-1},C_1,C_3,C_4,C_5,\dots,C_n| \times |C_0,C_2,C_3,C_4,C_5,\dots,C_n| \\
&+3|C_{-1},C_2,C_3,C_4,C_5,\dots,C_n|^2 + 3|C_0,C_1,C_3,C_4,C_5,\dots,C_n|^2. \qquad (2.10)
\end{aligned}
$$

The following equalities

$$
\begin{aligned}
&|C_0,C_1,C_2,C_3,C_5,\dots,C_n| = -|C_{-3},C_2,C_3,C_4,C_5,\dots,C_n|, \\
&|C_{-1},C_1,C_2,C_4,C_5,\dots,C_n| = -|C_{-2},C_1,C_3,C_4,C_5,\dots,C_n|, \\
&|C_0,C_1,C_3,C_4,C_5,\dots,C_n| = -|C_{-1},C_2,C_3,C_4,C_5,\dots,C_n| \qquad (2.11)
\end{aligned}
$$

further reduce the expression of H.
We get $H = 6\times$ with $\tilde{H}$ equaling to

$$
\tilde{H} = -|C_{-3},C_2,C_3,C_4,C_5,\dots,C_n| \times |C_1,C_2,C_3,C_4,C_5,\dots,C_n|
$$

$$
\begin{aligned}
&+|C_1, C_2, C_{-1}, C_4, C_5, \dots, C_n| \times |C_1, C_2, C_3, C_4, C_5, \dots, C_n| \\
&+|C_1, C_{-1}, C_3, C_4, C_5, \dots, C_n| \times |C_0, C_2, C_3, C_4, C_5, \dots, C_n| \\
&+|C_{-1}, C_2, C_3, C_4, C_5, \dots, C_n| \times |C_{-1}, C_2, C_3, C_4, C_5, \dots, C_n|. \qquad (2.12)
\end{aligned}
$$

The last expression $\tilde{H}$ can be rewritten as the following determinant of the order $2n$

$$
\tilde{H} = \begin{vmatrix} C_{-1} \; C_2 \; C_3 \; C_4 \dots\dots C_n \;\; C_1 \;\; 0 \;\dots\dots\dots\dots\; 0 \\ C_{-3} \; C_0 \; C_1 \;\; 0 \;\dots\dots\; 0 \;\; C_{-1} \; C_2 \; C_3 \; C_4 \; C_5 \dots C_n \end{vmatrix} \qquad (2.13)
$$

It can be rewritten as four blocks of $n \times n$ determinants

$$
\tilde{H} = \left|\begin{array}{cccccc|ccccc}
p_{n-2} & p_{n+1} & p_{n+2} & p_{n+3} & \dots & p_{2n-1} & p_n & 0 & 0 & \dots & 0 \\
p_{n-4} & p_{n-1} & p_n & p_{n+1} & \dots & p_{2n-3} & p_{n-2} & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{-n} & p_{-n+3} & p_{-n+4} & p_{-n+5} & \dots & p_1 & p_{-n+2} & 0 & 0 & \dots & 0 \\
\hline
p_{n-4} & p_{n-1} & p_n & 0 & \dots & 0 & p_{n-2} & p_{n+1} & p_{n+2} & \dots & p_{2n-1} \\
p_{n-6} & p_{n-3} & p_{n-2} & 0 & \dots & 0 & p_{n-4} & p_{n-1} & p_n & \dots & p_{2n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{-n-2} & p_{-n+1} & p_{-n+2} & 0 & \dots & 0 & p_{-n} & p_{-n+3} & p_{-n+4} & \dots & p_1
\end{array}\right|. \qquad (2.14)
$$

