Journal of Nonlinear Modeling and Analysis
Volume 4, Number 4, December 2022, 650–657
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.650

Existence and Uniqueness Theorems for a Three-step Newton-type Method under L-Average Conditions

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Abstract In this paper, we study the local convergence of a three-step Newton-type method for solving nonlinear equations in Banach spaces under weaker hypothesis. More precisely, we derive the existence and uniqueness theorems, when the first-order derivative of nonlinear operator satisfies the L-average conditions instead of the usual Lipschitz condition, which have been discussed in the earlier study.

Keywords Banach space, Nonlinear equation, Lipschitz condition, L-average, Convergence ball.

MSC(2010) 65H10.

1. Introduction

Let $T : \mathfrak{D} \subseteq M \to N$ be a nonlinear operator from a Banach space M to another Banach space N, where $\mathfrak{D}$ is a non-empty open convex subset, and T is Fréchet differentiable nonlinear operator. Nonlinear equations arise in many fields of science and engineering, and are defined by

$$T(\alpha) = 0. \tag{1.1}$$

The well-known iterative method for finding the solution of above type equation is Newton's method, which is given by

$$\alpha_{n+1} = \alpha_n - [T'(\alpha_n)]^{-1} T(\alpha_n),\ n \geq 0. \tag{1.2}$$

Newton's method [8] is a well established iterative method, which converges quadratically. It was first discussed by Kantorovich [6], and then investigated by Rall [10]. Some higher-order methods that do not require the computation of second-order derivatives have been developed in [4, 5, 7, 9] and other literature. Due to high operational cost, the methods of higher R-order convergence are not normally used despite fast speed of convergence. However, the methods of higher R-order are useful in the problems of stiff system [6], where fast convergence is required.

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Here, we discuss the local convergence of a three-step Newton-type method under the L-average conditions, which is expressed as

$$\begin{aligned}\beta_n &= \alpha_n - [T'(\alpha_n)]^{-1}T(\alpha_n),\\ \gamma_n &= \alpha_n - [T'(\alpha_n)]^{-1}[T(\alpha_n)+T(\beta_n)],\\ \alpha_{n+1} &= \alpha_n - [T'(\alpha_n)]^{-1}[T(\alpha_n)+T(\beta_n)+T(\gamma_n)],\ n\geq 0. \end{aligned}\tag{1.3}$$

Method (1.3) is characterized by the simplest fourth-order iterative method, which is not involved in the second derivative. The local convergence of this method has been studied by Argyros et al., [3] under Lipschitz and center Lipschitz conditions, which are given by

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\beta))|| \leq L||\alpha-\beta||, \forall\ \alpha,\beta\in B(\alpha^*,r) \tag{1.4}$$

and

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\alpha^*))|| \leq L_0||\alpha-\alpha^*||, \forall\ \alpha\in B(\alpha^*,r). \tag{1.5}$$

For some kinds of domain with center such as $B(\alpha^*,r)$, sometimes it is not necessary for the inequality to hold for any α,β in the domain, and it is only required to hold for any α and for points lying on the connecting line $\alpha^\tau=\alpha^*+\tau(\alpha-\alpha^*)$ between α and α^*, where $0\leq\tau\leq 1$. Recently, in [11], Wang has introduced the Lipschitz condition and center Lipschitz condition with L-average, which are given by for $[T'(\alpha^*)]^{-1}T'(\alpha)$:

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\alpha^\tau))|| \leq \int_{\tau\kappa(\alpha)}^{\kappa(\alpha)} L(u)du, \forall\ \alpha,\alpha^\tau\in B(\alpha^*,r), 0\leq\tau\leq 1 \tag{1.6}$$

and

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\alpha^*))|| \leq \int_{0}^{\kappa(\alpha)} L_0(u)du, \forall\ \alpha\in B(\alpha^*,r), \tag{1.7}$$

where L and L_0 are positive integrable function instead of constant. If L and L_0 are constants, then R. H. S. of the above two equations becomes $L||\alpha-\alpha^\tau||$ and $L_0||\alpha-\alpha^*||$.

Encouraged by the development discussed above, in this paper, we first mention the definitions of Lipschitz condition and center Lipschitz condition with L-average for algorithm (1.3), and some theorems are then derived. The first L-average conditions have been used to study the local convergence without additional hypotheses, along with an error estimate. In the second theorem, the domain of uniqueness of solution has been derived under center Lipschitz condition. Also, few corollaries are stated.

