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The Exact Solutions for the Benjamin-Bona-Mahony Equation*

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Abstract The Benjamin-Bona-Mahony (BBM) equation represents the unidirectional propagation of nonlinear dispersive long waves, which has a clear physical background, and is a more suitable mathematical and physical equation than the KdV equation. Therefore, the research on the BBM equation is very important. In this article, we put forward an effective algorithm, the modified hyperbolic function expanding method, to build the solutions of the BBM equation. We, by utilizing the modified hyperbolic function expanding method, obtain the traveling wave solutions of the BBM equation. When the parameters are taken as special values, the solitary waves are also derived from the traveling waves. The traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. The modified hyperbolic function expanding method is direct, concise, elementary and effective, and can be used for many other nonlinear partial differential equations.

Keywords Generalized hyperbolic tangent function method, The modified hyperbolic function expanding method, Traveling wave solution, Balance coefficient method, Partial differential equation.

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1. Introduction

Many phenomena in natural science can be described by nonlinear partial differential equations (NLPDE), for instance, physics, ecology, medicine, zoology, fiber communications, fluid dynamics, propagation of waves, marine engineering, plasma physics, incompressible fluid, ocean and rogue waves, photonics, optics, optical-fiber communications, superconductors, arterial mechanics and cosmic plasmas [1, 2, 4–7, 13, 17, 19, 22, 23, 26–29, 44]. Due to the wide application of NLPDE in real life, many scholars want to study their application mechanism by solving exact solutions of NLPDE, paving way for the in-depth research [44] in the future.

So far, we have generally obtained the exact solutions of NLPDE by using mathematical software such as Maple and Mathematica [3, 28, 30, 44, 45]. Common meth-

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ods to solving NLPDE include the tanh method [18, 24, 38, 47], the Hirota's bilinear method [40], the sine-cosine method [33], the Exp-function method [10, 16], the inverse scattering method [20], the $\frac{G'}{G}$-expansion method [25, 36], the *sn*-*ns* method [15], the Darboux transformation method [9], the F-expansion method [21], the polynomial expansion method [22], the modified polynomial expansion method [23], the Bäcklund transformation [35], etc. Specifically, using the Hirota's bilinear method and the tanh-coth method, in [39], Wazwaz yielded the N-soliton solutions, $N = 1, 2, 3, 4, 5$, for the Kadomtsev-Petviashvili equation. In [40], the author obtained the single-soliton solution and the N-soliton solution about the Sawada-Kotera-Kadomtsev-Petviashvili equation. In [37], the author obtained the traveling wave solutions to the KdV equation, the generalized KdV equation, the $K(n, n)$ equations, the Boussinesq equation, the RLW equation, the BBM equation and the Phi-four equation via the sine-cosine method. In [49], Zhang and Huang acquired N-soliton solutions to the KdV equation with the variable coefficients Exp-function method. In [1], Abdel-Gawad and Osman obtained a wide class of exact solutions to the variable coefficient KdV equation by using the extended unified method, and presented a new algorithm for treating the coupled NLPDE. In [34], Vakhnenko and Parkes obtained the exact N-soliton solutions to the Vakhnenko equation utilizing the inverse scattering method. In [41], the author yielded the traveling wave solutions to the Zhiber-Shabat equation, the Liouville equation, the sinh-Gordon equation, the Dodd-Bullough-Mikhailov equation and the Tzitzeica-Dodd-Bullough equation via the tanh method and the extended tanh method, which had showed their different physical structures. Employing the modified extended tanh method, in [31], Raslan, Ali and Shallal acquired the traveling wave solutions for the space-time fractional nonlinear partial differential equations, for instance, the fractional equal width wave equation and the fractional modified equal width wave equation. In [46], Zahran and Khater obtained the traveling wave solutions to the Bogoyavlenskii equation, which had showed broad applicability. In [12], by using the first integral method, the tanh-coth method, the sech-csch method, the tan-cot method and the sec-csc ansatz, Darvishi et al., acquired the traveling wave solutions to the $(2 + 1)$-dimensional Zakharov-like soliton equation. In [4], Akbar, Kayum and Osman obtained special solutions, including the bright solitons, the periodic solutions, the compaction solutions, the bell-shape solutions, the parabolic shape solutions, the singular periodic solutions, the plane shape solutions and some new types of solitons, for the $(3 + 1)$-dimensional Zakharov-Kuznetsov (ZK) and the new extended quantum ZK equations by using the enhanced modified simple equation method. In [32], the exact traveling wave solutions to the M-fractional generalized reaction Duffing model and the density dependent M-fractional diffusion reaction equation by using three fertile methods, including the $(\frac{G'}{G}, \frac{1}{G})$ method, the modified $\frac{G'}{G^2}$ method and the $\frac{1}{G'}$-expansion method. In [36], Wang, Li and Zhang yielded solitary wave solutions and periodic wave solutions for the BBM equation and the modified Benjamin-Bona-Mahony equation by employing a generalized $\frac{G'}{G}$-expansion method. In [14], Ghanbari, Baleanu and Qurashi obtained some new exact optical solitons for the generalized Benjamin-Bona-Mahony equation via the generalized exponential rational function method, and detailed the physical interpretation of these solutions. In [43], Yan et al., obtained non-local symmetry and

Bäcklund transformation for the BBM equation via the truncated Painlevé expansion method. By employing the Jacobian elliptic function, they yielded solitary wave solutions and conoidal periodic wave solutions. In [11], Cheng, Luo and Hong employed the theory of the planar dynamical system to investigate the dynamical behavior and traveling solutions of the Drinfel'd-Sokolov $D(m,n)$ system. In [50], Zhou and Zhuang obtained bifurcations of phase portraits and different traveling wave solutions for planar dynamic systems of the Raman soliton model by using the bifurcation theory method of dynamic systems.

The BBM equation (1.1) was first proposed by Benjamin, Bona and Mahony, representing the unidirectional propagation of a nonlinear dispersive long wave $u(x,t)$ [8]. The BBM equation is a more suitable mathematical and physical equation than the KdV equation, and has a clear physical background

$$u_t + \alpha u_x + \beta u u_x - \delta u_{xxt} = 0, \tag{1.1}$$

where $u(x,t)$ depends on spatial variable x and temporal variable t. α, β, δ are arbitrary constants with the nonlinear and dispersion coefficients and $\beta \neq 0$, $\delta > 0$ [48].

In this article, we will study the BBM equation (1.1) by utilizing the modified hyperbolic function expanding method [13]. The article is arranged as follows. First, we give a brief introduction in Section 1. Secondly, the algorithm of the modified hyperbolic function expanding method is given in Section 2. Thirdly, we give the exact solutions of the BBM equation in Section 3. At the same time, when the appropriate values of the free parameters are selected, the special solutions and the corresponding figures are given in Section 4. Finally, we make a conclusion on the article in the last section.

2. The modified hyperbolic function expanding method

In this part, we will introduce the specific steps of the modified hyperbolic function method.

First, we give a generalized NLPDE

$$P(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{2.1}$$

where $u(x,t)$ is an unknown function about the spatial variable x and the temporal variable t.

Next, we will discuss the specific steps of the modified hyperbolic function expanding method.