We denote by $\mathcal{L}$, the rows and by $\mathcal{C}$, the columns of this determinant of order $2n$, combine the lines of the previous determinant in the following way and replace $\mathcal{L}_{n+j}$ by $\mathcal{L}_{n+j} - \mathcal{L}_{j+1}$, for $1 \leq j \leq n-1$. Then, we obtain the following determinant

$$
\tilde{H} = \left|\begin{array}{cccccc|ccccc}
p_{n-2} & p_{n+1} & p_{n+2} & p_{n+3} & \dots & p_{2n-1} & p_n & 0 & 0 & \dots & 0 \\
p_{n-4} & p_{n-1} & p_n & p_{n+1} & \dots & p_{2n-3} & p_{n-2} & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{-n} & p_{-n+3} & p_{-n+4} & p_{-n+5} & \dots & p_1 & p_{-n+2} & 0 & 0 & \dots & 0 \\
\hline
0 & 0 & 0 & -p_{n+1} & \dots & -p_{2n-3} & 0 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} \\
0 & 0 & 0 & -p_{n-1} & \dots & -p_{2n-5} & 0 & p_{n-1} & p_n & \dots & p_{2n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -p_{-n+5} & \dots & -p_1 & 0 & p_{n+1} & p_{-n+2} & \dots & p_3 \\
p_{-n-2} & p_{-n+1} & p_{-n+2} & 0 & \dots & 0 & p_{-n} & p_{-n+3} & p_{-n+4} & \dots & p_1
\end{array}\right|. \qquad (2.15)
$$

Then, by $\mathcal{C}_j + \mathcal{C}_{n+j-2}$, we replace $\mathcal{C}_j$, for $4 \leq j \leq n$, when we obtain the following determinant

$$\tilde{H}=\left|\begin{array}{cccccc|ccccc}
p_{n-2} & p_{n+1} & p_{n+2} & p_{n+3} & \cdots & p_{2n-1} & p_n & 0 & 0 & \cdots & 0\\
p_{n-4} & p_{n-1} & p_n & p_{n+1} & \cdots & p_{2n-3} & p_{n-2} & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
p_{-n} & p_{-n+3} & p_{-n+4} & p_{-n+5} & \cdots & p_1 & p_{-n+2} & 0 & 0 & \cdots & 0\\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n+1} & p_{n+2} & \cdots & p_{2n-1}\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n-1} & p_n & \cdots & p_{2n-3}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n+1} & p_{-n+4} & \cdots & p_3\\
p_{-n-2} & \cdots & \cdots & p_{-1} & 0 & 0 & p_{-n} & p_{-n+2} & p_{-n+4} & \cdots & p_1
\end{array}\right|. \tag{2.16}$$

However, using the fact that $p_n = 0$ for $n < 0$, this last determinant can be rewritten as

$$\tilde{H}=\left|\begin{array}{cccccc|ccccc}
p_{n-2} & p_{n+1} & p_{n+2} & p_{n+3} & \cdots & p_{2n-1} & p_n & 0 & 0 & \cdots & 0\\
p_{n-4} & p_{n-1} & p_n & p_{n+1} & \cdots & p_{2n-3} & p_{n-2} & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
p_{-n} & p_{-n+3} & p_{-n+4} & p_{-n+5} & \cdots & p_1 & p_{-n+2} & 0 & 0 & \cdots & 0\\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n+1} & 0 & \cdots & p_{2n-1}\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n-1} & 0 & \cdots & p_{2n-3}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & p_{n+1} & 0 & \cdots & p_{2n-1}\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & p_0 & p_1
\end{array}\right|. \tag{2.17}$$

It can be easily seen that this last determinant is clearly equal to 0, which proves

$$H = (D_x^4 - 4D_xD_x)f \cdot f = 0,$$

and so that $v_n(x,t) = 2\partial_x^2(\ln f(x,t)))$ is a solution to the KdV equation.

Therefore, we get the result. □

Remark 2.1. The presented rational solutions can be expressed as

$$\begin{aligned}
v(x,t) &= -2(\ln W(f_1, f_2, ..., f_k))_{xx}(x,t)\\
&= -2(\ln W(f_k, , ..., f_2, f_1))_{xx}(x,t)\\
&= 2((\ln W(\phi_0, \phi_1, ..., \phi_{k-1}))_{xx}(x,t))_{t->-1/4t, x->x},
\end{aligned} \tag{2.18}$$

where the ϕ_i's are specific polynomials determined in Theorem 2.1. This corresponds to the subsequent discussion made in [16] by Ma and You. This is because $f_1, f_2, ..., f_k$ satisfy $f_{j,xx} = f_{j+1}$ and $f_{j,t} = f_{j,xxx}$.