(i) To find the exact solution of equation (2.1), we need to transform equation (2.1) into an ordinary differential equation (ODE).

Suppose $u(x,t) = u(x-ct) = u(\xi)$, where $\xi = x - ct$, and $c \neq 0$ is any real number. We can obtain

$$P(u, u', u'', u''', \cdots) = 0, \tag{2.2}$$

where the prime is the derivative with respect to ξ.

(ii) Assume that the expression of the exact solution of equation (1.1) is

$$u(\xi) = a_0 + \sum_{i=1}^{M} (a_i \varphi^{-i}(\xi) + b_i \varphi^{i}(\xi) + c_i \varphi^{i}(\xi)\varphi'(\xi) + d_i \varphi^{-i}(\xi)\varphi'(\xi)), \tag{2.3}$$

where $\varphi(\xi)$ meets the following equation

$$\varphi'(\xi) = \gamma + \varphi^2(\xi), \tag{2.4}$$

and γ is an arbitrary real number.

(iii) Confirm the value of M in equation (2.3). To confirm the parameter M, we use the balance coefficient method to balance the highest order of the derivative term and the highest order of the nonlinear term in equation (2.2) [13, 42].

(iv) Substituting the value of M and equation (2.4) into equation (2.3), with the help of Maple, we can set the coefficient of all terms with the same power exponent about $\varphi(\xi)$ in equation (2.3) to zeros, to solve for the values of $a_0, a_i, b_i, c_i, d_i, (i = 1, 2, \cdots, M), c, \gamma$.

(v) Substituting the values of $a_0, a_i, b_i, c_i, d_i, (i = 1, 2, \cdots, M), c, \gamma$ into equation (2.3), we get the exact solutions of equation (1.1).

3. Solutions to the BBM equation

In order to obtain the solutions to the BBM equation, first, we need to give the solutions $\varphi(\xi)$ for equation (2.4) as follows.

$$\varphi_1(\xi) = \sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p), \gamma > 0, \tag{3.1}$$

$$\varphi_2(\xi) = -\frac{1}{\xi + p}, \gamma = 0, \tag{3.2}$$

$$\varphi_3(\xi) = -\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi + p), \gamma < 0, \tag{3.3}$$

where p is a constant of integration.

Next, we give the solutions to the BBM equation. First, we need to transform equation (1.1) into an ODE.

Set $u(x, t) = u(\xi)$, where $\xi = x - ct$, and $c \neq 0$ is any real number. Substituting $u(x, t) = u(\xi)$ into equation (1.1), we can obtain the following equation

$$-cu' + \alpha u' + \beta uu' + c\delta u''' = 0. \tag{3.4}$$

Integrating both sides of equation (3.4) and letting the integral constant be 0, we can get

$$-cu + \alpha u + \frac{\beta}{2}u^2 - c\delta u'' = 0. \tag{3.5}$$

We apply the balance coefficient method to determine M by balancing the highest nonlinear term and the highest derivative terms [13, 42]. From the highest nonlinear term u^2 with the exponent $2M$, the highest derivative term u'' and the exponent $M + 2$ in equation (3.5), we can obtain $M = 2$.

Consequently, we yield the exact solution expression of equation (1.1) as

$$u(\xi) = a_0 + \sum_{i=1}^{2}(a_i\varphi^{-i}(\xi) + b_i\varphi^i(\xi) + c_i\varphi^i(\xi)\varphi'(\xi) + d_i\varphi^{-i}(\xi)\varphi'(\xi)). \tag{3.6}$$

Substituting equation (3.6) and equation (2.4) into equation (3.5), combining terms with the same power of $\varphi(\xi)$ and setting the coefficient of the same power of $\varphi(\xi)$ to 0, we yield the system of algebraic equations about $a_0, a_i, b_i, c_i, d_i (i = 1, 2), c, \gamma$.

Using Maple, we get eight sets of solutions for the values of a_0, a_i, b_i, c_i, $d_i (i = 1, 2)$, c, γ in the above algebraic equation system, as shown below.

$$\begin{aligned} 1)\quad & a_0 = a_0, a_1 = -d_1\gamma, a_2 = -d_2\gamma, b_1 = -d_1, b_2 = 0, \\ & c = \frac{1}{2}a_0\beta + \frac{1}{2}\beta d_2 + \alpha, c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma; \end{aligned} \tag{3.7}$$

$$\begin{aligned} 2)\quad & a_0 = -d_2, a_1 = -d_1\gamma, a_2 = -d_2\gamma, b_1 = -d_1, b_2 = 0, c = c, c_1 = 0, \\ & c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma; \end{aligned} \tag{3.8}$$

$$\begin{aligned} 3)\quad & a_0 = \frac{-4d_2\beta\delta\gamma + 4\alpha\delta\gamma + d_2\beta}{\beta(4\delta\gamma - 1)}, a_1 = -\frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)}, \\ & a_2 = \frac{(3a_0 + 2d_2)\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)}, b_1 = -d_1, b_2 = 0, c = -a_0\beta - d_2\beta + \alpha, \\ & c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma; \end{aligned} \tag{3.9}$$

$$\begin{aligned} 4)\quad & a_0 = \frac{-4d_2\beta\delta\gamma + 12\alpha\delta\gamma + d_2\beta}{\beta(4\delta\gamma + 1)}, a_1 = \frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta + 3\alpha)}, \\ & a_2 = -\frac{a_0\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta + 3\alpha)}, b_1 = -d_1, b_2 = 0, c = \frac{1}{3}a_0\beta + \frac{1}{3}d_2\beta + \alpha, \\ & c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma; \end{aligned} \tag{3.10}$$

$$\begin{aligned} 5)\quad & a_0 = -\frac{d_2\beta + 3\alpha - 3c}{\beta}, a_1 = -\frac{d_1(\alpha - c)}{4c\delta}, a_2 = -\frac{d_2(\alpha - c)}{4c\delta}, \\ & b_1 = -d_1, b_2 = -\frac{12c\delta}{\beta}, c = \frac{\alpha}{4\delta\gamma + 1}, c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \\ & \gamma = \gamma; \end{aligned} \tag{3.11}$$

$$\begin{aligned} 6)\quad & a_0 = \frac{-d_2\beta + \alpha - c}{\beta}, a_1 = \frac{d_1(\alpha - c)}{4c\delta}, a_2 = \frac{d_2(\alpha - c)}{4c\delta}, b_1 = -d_1, \\ & b_2 = -\frac{12c\delta}{\beta}, c = -\frac{\alpha}{4\delta\gamma - 1}, c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma; \end{aligned} \tag{3.12}$$

$$\begin{aligned} 7)\quad & a_0 = -\frac{2d_2\beta + \alpha - c}{2\beta}, a_1 = \frac{d_1(\alpha - c)}{16c\delta}, \\ & a_2 = -\frac{-4d_2\alpha\beta + 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta}, b_1 = -d_1, b_2 = -\frac{12c\delta}{\beta}, \\ & c = -\frac{\alpha}{16\delta\gamma - 1}, c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma; \end{aligned} \tag{3.13}$$

$$\begin{aligned} 8)\quad & a_0 = -\frac{2d_2\beta + 3\alpha - 3c}{2\beta}, a_1 = -\frac{d_1(\alpha - c)}{16c\delta}, \\ & a_2 = -\frac{4d_2\alpha\beta - 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta}, b_1 = -d_1, b_2 = -\frac{12c\delta}{\beta}, \\ & c = \frac{\alpha}{16\delta\gamma + 1}, c_1 = 0, c_2 = 0, d_1 = d_1, d_2 = d_2, \gamma = \gamma. \end{aligned} \tag{3.14}$$

Then, we substitute the above eight sets of solutions about a_0, a_i, b_i, c_i, $d_i(i = 1,2)$, c, γ and equation (2.4) into equation (3.6) respectively, and yield the exact solutions of equation (1.1) as follows.