A general solution to the KdV equation was analyzed and presented in [16]. Therefore, all rational solutions in the manuscript will become special cases of the ones in [16]. Actually, a combination of Theorem 2.1 and Theorem 3.1 in [16] presents a larger class of rational solutions to the KdV equation in Wronskian form than the class given in the manuscript.

3. Explicit rational solutions to the KdV equation for the first orders

In this section, we use the previous method to construct explicit rational solutions to the KdV equation.

In the following, we will call

$$v_k(x,t) = 2\partial_x^2(\ln(A_k(x,t)))$$

as a rational solution to the KdV equation of order k.

We present some examples of these solutions for the first simplest orders.

3.1. Rational solutions of order 1 to the KdV equation

Example 3.1. The function $v_k(x,t)$ defined by

$$v_k(x,t) = -\frac{2}{x^2} \tag{3.1}$$

is a rational solution to the KdV equation.

3.2. Rational solutions of order 2 to the KdV equation

Example 3.2. The function $v_k(x,t)$ defined by

$$v_k(x,t) = -6\,\frac{\left(x^3+6\,t\right)x}{\left(-x^3+3\,t\right)^2} \tag{3.2}$$

is a rational solution to the KdV equation.

3.3. Rational solutions of order 3 to the KdV equation

Example 3.3. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{6\,\left(-2\,x^9-675\,t^2x^3+1350\,t^3\right)x}{\left(-x^6+15\,tx^3+45\,t^2\right)^2} \tag{3.3}$$

is a rational solution to the KdV equation.

3.4. Rational solutions of order 4 to the KdV equation

Example 3.4. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{-20\,x^{18} + 720\,tx^{15} - 28350\,t^2x^{12} - 661500\,t^3x^9 + 11907000\,t^4x^6 - 44651250\,t^6}{\left(-x^9 + 45\,tx^6 + 4725\,t^3\right)^2 x^2} \tag{3.4}$$

is a rational solution to the KdV equation.

3.5. Rational solutions of order 5 to the KdV equation

Example 3.5. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3.5}$$

with

$$\begin{aligned} n(x,t) = -30\,(&x^{27} - 126\,tx^{24} + 7560\,t^2x^{21} + 5655825\,t^4x^{15} - 500094000\,t^5x^{12} \\ &+ 4313310750\,t^6x^9 - 11252115000\,t^7x^6 + 295368018750\,t^8x^3 + 590736037500\,t^9)x \end{aligned} \tag{3.6}$$

and

$$d(x,t) = (x^{15} - 105\,tx^{12} + 1575\,t^2x^9 - 33075\,t^3x^6 - 992250\,t^4x^3 + 1488375\,t^5)^2 \tag{3.7}$$

is a rational solution to the KdV equation.

3.6. Rational solutions of order 6 to the KdV equation

Example 3.6. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3.8}$$

with

$$\begin{aligned} n(x,t) = 42\,(&-x^{39} + 300\,tx^{36} - 37800\,t^2x^{33} + 1890000\,t^3x^{30} - 90932625\,t^4x^{27} \\ &+ 2062887750\,t^5x^{24} + 389269597500\,t^6x^{21} - 17947123425000\,t^7x^{18} \\ &+ 263869129321875\,t^8x^{15} - 2859162421500000\,t^9x^{12} \\ &- 63795061529718750\,t^{10}x^9 - 96496731725625000\,t^{11}x^6 \\ &- 4221732012996093750\,t^{12}x^3 + 5066078415595312500\,t^{13})x \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} d(x,t) = (&x^{21} - 210\,tx^{18} + 10395\,t^2x^{15} - 264600\,t^3x^{12} - 5457375\,t^4x^9 \\ &- 343814625\,t^5x^6 + 3438146250\,t^6x^3 + 5157219375\,t^7)^2 \end{aligned} \tag{3.10}$$

is a rational solution to the KdV equation.

3.7. Rational solutions of order 7 to the KdV equation

Example 3.7. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3.11}$$

with

$$\begin{aligned} n(x,t) = & -56\,x^{54} + 33264\,tx^{51} - 8334900\,t^2x^{48} + 1073217600\,t^3x^{45} - 88841699100\,t^4x^{42} \\ & + 4208291010000\,t^5x^{39} - 20822788948500\,t^6x^{36} + 16700107780125000\,t^7x^{33} \\ & - 2828512200341520000\,t^8x^{30} + 146885752493414550000\,t^9x^{27} \\ & - 3822896546264326781250\,t^{10}x^{24} + 106059364846170986250000\,t^{11}x^{21} \\ & - 1400650109245812737812500\,t^{12}x^{18} + 17562387611658575936250000\,t^{13}x^{15} \\ & - 80365723040511963194765625 0\,t^{14}x^{12} - 4797023778387148190695312500\,t^{15}x^9 \\ & + 86346428010968667432515625000\,t^{16}x^6 - 10793303501371083429064453125 0\,t^{18} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} d(x,t) = & (x^{27} - 378\,tx^{24} + 42525\,t^2x^{21} - 2182950\,t^3x^{18} + 19646550\,t^4x^{15} \\ & - 3094331625\,t^5x^{12} - 147496474125\,t^6x^9 + 6637341335625\,t^7x^6 \\ & + 232306946746875\,t^9)^2x^2 \end{aligned} \tag{3.13}$$

is a rational solution to the KdV equation.

3.8. Rational solutions of order 8 to the KdV equation

Example 3.8. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3.14}$$

with

$$\begin{aligned} & n(x,t) \\ = & -18\,(4\,x^{69} - 4200\,tx^{66} + 1905750\,t^2x^{63} - 483159600\,t^3x^{60} + 78138695250\,t^4x^{57} \\ & - 8340851118600\,t^5x^{54} + 595545378986250\,t^6x^{51} - 39921395686672500\,t^7x^{48} \\ & + 1536199377447735000\,t^8x^{45} + 36819907677150097500 0\,t^9x^{42} \\ & - 6587305078878396309937 5\,t^{10}x^{39} + 4801800716006149418062500\,t^{11}x^{36} \\ & - 168255232705748229729187500\,t^{12}x^{33} + 4958157366226547182074375000\,t^{13}x^{30} \\ & - 51649152550511657393517890625\,t^{14}x^{27} - 1919124744346010431730656406250\,t^{15}x^{24} \\ & + 26989261886041804489539774375 0000\,t^{16}x^{21} \\ & - 6914621955118370887995851250000 00\,t^{17}x^{18} \\ & - 127735293081606333636280502109375 00\,t^{18}x^{15} \\ & - 314528866183446895767711283734375 0000\,t^{19}x^{12} \\ & + 19883967800655314009338194685957031250\,t^{20}x^9 \end{aligned}$$

$$
\begin{aligned}
&- 3337169421089003749818997709531250000 0\, t^{21} x^{6} \\
&+ 5475043581474146777046793117199707031 25\, t^{22} x^{3} \\
&+ 6570052297768976132456151740639648437 50\, t^{23}) x \qquad (3.15)
\end{aligned}
$$

and

$$
\begin{aligned}
d(x,t) = (&-x^{36} + 630\, t x^{33} - 135135\, t^{2} x^{30} + 13825350\, t^{3} x^{27} - 638512875\, t^{4} x^{24} \\
&+ 29499294825\, t^{5} x^{21} + 1179971793000\, t^{6} x^{18} + 54426198952125\, t^{7} x^{15} \\
&- 1268395929237937 5\, t^{8} x^{12} + 132879573539212500\, t^{9} x^{9} \\
&- 2092853283242596875\, t^{10} x^{6} - 31392799248638953125\, t^{11} x^{3} \\
&+ 31392799248638953125\, t^{12})^{2} \qquad (3.16)
\end{aligned}
$$

is a rational solution to the KdV equation.