For 1), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned} u_{11}(\xi) =& a_0 - d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p) - d_2\cot^2(\sqrt{\gamma}\xi + p) \\ & - d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p) + d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p)\sec^2(\sqrt{\gamma}\xi + p) \\ & + d_2\csc^2(\sqrt{\gamma}\xi + p), \end{aligned} \tag{3.15}$$

where $\xi = x - ct, c = \frac{1}{2}a_0\beta + \frac{1}{2}\beta d_2 + \alpha$, and a_0, d_1, d_2, p are arbitrary constants.

If $\gamma = 0$,

$$u_{12}(\xi) = a_0 + d_1\gamma(\xi + p) - d_2\gamma(\xi + p)^2 + d_2 = C, \tag{3.16}$$

where $\xi = x - ct, c = \dfrac{1}{2}a_0\beta + \dfrac{1}{2}\beta d_2 + \alpha$, and $C = a_0 + d_2$ is an arbitrary constant, which is a constant solution.

If $\gamma < 0$,

$$\begin{aligned} u_{13}(\xi) =& a_0 - d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi + p) + d_2\coth^2(\sqrt{-\gamma}\xi + p) \\ & + d_1\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi + p) \\ & + d_1\sqrt{-\gamma}\text{coth}(\sqrt{-\gamma}\xi + p)\text{sech}^2(\sqrt{-\gamma}\xi + p) \\ & + d_2\text{csch}^2(\sqrt{-\gamma}\xi + p), \end{aligned} \tag{3.17}$$

where $\xi = x - ct, c = \dfrac{1}{2}a_0\beta + \dfrac{1}{2}\beta d_2 + \alpha$, and a_0, d_1, d_2, p are arbitrary constants.

For 2), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned} u_{21}(\xi) =& - d_2 - d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p) - d_2\cot^2(\sqrt{\gamma}\xi + p) \\ & - d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p) + d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p)\sec^2(\sqrt{\gamma}\xi + p) \\ & + d_2\csc^2(\sqrt{\gamma}\xi + p), \end{aligned} \tag{3.18}$$

where $\xi = x - ct$, and $c(\neq 0), d_1, d_2, p$ are arbitrary constants.

If $\gamma = 0$,

$$u_{22}(\xi) = d_1\gamma(\xi + p) - d_2\gamma(\xi + p)^2 = 0, \tag{3.19}$$

which is a trivial solution.

If $\gamma < 0$,

$$\begin{aligned} u_{23}(\xi) =& - d_2 - d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi + p) + d_2\coth^2(\sqrt{-\gamma}\xi + p) \\ & + d_1\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi + p) \\ & + d_1\sqrt{-\gamma}\text{coth}(\sqrt{-\gamma}\xi + p)\text{sech}^2(\sqrt{-\gamma}\xi + p) \\ & + d_2\text{csch}^2(\sqrt{-\gamma}\xi + p), \end{aligned} \tag{3.20}$$

where $\xi = x - ct$, and $c(\neq 0), d_1, d_2, p$ are arbitrary constants.

For 3), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned} u_{31}(\xi) =& \frac{-4d_2\beta\delta\gamma + 4\alpha\delta\gamma + d_2\beta}{\beta(4\delta\gamma - 1)} - \frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)\sqrt{\gamma}} \cot(\sqrt{\gamma}\xi + p) \\ &+ \frac{(3a_0 + 2d_2)\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)\gamma} \cot^2(\sqrt{\gamma}\xi + p) - d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p) \\ &+ d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p)\sec^2(\sqrt{\gamma}\xi + p) + d_2\csc^2(\sqrt{\gamma}\xi + p), \end{aligned} \tag{3.21}$$

where $\xi = x - ct, c = -a_0\beta - d_2\beta + \alpha \neq 0$, and a_0, d_1, d_2, p are arbitrary constants.

If $\gamma = 0$,

$$\begin{aligned} u_{32}(\xi) =& \frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)}(\xi + p) \\ &+ \frac{(3a_0 + 2d_2)\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)}(\xi + p)^2 + 2d_2, \end{aligned} \tag{3.22}$$

where $\xi = x - ct, c = -a_0\beta - d_2\beta + \alpha \neq 0$, and a_0, d_1, d_2, p are arbitrary constants.

If $\gamma < 0$,

$$\begin{aligned} u_{33}(\xi) =& \frac{-4d_2\beta\delta\gamma + 4\alpha\delta\gamma + d_2\beta}{\beta(4\delta\gamma - 1)} + \frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)\sqrt{-\gamma}} \coth(\sqrt{-\gamma}\xi + p) \\ &- \frac{(3a_0 + 2d_2)\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta - \alpha)\gamma} \coth^2(\sqrt{-\gamma}\xi + p) + d_1\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi + p) \\ &+ d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi + p)\mathrm{sech}^2(\sqrt{-\gamma}\xi + p) + d_2\mathrm{csch}^2(\sqrt{-\gamma}\xi + p), \end{aligned} \tag{3.23}$$

where $\xi = x - ct, c = -a_0\beta - \beta d_2 + \alpha \neq 0$, and a_0, d_1, d_2, p are arbitrary constants.

For 4), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned} u_{41}(\xi) =& \frac{-4d_2\beta\delta\gamma + 12\alpha\delta\gamma + d_2\beta}{\beta(4\delta\gamma + 1)} + \frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta + 3\alpha)\sqrt{\gamma}} \cot(\sqrt{\gamma}\xi + p) \\ &- \frac{a_0\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta + 3\alpha)\gamma} \cot^2(\sqrt{\gamma}\xi + p) - d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p) \\ &+ d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p)\sec^2(\sqrt{\gamma}\xi + p) + d_2\csc^2(\sqrt{\gamma}\xi + p), \end{aligned} \tag{3.24}$$

where $\xi = x - ct, c = \frac{1}{3}a_0\beta + \frac{1}{3}\beta d_2 + \alpha \neq 0$, and a_0, d_1, d_2, p are arbitrary constants.

If $\gamma = 0$,

$$\begin{aligned} u_{42}(\xi) =& - \frac{d_1\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta + 3\alpha)}(\xi + p) \\ &- \frac{a_0\beta(a_0 + d_2)}{4\delta(a_0\beta + d_2\beta + 3\alpha)}(\xi + p)^2 + 2d_2, \end{aligned} \tag{3.25}$$

where $\xi = x - ct, c = \frac{1}{3}a_0\beta + \frac{1}{3}d_2\beta + \alpha \neq 0$, and a_0, d_1, d_2, p are arbitrary constants.