3.9. Rational solutions of order 9 to the KdV equation

Example 3.9. The function $v_k(x,t)$ defined by

$$
v_k(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (3.17)
$$

with

$$
\begin{aligned}
n(x,t) = 90\, (&-x^{87} + 1716\, t x^{84} - 1302840\, t^{2} x^{81} + 576298800\, t^{3} x^{78} - 167162670675\, t^{4} x^{75} \\
&+ 33671337950250\, t^{5} x^{72} - 4883313265330500\, t^{6} x^{69} + 532930720393071000\, t^{7} x^{66} \\
&- 41982352119903195000\, t^{8} x^{63} + 1899274320510672105000\, t^{9} x^{60} \\
&- 20773912831860005692875 0\, t^{10} x^{57} + 7765442365922870262133500 0\, t^{11} x^{54} \\
&- 1407484319855266711446825937 5\, t^{12} x^{51} \\
&+ 1353653968207045875528376968750\, t^{13} x^{48} \\
&- 8368710526749129562397942550000 0\, t^{14} x^{45} \\
&+ 3232758461572985661403193106000000\, t^{15} x^{42} \\
&- 7110612857387604995445419574482812 5\, t^{16} x^{39} \\
&+ 1019021984277855554839081692265312500\, t^{17} x^{36} \\
&- 3363383055435568030284116331438492187 50\, t^{18} x^{33} \\
&+ 1417918791428436506581427873477034375000 0\, t^{19} x^{30} \\
&- 3235392177796924731781736680961137722656 25\, t^{20} x^{27} \\
&+ 2132534440692278750097305649817877671171875 0\, t^{21} x^{24} \\
&+ 3573188882940833677805872672632114285468750 00\, t^{22} x^{21} \\
&- 2306838603928897457414626693844083724296875 0000\, t^{23} x^{18} \\
&+ 2699587650987639405884625425024826453289453125 00\, t^{24} x^{15} \\
&- 2537131474727900949349747901076993099485156250 000\, t^{25} x^{12} \\
&- 2906000832340039946597441831354934571958847656 2500\, t^{26} x^{9} \\
&- 3522425251321260541330232522854466147828906250 0000\, t^{27} x^{6} \\
&- 8090570499128520305867877825931351933294519042968 75\, t^{28} x^{3}
\end{aligned}
$$

$$+ 693477471353873169074389527936973022853815917968750\, t^{29})x \tag{3.18}$$

and

$$\begin{aligned} d(x,t) = (&-x^{45} + 990\, tx^{42} - 363825\, t^2x^{39} + 67567500\, t^3x^{36} - 6810804000\, t^4x^{33} \\ &+ 445298879025\, t^5x^{30} - 8210637059625\, t^6x^{27} + 1121710685720625\, t^7x^{24} \\ &- 48406130360713125\, t^8x^{21} - 2388510334367344687 5\, t^9x^{18} \\ &+ 103787584106405011312 5\, t^{10}x^{15} - 2358854935542730937812 5\, t^{11}x^{12} \\ &- 305263579893765180187500\, t^{12}x^9 - 9615802766653603175906250\, t^{13}x^6 \\ &+ 7211852074990202381929687 5\, t^{14}x^3 + 7211852074990202381929687 5\, t^{15})^2 \end{aligned} \tag{3.19}$$

is a rational solution to the KdV equation.