If $\gamma < 0$,

$$
\begin{aligned}
u_{43}(\xi) =& \frac{-4d_2\beta\delta\gamma + 12\alpha\delta\gamma + d_2\beta}{\beta(4\delta\gamma+1)} - \frac{d_1\beta(a_0+d_2)}{4\delta(a_0\beta+d_2\beta+3\alpha)\sqrt{-\gamma}}\coth(\sqrt{-\gamma}\xi+p) \\
&+ \frac{a_0\beta(a_0+d_2)}{4\delta(a_0\beta+d_2\beta+3\alpha)\gamma}\coth^2(\sqrt{-\gamma}\xi+p) + d_1\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi+p) \\
&+ d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi+p)\text{sech}^2(\sqrt{-\gamma}\xi+p) + d_2\text{csch}^2(\sqrt{-\gamma}\xi+p),
\end{aligned} \tag{3.26}
$$

where $\xi = x - ct, c = \frac{1}{3}a_0\beta + \frac{1}{3}d_2\beta + \alpha \neq 0$, and a_0, d_1, d_2, p are arbitrary constants.

For 5), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$
\begin{aligned}
u_{51}(\xi) =& -\frac{d_2\beta+3\alpha-3c}{\beta} - \frac{d_1(\alpha-c)}{4c\delta\sqrt{\gamma}}\cot(\sqrt{\gamma}\xi+p) \\
&- \frac{d_2(\alpha-c)}{4c\delta\gamma}\cot^2(\sqrt{\gamma}\xi+p) - d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi+p) \\
&- \frac{12c\delta\gamma}{\beta}\tan^2(\sqrt{\gamma}\xi+p) + d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi+p)\sec^2(\sqrt{\gamma}\xi+p) \\
&+ d_2\csc^2(\sqrt{\gamma}\xi+p),
\end{aligned} \tag{3.27}
$$

where $\xi = x - ct, c = \frac{\alpha}{4\delta\gamma+1}$, and d_1, d_2, p are arbitrary constants.

If $\gamma = 0$, then $c = \frac{\alpha}{4\delta\gamma+1} = \alpha$,

$$
\begin{aligned}
u_{52}(\xi) =& -\frac{d_2\beta+3\alpha-3c}{\beta} + \frac{d_1(\alpha-c)}{4c\delta}(\xi+p) - \frac{d_2(\alpha-c)}{4c\delta}(\xi+p)^2 \\
&+ \frac{12c\delta}{\beta(\xi+p)^2} + d_2 = \frac{12\alpha\delta}{\beta(\xi+p)^2},
\end{aligned} \tag{3.28}
$$

where $\xi = x - \alpha t$, and p is an arbitrary constant.

If $\gamma < 0$,

$$
\begin{aligned}
u_{53}(\xi) =& -\frac{d_2\beta+3\alpha-3c}{\beta} + \frac{d_1(\alpha-c)}{4c\delta\sqrt{-\gamma}}\coth(\sqrt{-\gamma}\xi+p) \\
&+ \frac{d_2(\alpha-c)}{4c\delta\gamma}\coth^2(\sqrt{-\gamma}\xi+p) + d_1\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi+p) \\
&+ \frac{12c\delta\gamma}{\beta}\tanh^2(\sqrt{-\gamma}\xi+p) \\
&+ d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi+p)\text{sech}^2(\sqrt{-\gamma}\xi+p) \\
&+ d_2\text{csch}^2(\sqrt{-\gamma}\xi+p),
\end{aligned} \tag{3.29}
$$

where $\xi = x - ct, c = \frac{\alpha}{4\delta\gamma+1}$, and d_1, d_2, p are arbitrary constants.

For 6), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned}u_{61}(\xi) =& \frac{-d_2\beta + \alpha - c}{\beta} + \frac{d_1(\alpha - c)}{4c\delta\sqrt{\gamma}}\cot(\sqrt{\gamma}\xi + p)\\&+ \frac{d_2(\alpha - c)}{4c\delta\gamma}\cot^2(\sqrt{\gamma}\xi + p) - d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p)\\&- \frac{12c\delta}{\beta}\tan^2(\sqrt{\gamma}\xi + p) + d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p)\sec^2(\sqrt{\gamma}\xi + p)\\&+ d_2\csc^2(\sqrt{\gamma}\xi + p),\end{aligned} \tag{3.30}$$

where $\xi = x - ct, c = -\frac{\alpha}{4\delta\gamma - 1}$, and d_1, d_2, p are arbitrary constants.

If $\gamma = 0$, then $c = -\frac{\alpha}{4\delta\gamma - 1} = \alpha$,

$$\begin{aligned}u_{62}(\xi) =& \frac{-d_2\beta + \alpha - c}{\beta} - \frac{d_1(\alpha - c)}{4c\delta}(\xi + p) + \frac{d_2(\alpha - c)}{4c\delta}(\xi + p)^2\\&- \frac{12c\delta}{\beta(\xi + p)^2} + d_2 = -\frac{12\alpha\delta}{\beta(\xi + p)^2},\end{aligned} \tag{3.31}$$

where $\xi = x - \alpha t$, and p is an arbitrary constant.

If $\gamma < 0$,

$$\begin{aligned}u_{63}(\xi) =& \frac{-d_2\beta + \alpha - c}{\beta} - \frac{d_1(\alpha - c)}{4c\delta\sqrt{-\gamma}}\coth(\sqrt{-\gamma}\xi + p)\\&- \frac{d_2(\alpha - c)}{4c\delta\gamma}\coth^2(\sqrt{-\gamma}\xi + p) + d_1\sqrt{-\gamma}\tanh(\sqrt{-\gamma}\xi + p)\\&+ \frac{12c\delta\gamma}{\beta}\tanh^2(\sqrt{-\gamma}\xi + p)\\&+ d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi + p)\mathrm{sech}^2(\sqrt{-\gamma}\xi + p)\\&+ d_2\mathrm{csch}^2(\sqrt{-\gamma}\xi + p),\end{aligned} \tag{3.32}$$

where $\xi = x - ct, c = -\frac{\alpha}{4\delta\gamma - 1}$, and d_1, d_2, p are arbitrary constants.

For 7), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned}u_{71}(\xi) =& -\frac{2d_2\beta + \alpha - c}{2\beta} + \frac{d_1(\alpha - c)}{16c\delta\sqrt{\gamma}}\cot(\sqrt{\gamma}\xi + p)\\&- \frac{-4d_2\alpha\beta + 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta\gamma}\cot^2(\sqrt{\gamma}\xi + p)\\&- d_1\sqrt{\gamma}\tan(\sqrt{\gamma}\xi + p) - \frac{12c\delta\gamma}{\beta}\tan^2(\sqrt{\gamma}\xi + p)\\&+ d_1\sqrt{\gamma}\cot(\sqrt{\gamma}\xi + p)\sec^2(\sqrt{\gamma}\xi + p) + d_2\csc^2(\sqrt{\gamma}\xi + p),\end{aligned} \tag{3.33}$$

where $\xi = x - ct, c = -\frac{\alpha}{16\delta\gamma - 1}$, and d_1, d_2, p are arbitrary constants.