3.10. Rational solutions of order 10 to the KdV equation

Example 3.10. The function $v_k(x,t)$ defined by

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3.20}$$

with

$$\begin{aligned} &n(x,t) \\ =&- 110\, x^{108} + 291060\, tx^{105} - 347490000\, t^2x^{102} + 248154192000\, t^3x^{99} \\ &- 119078820110250\, t^4x^{96} + 40841520834114000\, t^5x^{93} - 10403875101641521500\, t^6x^{90} \\ &+ 2022895713150642330000\, t^7x^{87} - 304310886232135639106250\, t^8x^{84} \\ &+ 35771706849366129904762500\, t^9x^{81} - 3519939926423867815118250000\, t^{10}x^{78} \\ &+ 296754986088067433541615000000\, t^{11}x^{75} \\ &+ 4582775043052251449589779343750\, t^{12}x^{72} \\ &- 1140751472244820566709653762187 5000\, t^{13}x^{69} \\ &+ 2793940550549247930228715261632187500\, t^{14}x^{66} \\ &- 389645507719688878112158874570400000000\, t^{15}x^{63} \\ &+ 35750508746600149120830492075248360156250\, t^{16}x^{60} \\ &- 2267111177718461993872185235868925998437500\, t^{17}x^{57} \\ &+ 105602095963586360431772913336582758203125000\, t^{18}x^{54} \\ &- 4746298210770147629947127050066564488468750000\, t^{19}x^{51} \\ &+ 179140091527863148176036798525000521806640625 00\, t^{20}x^{48} \\ &+ 2151782260775542742321433560066322377739468750000 0\, t^{21}x^{45} \\ &- 1866304090193719979598090753176960710397522343750000\, t^{22}x^{42} \\ &+ 124826969096794011744700655966112731665388906250000000\, t^{23}x^{39} \\ &- 1077606584700530778623127008158745262625547976269531250\, t^{24}x^{36} \\ &- 576299849314040393346961263683711055700791986583984375 00\, t^{25}x^{33} \\ &- 8955758855718045099509061569538284662355447568285937500000\, t^{26}x^{30} \end{aligned}$$

$$
\begin{aligned}
&+ 53595540887959585328211746985528542837610416510486718750000\,t^{27}x^{27}\\
&- 13703065967666242029463436434276849693966219703293088378906250\,t^{28}x^{24}\\
&+ 19427222931444129856882917267638979035967641555006220703125000\,t^{29}x^{21}\\
&- 208150784481608941651057339544146316997551027469380490234375000\,t^{30}x^{18}\\
&+ 1324754444366795110024227171961826994615826516909729171875000000\,t^{31}x^{15}\\
&- 6245737853252332537638893402020685986951096563687070182861328125\,t^{32}x^{12}\\
&- 176161836886604251061609813903147553478107851796301979516601562500\,t^{33}x^{9}\\
&+ 31709130639588765191089766502566559626059413323334356312988281250000\,t^{34}x^{6}\\
&- 1981820664974297824443110406410409976628713332708397269561767578125\,t^{36}
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
d(x,t) = (&-x^{54} + 1485\,tx^{51} - 868725\,t^{2}x^{48} + 269594325\,t^{3}x^{45} - 49165491375\,t^{4}x^{42}\\
&+ 5714434826100\,t^{5}x^{39} - 396365167823625\,t^{6}x^{36} + 24418778773764375\,t^{7}x^{33}\\
&- 435655173246418125\,t^{8}x^{30} - 194269936514328675000\,t^{9}x^{27}\\
&- 4035822393920168226375\,t^{10}x^{24} + 5744505548909944754437500\,t^{11}x^{21}\\
&- 290076716794050362473171875\,t^{12}x^{18} + 1331101840126763068207593750\,t^{13}x^{15}\\
&- 20964853981996518324269601562 5\,t^{14}x^{12}\\
&- 39972988259006694938274040312 50\,t^{15}x^{9}\\
&+ 17987844716553012722223318140625 0\,t^{16}x^{6}\\
&+ 31478728253967772263890806746093 75\,t^{18})^{2}x^{2}
\end{aligned}
\tag{3.22}
$$

is a rational solution to the KdV equation.

We can go on and present more explicit rational solutions, but they become very complicated. For example, in the case of the order 10, the numerator is a polynomial of degree 108 in x, 36 in t containing 36 terms, and the denominator is a polynomial of degree 110 in x, 36 in t containing 36 terms. In the case of the order 20, the numerator is a polynomial of degree 418 in x, 139 in t containing 140 terms, the denominator is a polynomial of degree 420 in x and 140 in t containing 141 terms. It will be interesting to study the structure of these solutions in details.