If $\gamma = 0$, then $c = -\dfrac{\alpha}{16\delta\gamma - 1} = \alpha$,

$$\begin{aligned} u_{72}(\xi) = &- \frac{2d_2\beta + \alpha - c}{2\beta} - \frac{d_1(\alpha - c)}{16c\delta}(\xi + p) \\ &- \frac{-4d_2\alpha\beta + 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta}(\xi + p)^2 \\ &- \frac{12c\delta}{\beta(\xi + p)^2} + d_2 = -\frac{12\alpha\delta}{\beta(\xi + p)^2}, \end{aligned} \tag{3.34}$$

where $\xi = x - \alpha t$, and p is an arbitrary constant.

If $\gamma < 0$,

$$\begin{aligned} u_{73}(\xi) = &- \frac{2d_2\beta + \alpha - c}{2\beta} - \frac{d_1(\alpha - c)}{16c\delta\sqrt{-\gamma}} \coth(\sqrt{-\gamma}\xi + p) \\ &+ \frac{-4d_2\alpha\beta + 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta\gamma} \coth^2(\sqrt{-\gamma}\xi + p) \\ &+ d_1\sqrt{-\gamma} \tanh(\sqrt{-\gamma}\xi + p) + \frac{12c\delta\gamma}{\beta} \tanh^2(\sqrt{-\gamma}\xi + p) \\ &+ d_1\sqrt{-\gamma}\coth(\sqrt{-\gamma}\xi + p)\mathrm{sech}^2(\sqrt{-\gamma}\xi + p) \\ &+ d_2\mathrm{csch}^2(\sqrt{-\gamma}\xi + p), \end{aligned} \tag{3.35}$$

where $\xi = x - ct, c = -\dfrac{\alpha}{16\delta\gamma - 1}$, and d_1, d_2, p are arbitrary constants.

For 8), the solutions of equation (1.1) admit:

If $\gamma > 0$,

$$\begin{aligned} u_{81}(\xi) = &- \frac{2d_2\beta + 3\alpha - 3c}{2\beta} - \frac{d_1(\alpha - c)}{16c\delta\sqrt{\gamma}} \cot(\sqrt{\gamma}\xi + p) \\ &- \frac{4d_2\alpha\beta - 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta\gamma} \cot^2(\sqrt{\gamma}\xi + p) \\ &- d_1\sqrt{\gamma} \tan(\sqrt{\gamma}\xi + p) - \frac{12c\delta\gamma}{\beta} \tan^2(\sqrt{\gamma}\xi + p) \\ &+ d_1\sqrt{\gamma} \cot(\sqrt{\gamma}\xi + p) \sec^2(\sqrt{\gamma}\xi + p) + d_2 \csc^2(\sqrt{\gamma}\xi + p), \end{aligned} \tag{3.36}$$

where $\xi = x - ct, c = \dfrac{\alpha}{16\delta\gamma + 1}$, and d_1, d_2, p are arbitrary constants.

If $\gamma = 0$, then $c = \dfrac{\alpha}{16\delta\gamma + 1} = \alpha$,

$$\begin{aligned} u_{82}(\xi) = &- \frac{2d_2\beta + 3\alpha - 3c}{2\beta} + \frac{d_1(\alpha - c)}{16c\delta}(\xi + p) \\ &- \frac{4d_2\alpha\beta - 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta}(\xi + p)^2 \\ &- \frac{12c\delta}{\beta(\xi + p)^2} + d_2 = -\frac{12\alpha\delta}{\beta(\xi + p)^2}, \end{aligned} \tag{3.37}$$

where $\xi = x - \alpha t$, and p is an arbitrary constant.

If $\gamma < 0$,

$$\begin{aligned} u_{83}(\xi) = & -\frac{2d_2\beta + 3\alpha - 3c}{2\beta} + \frac{d_1(\alpha - c)}{16c\delta\sqrt{-\gamma}} \coth(\sqrt{-\gamma}\xi + p) \\ & + \frac{4d_2\alpha\beta - 4cd_2\beta + 3\alpha^2 - 6c\alpha + 3c^2}{64c\beta\delta\gamma} \coth^2(\sqrt{-\gamma}\xi + p) \\ & + d_1\sqrt{-\gamma} \tanh(\sqrt{-\gamma}\xi + p) + \frac{12c\delta\gamma}{\beta} \tanh^2(\sqrt{-\gamma}\xi + p) \\ & + d_1\sqrt{-\gamma} \coth(\sqrt{-\gamma}\xi + p) \mathrm{sech}^2(\sqrt{-\gamma}\xi + p) \\ & + d_2 \mathrm{csch}^2(\sqrt{-\gamma}\xi + p), \end{aligned} \tag{3.38}$$

where $\xi = x - ct, c = \dfrac{\alpha}{16\delta\gamma + 1}$, and d_1, d_2, p are arbitrary constants.

Note: For the solution u_{12} is a constant solution, the solution u_{22} is a trivial solution, and they have no physical meanings. The solutions u_{62}, u_{72} and u_{82} are the same solutions.

4. Examples and corresponding graphics

In this part, we will investigate the exact solutions of the BBM equation and the corresponding figures when taking specific parameter values.

For simplicity, we take $\alpha = \beta = \delta = 1$ to study all the solutions.

We will select some values for $a_0, a_i, b_i, c_i (i = 1, 2), c, p, \gamma$ to yield the exact solutions of the BBM equation.

We assume that the values of the above parameters are as follows

$$a_0 = 1,\ d_1 = -1,\ d_2 = 2,\ p = 0.$$

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.15) becomes

$$\begin{aligned} u_{11}(\xi) = & 1 + 2\cot(2\xi) - 2\cot^2(2\xi) + 2\tan(2\xi) - 2\cot(2\xi)\sec^2(2\xi) \\ & + 2\csc^2(2\xi), \end{aligned} \tag{4.1}$$

where $\xi = x - \frac{5}{2}t$. The figure of solution (4.1) is shown as Figure (a) in Figure 1.

When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.17) becomes

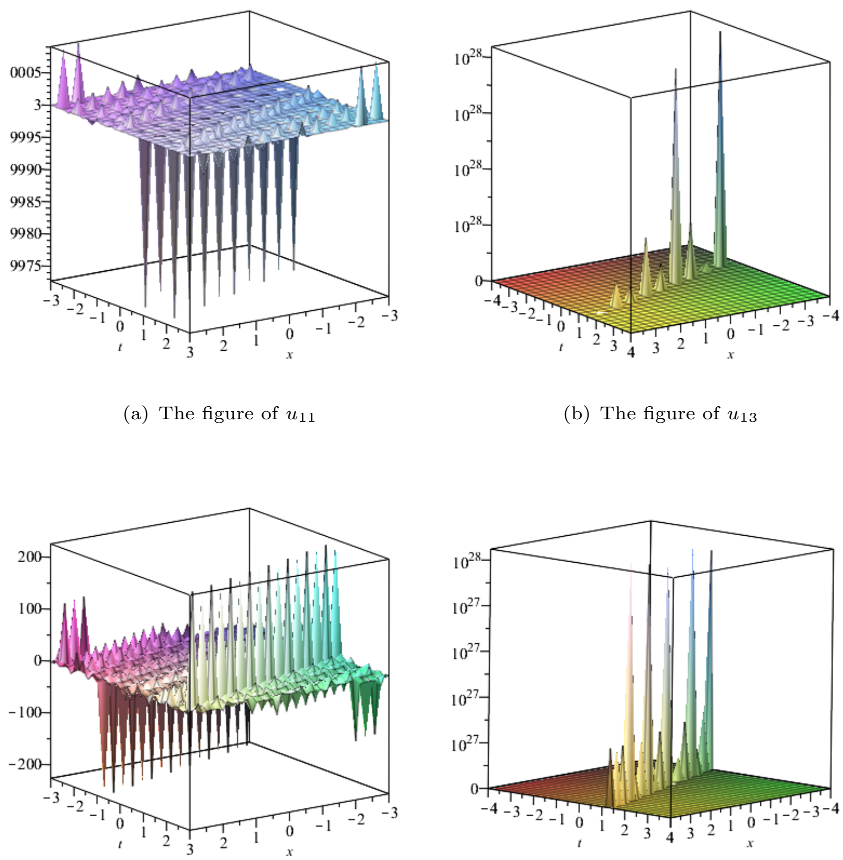
$$\begin{aligned} u_{13}(\xi) = & 1 + 2\coth(2\xi) + 2\coth^2(2\xi) - 2\tanh(2\xi) - 2\coth(2\xi)\mathrm{sech}^2(2\xi) \\ & + 2\mathrm{csch}^2(2\xi), \end{aligned} \tag{4.2}$$

where $\xi = x - \frac{5}{2}t$. The figure of solution (4.2) is shown as Figure (b) in Figure 1.

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.18) becomes

$$\begin{aligned} u_{21}(\xi) = & -2 + 2\cot(2\xi) - 2\cot^2(2\xi) + 2\tan(2\xi) - cot(2\xi)\sec^2(2\xi) \\ & + 2\csc^2(2\xi), \end{aligned} \tag{4.3}$$

where $\xi = x - 3t$. The figure of solution (4.3) is shown as Figure (c) in Figure 1.



(a) The figure of u_{11}

(b) The figure of u_{13}

(c) The figure of u_{21}

(d) The figure of u_{23}

Figure 1. The figure of (4.1) shown on (a); the figure of (4.2) shown on (b); the figure of (4.3) shown on (c); the figure of (4.4) shown on (d)

When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.20) becomes

$$\begin{aligned} u_{23}(\xi) = &-2 + 2\coth(2\xi) + 2\coth^2(2\xi) - 2\tanh(2\xi) \\ &- 2\coth(2\xi)\mathrm{sech}^2(2\xi) + 2\mathrm{csch}^2(2\xi), \end{aligned} \tag{4.4}$$

where $\xi = x - 3t$. The figure of solution (4.4) is shown as Figure (d) in Figure 1.

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.21) becomes

$$\begin{aligned} u_{31}(\xi) = &-\frac{14}{15} + \frac{3}{16}\cot(2\xi) + \frac{21}{32}\cot^2(2\xi) + 2\tan(2\xi) - 2\cot(2\xi)\sec^2(2\xi) \\ &+ 2\csc^2(2\xi), \end{aligned} \tag{4.5}$$

where $\xi = x + 2t$. The figure of solution (4.5) is shown as Figure (a) in Figure 2.

When $\gamma = 0$, then solution (3.22) becomes

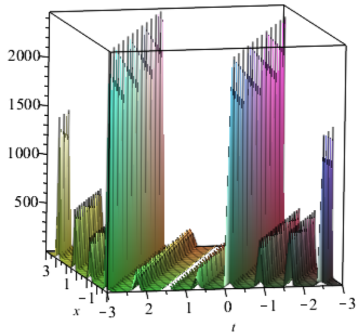
$$u_{32}(\xi) = -\frac{3}{8}\xi + \frac{21}{8}\xi^2, \tag{4.6}$$

where $\xi = x + 2t$. The figure of solution (4.6) is shown as Figure (b) in Figure 2.

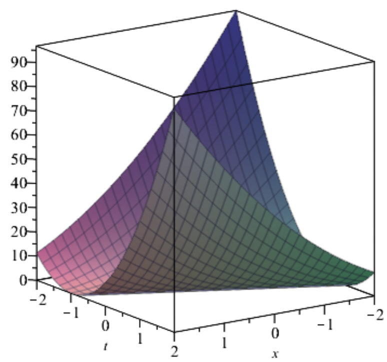
When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.23) becomes

$$\begin{aligned} u_{33}(\xi) = & -\frac{18}{17} - \frac{3}{16}\coth(2\xi) + \frac{21}{32}\coth^2(2\xi) - 2\tanh(2\xi) \\ & - 2\coth(2\xi)\text{sech}^2(2\xi) + 2\text{csch}^2(2\xi), \end{aligned} \tag{4.7}$$

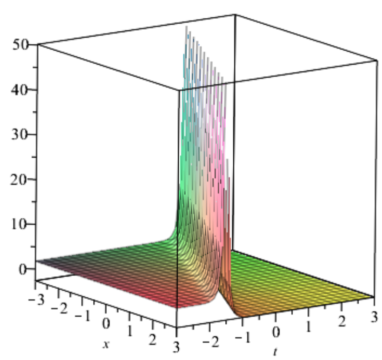
where $\xi = x + 2t$. The figure of solution (4.7) is shown as Figure (c) in Figure 2.



(a) The figure of u_{31}



(b) The figure of u_{32}



(c) The figure of u_{33}

Figure 2. The figure of (4.5) shown on (a); the figure of (4.6) shown on (b); the figure of (4.7) shown on (c)

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.24) becomes

$$\begin{aligned} u_{41}(\xi) = & \frac{18}{17} - \frac{1}{16}\cot(2\xi) - \frac{1}{32}\cot^2(2\xi) + 2\tan(2\xi) - 2\cot(2\xi)\sec^2(2\xi) \\ & + 2\csc^2(2\xi), \end{aligned} \tag{4.8}$$

where $\xi = x - 2t$. The figure of solution (4.8) is shown as Figure (a) in Figure 3.

When $\gamma = 0$, then solution (3.25) becomes

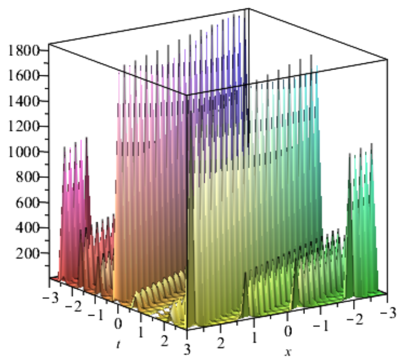
$$u_{42}(\xi) = 4 + \frac{1}{8}\xi - \frac{1}{8}\xi^2, \tag{4.9}$$

where $\xi = x - 2t$. The figure of solution (4.9) is shown as Figure (b) in Figure 3.