4. Conclusion

Solutions to the KdV equation as the second derivative with respect to x of a logarithm of a determinant involving particular polynomials have been given. We get rational solutions as the quotient of two polynomials in x and t, the numerator is a polynomial of the degree $n(n+1)-2$ in x and the denominator is a polynomial of the degree $n(n+1)$ in x. The structure in t is more complex and should be studied in more details. This representation gives a very efficient method to construct rational solutions to the KdV equation. Some explicit expressions are given for some orders, and more recent works can be cited as [15], [18] or [13].

It can be noted that the solutions presented in this paper are different from those presented in a previous work of [6]. It will be relevant to study the structure of polynomial in details by entering in these last solutions.

Acknowledgements

The author would like to thank the reviewer for his very attentive reading of the manuscript and various constructive remarks.

References

[1] M. J. Ablowitz and H. Cornille, *On solutions of the KdV equation*, Physics Letters, 1979, 72A(4), 277–280.

[2] M. Adler and J. Moser, *On a class of polynomials connected with the Korteweg-deVries equation*, Communications in Mathematical Physics, 1978, 61, 1–30.

[3] H. Airault, H. P McKean and J. Moser, *Rational and elliptic solutions of the KdV equation and a related many-body problem*, Communications on Pure and Applied Mathematics, 1977, 30(2), 95–148.

[4] J. Boussinesq, *Essai sur la théorie des eaux courantes*, XXIII, Mémoire à l'Acadḿie des Sciences de l'Institut de France, Paris, 1872.

[5] N. C. Freeman and J. J. C. Nimmo, *Rational solutions of the KdV equation in wronskian form*, Physics Letters A, 1983, 96(9), 443–446.

[6] P. Gaillard, *Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case*, Journal of Geometry and Physics, 2021, 161, Article ID 104059, 12 pages.

[7] C. S. Gardner, J. M. Green, M. D. Kruskall anf R. M. Miura, *Method for Solving the Korteweg-deVries Equation*, Physical Review Letters, 1967, 19(19), 1095–1097.

[8] R. Hirota, *Exact Solution of the Korteweg–de Vries Equation for Multiple Collisions of Solitons*, Physical Review Letters, 1971, 27(18), 1192–1194.

[9] A. R. Its and V. B. Matveev, *Hill's operator with finitely many gaps*, Functional Analysis and Its Applications, 1975, 9, 65–66.

[10] D. J. Korteweg and G. D. Vries, *On the change of form of long wawes adwancing in a rectangular canal, and on a new type of long stationary waves*, Philosophical Magazine, 1995, 39, 442–443.

[11] M. Kovalyov, *On a class of solutions of KdV*, Journal of Differential Equations, 2005, 213(1), 1–80.

[12] P. D. Lax, *Periodic solutions of the KdV equation*, Communications on Pure and Applied Mathematics, 1975, 28(1), 141–188.

[13] J. Liu, *Exact solution of rational expansion method to the variable coefficient combined KdV equation with forced term*, Pure and Applied Mathematics, 2012, 28(5), 705–710.

[14] W. X. Ma, *Lump solutions to the Kadomtsev–Petviashvili equation*, Physics Letters A, 2015, 379(36), 1975–1978.

[15] W. X. Ma, *Linear superposition of Wronskian rational solutions to the KdV equation*, Communications in Theoretical Physics, 2021, 73(6), Article ID 065001, 5 pages.

[16] W. X. Ma and Y. You, *Solving the KdV equation by its bilinear form wronskian solutions*, Transactions of the American Mathematical Society, 2004, 357(5), 1753–1778.

[17] V. B. Matveev, *Generalized Wronskian Formula for solutions of the KdV equation*, Physics Letters A, 1992, 166(3–4), 205–208.

[18] M. S. Osman and A. M. Wazwaz, *An efficient algorithm to construct multi-soliton rational solutionsof the (2+ 1)-dimensional KdV equation with variable coefficients*, Applied Mathematics and Computation, 2018, 321, 282–289.

[19] V. E. Zakharov and L. D. Faddeev, *Korteweg-de Vries equation: A completely integrable Hamiltonian system*, Functional Analysis and Its Applications, 1971, 5(4), 280–287.