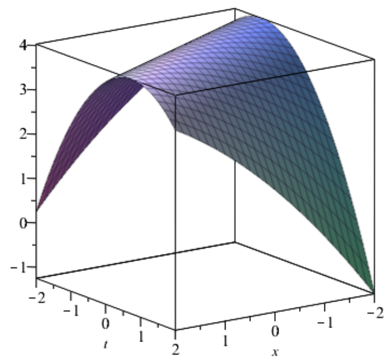
When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.26) becomes

$$\begin{aligned} u_{43}(\xi) =& \frac{14}{15} + \frac{1}{16}\coth(2\xi) - \frac{1}{32}\coth^2(2\xi) - 2\tanh(2\xi) \\ & - 2\coth(2\xi)\mathrm{sech}^2(2\xi) + 2\mathrm{csch}^2(2\xi), \end{aligned} \tag{4.10}$$

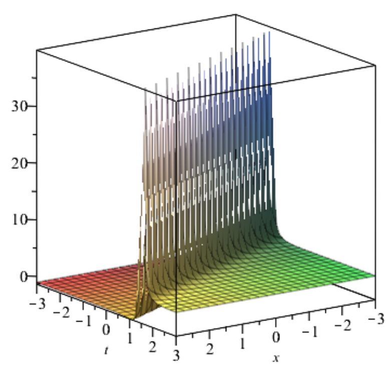
where $\xi = x - 2t$. The figure of solution (4.10) is shown as Figure (c) in Figure 3.



(a) The figure of u_{41}



(b) The figure of u_{42}



(c) The figure of u_{43}

Figure 3. The figure of (4.8) shown on (a); the figure of (4.9) shown on (b); the figure of (4.10) shown on (c)

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.27) becomes

$$\begin{aligned} u_{51}(\xi) = &-\frac{82}{17} + 2\cot(2\xi) - 2\cot^2(2\xi) + 2\tan(2\xi) - \frac{48}{17}\tan^2(2\xi) \\ &- 2\cot(2\xi)\sec^2(2\xi) + 2\csc^2(2\xi), \end{aligned} \tag{4.11}$$

where $\xi = x - \frac{1}{17}t$. The figure of solution (4.11) is shown as Figure (a) in Figure 4.

When $\gamma = 0$, then solution (3.28) becomes

$$u_{52}(\xi) = \frac{12}{\xi^2}, \tag{4.12}$$

where $\xi = x - t$. The figure of solution (4.12) is shown as Figure (b) in Figure 4.

When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.29) becomes

$$\begin{aligned} u_{53}(\xi) = &-\frac{26}{5} + 2\coth(2\xi) + 2\coth^2(2\xi) - 2\tanh(2\xi) + \frac{16}{5}\tanh^2(2\xi) \\ &- 2\coth(2\xi)\text{sech}^2(2\xi) + 2\text{csch}^2(2\xi)^2, \end{aligned} \tag{4.13}$$

where $\xi = x + \frac{1}{15}t$. The figure of solution (4.13) is shown as Figure (c) in Figure 4.

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.30) becomes

$$\begin{aligned} u_{61}(\xi) = &-\frac{14}{15} + 2\cot(2\xi) - 2\cot^2(2\xi) + 2\tan(2\xi) + \frac{4}{5}\tan^2(2\xi) \\ &- 2\cot(2\xi)\sec^2(2\xi) + 2\csc^2(2\xi), \end{aligned} \tag{4.14}$$

where $\xi = x + \frac{1}{15}t$. The figure of solution (4.14) is shown as Figure (a) in Figure 5.

When $\gamma = 0$, then solution (3.31) becomes

$$u_{62}(\xi) = -\frac{12}{\xi^2}, \tag{4.15}$$

where $\xi = x - t$. The figure of solution (4.15) is shown as Figure (b) in Figure 5.

When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.32) becomes

$$\begin{aligned} u_{63}(\xi) = &-\frac{18}{17} +, +2\coth(2\xi) + 2\coth^2(2\xi) - 2\tanh(2\xi) - \frac{48}{17}\tanh^2(2\xi) \\ &- 2\coth(2\xi)\text{sech}^2(2\xi) + 2\text{csch}^2(2\xi), \end{aligned} \tag{4.16}$$

where $\xi = x - \frac{1}{17}t$. The figure of solution (4.16) is shown as Figure (c) in Figure 5.

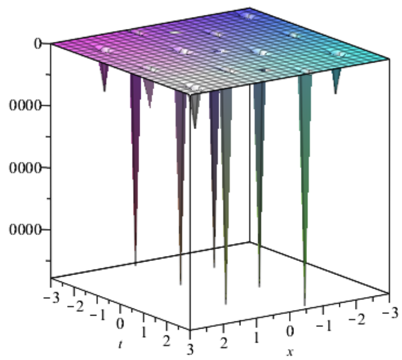
When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.33) becomes

$$\begin{aligned} u_{71}(\xi) = &-\frac{158}{63} + 2\cot(2\xi) - \frac{26}{21}\cot^2(2\xi) + 2\tan(2\xi) + \frac{16}{21}\tan^2(2\xi) \\ &- 2\cot(2\xi)\sec^2(2\xi) + 2\csc^2(2\xi), \end{aligned} \tag{4.17}$$

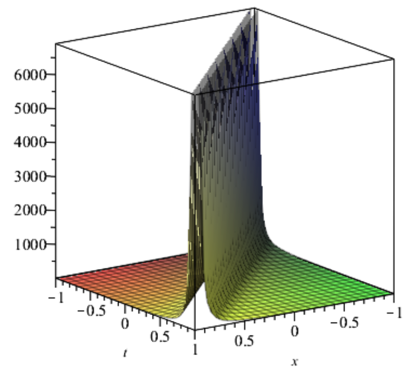
where $\xi = x + \frac{1}{63}t$. The figure of solution (4.17) is shown as Figure (a) in Figure 6.

When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.35) becomes

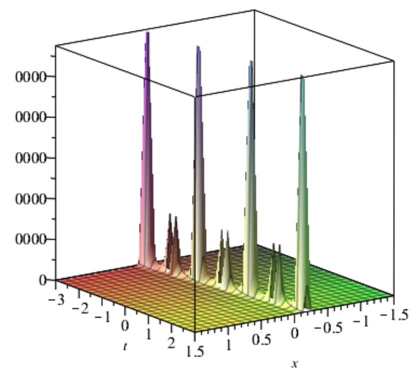
$$\begin{aligned} u_{73}(\xi) = &-\frac{162}{65} + 2\coth(2\xi) + \frac{82}{65}\coth^2(2\xi) - 2\tanh(2\xi) - \frac{48}{65}\tanh^2(2\xi) \\ &- 2\coth(2\xi)\text{sech}^2(2\xi) + 2\text{csch}^2(2\xi), \end{aligned} \tag{4.18}$$



(a) The figure of u_{51}



(b) The figure of u_{52}



(c) The figure of u_{53}

Figure 4. The figure of (4.11) shown on (a); the figure of (4.12) shown on (b); the figure of (4.13) shown on (c)

where $\xi = x - \frac{1}{65}t$. The figure of solution (4.18) is shown as Figure (b) in Figure 6.

When $\gamma > 0$, we adopt $\gamma = 4$. Then, solution (3.36) becomes

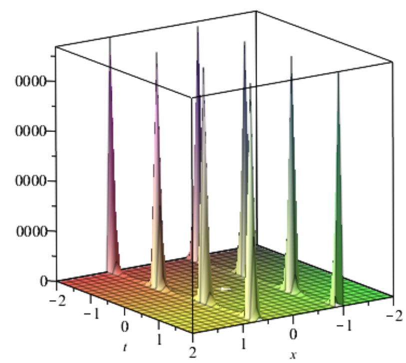
$$\begin{aligned} u_{81}(\xi) = &-\frac{226}{65} + 2\cot(2\xi) - \frac{178}{65}\cot^2(2\xi) + 2\tan(2\xi) - \frac{48}{65}\tan^2(2\xi) \\ &- 2\cot(2\xi)\sec^2(2\xi) + 2\csc^2(2\xi), \end{aligned} \tag{4.19}$$

where $\xi = x - \frac{1}{65}t$. The figure of solution (4.19) is shown as Figure (c) in Figure 6.

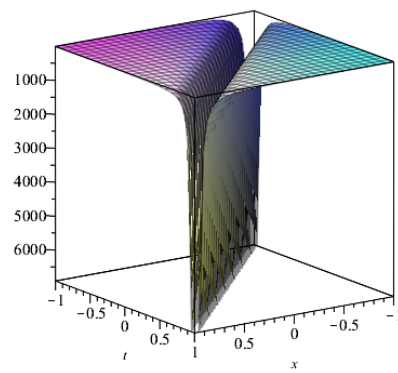
When $\gamma < 0$, we adopt $\gamma = -4$. Then, solution (3.38) becomes

$$\begin{aligned} u_{83}(\xi) = &-\frac{74}{21} + 2\coth(2\xi) + \frac{58}{21}\coth^2(2\xi) - 2\tanh(2\xi) + \frac{16}{21}\tanh^2(2\xi) \\ &- 2\coth(2\xi)\mathrm{sech}^2(2\xi) + 2\mathrm{csch}^2(2\xi), \end{aligned} \tag{4.20}$$

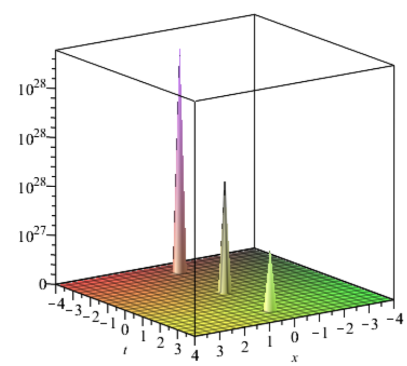
where $\xi = x + \frac{1}{63}t$. The figure of solution (4.20) is shown as Figure (d) in Figure 6.



(a) The figure of u_{61}



(b) The figure of u_{62}



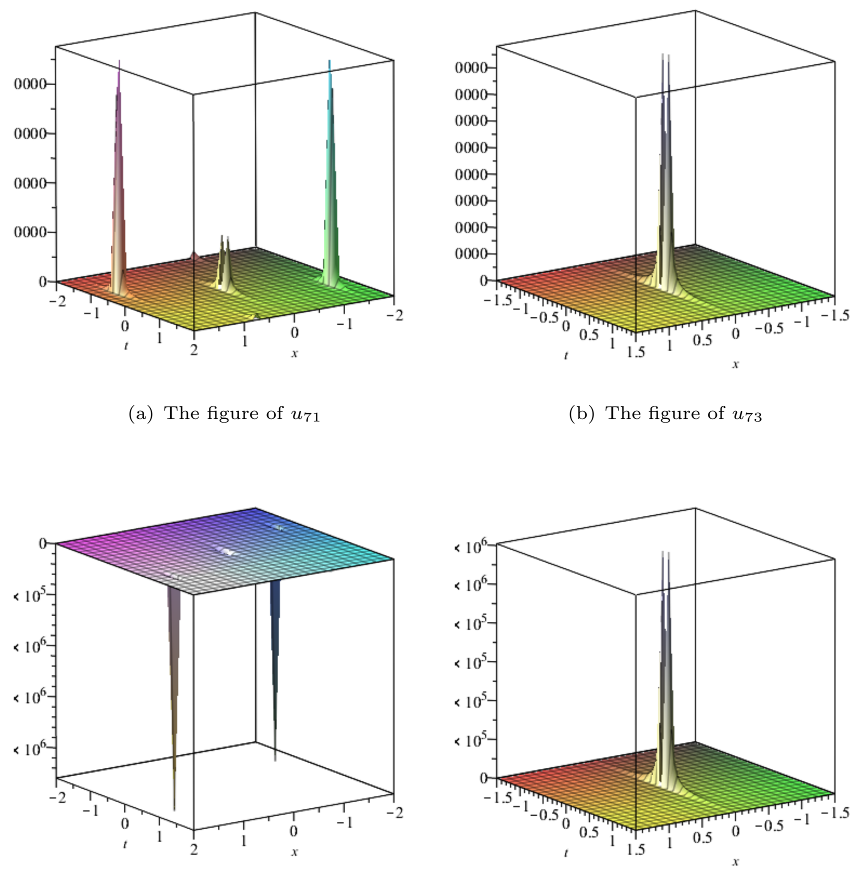
(c) The figure of u_{63}

Figure 5. The figure of (4.14) shown on (a); the figure of (4.15) shown on (b); the figure of (4.16) shown on (c)

5. Conclusion

In summary, this paper introduces the BBM equation and acquires the traveling wave solutions of the BBM equation by utilizing the modified hyperbolic function expanding method. Due to the different selection of the sign of the parameter γ, the traveling wave solutions have different structures. When $\gamma > 0$, and the solutions are trigonometric function solutions as shown in equations (3.15), (3.18), (3.21), (3.24), (3.27), (3.30), (3.33) and (3.36). When $\gamma = 0$, the solutions are rational function solutions as shown in equations (3.22), (3.25), (3.28) and (3.31). When $\gamma < 0$, the solutions are hyperbolic trigonometric function solutions as shown in equations (3.17), (3.20), (3.23), (3.26), (3.29), (3.32), (3.35) and (3.38).

Compared with the previous literature, the traveling wave solutions we have obtained are different from the previous ones. When $\gamma > 0$ and $\gamma < 0$, all of the solutions are trigonometric function solutions and hyperbolic trigonometric function solutions, which are important in physical sciences and mathematical sciences. The solutions obtained in this way reveal that the concerned model governs pulse dynamics in NLPDE effectively. We have supplemented and improved the exact solutions of this kind of equation, and provided a new idea for obtaining the exact solutions of this kind of equation in the future, laying a new foundation for our in-depth research. At the same time, it manifests that this method is very convenient and a more powerful tool to obtain the analytical solutions of NLPDE. In addition, studying the dynamical properties and all solutions of the BBM equation is also one of our main tasks in the future. The further study in this regard will procure other novel and marvelous results for NLPDE.



(a) The figure of u_{71} (b) The figure of u_{73}

(c) The figure of u_{81} (d) The figure of u_{83}

Figure 6. The figure of (4.17) shown on (a); the figure of (4.18) shown on (b); the figure of (4.19) shown on (c); the figure of (4.20) shown on (d)

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