The rest parts of this paper are organized as follows. Section 2 includes the definitions related to L-average conditions. The local convergence and its domain of uniqueness are respectively mentioned in Section 3 and Section 4.

2. L-average conditions

Here, we denote by $B(\alpha^*, r) = \{\alpha : ||\alpha - \alpha^*|| < r\}$, a ball with radius r and center α^*. The condition imposed on the function T

$$||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\beta^\tau))|| \leq L(1-\tau)(||\alpha - \alpha^*|| + ||\beta - \alpha^*||), \forall\ \alpha,\ \beta \in B(\alpha^*, r), \tag{2.1}$$

where $\beta^\tau = \alpha^* + \tau(\beta - \alpha^*), 0 \leq \tau \leq 1$, is usually called radius Lipschitz condition in the ball $B(\alpha^*, r)$ with constant L. Sometimes, if it is only required to satisfy

$$||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\alpha^*))|| \leq 2L_0||\alpha - \alpha^*||, \forall\ \alpha \in B(\alpha^*, r), \tag{2.2}$$

and we call it the center Lipschitz condition in the ball $B(\alpha^*, r)$ with the constant L_0, where $L_0 \leq L$. Replacing L by L_0 in the case $L_0 < L$ leads to wider choice of initial guesses (larger radius of convergence than in traditional studies) and fewer iterates to achieve an error tolerance [1,2]. Furthermore, L and L_0 in the Lipschitz conditions do not necessarily have to be constant, but can be a positive integrable function. In this case, conditions (2.1)-(2.2) are respectively replaced by

$$||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\beta^\tau))|| \leq \int_{\tau(\kappa(\alpha)+\kappa(\beta))}^{\kappa(\alpha)+\kappa(\beta)} L(u)du, \forall\ \alpha,\ \beta \in B(\alpha^*, r), 0 \leq \tau \leq 1, \tag{2.3}$$

and

$$||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\alpha^*))|| \leq \int_0^{2\kappa(\alpha)} L_0(u)du, \forall\ \alpha \in B(\alpha^*, r), \tag{2.4}$$

where $\kappa(\alpha) = ||\alpha - \alpha^*||$, and we have $L_0(u) \leq L(u)$. At the same time, the corresponding Lipschitz conditions are referred to having the L-average or generalized Lipschitz conditions. Next, we start with the following lemmas, which will be used in the main theorems later.

Lemma 2.1. *Suppose that T has a continuous derivative in $B(\alpha^*, r)$, and $[T'(\alpha^*)]^{-1}$ exists.*
(i) If $[T'(\alpha^*)]^{-1}T'$ satisfies the radius Lipschitz condition with the L-average:

$$||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\beta^\tau))|| \leq \int_{\tau(\kappa(\alpha)+\kappa(\beta))}^{\kappa(\alpha)+\kappa(\beta)} L(u)du, \forall\ \alpha,\ \beta \in V(\alpha^*, r), 0 \leq \tau \leq 1, \tag{2.5}$$

where $\beta^\tau = \alpha^* + \tau(\beta - \alpha^*)$, $\kappa(\alpha) = ||\alpha - \alpha^*||$ and L is non-decreasing, then we have

$$\int_0^1 ||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\beta^\tau))||\kappa(\beta)d\tau \leq \int_0^{\kappa(\alpha)+\kappa(\beta)} L(u)\frac{u}{\kappa(\alpha)+\kappa(\beta)}\kappa(\beta)du. \tag{2.6}$$

(ii) If $[T'(\alpha^*)]^{-1}T'$ satisfies the center Lipschitz condition with the L_0-average:

$$||[T'(\alpha^*)]^{-1}(T'(\alpha^\tau) - T'(\alpha^*))|| \leq \int_0^{2\tau\kappa(\alpha)} L_0(u)du, \forall\ \alpha \in V(\alpha^*, r), 0 \leq \tau \leq 1, \tag{2.7}$$

where $\kappa(\alpha) = ||\alpha - \alpha^*||$ and L_0 is non-decreasing, then we have

$$\int_0^1 ||[T'(\alpha^*)]^{-1}(T'(\alpha^\tau) - T'(\alpha^*))||\kappa(\alpha)d\tau \leq \int_0^{2\kappa(\alpha)} L_0(u)\left(\kappa(\alpha) - \frac{u}{2}\right)du. \quad (2.8)$$

Proof. The Lipschitz conditions (2.5) and (2.7) respectively imply

$$\begin{aligned}\int_0^1 ||[T'(\alpha^*)]^{-1}(T'(\alpha) - T'(\beta^\tau))||\kappa(\beta)d\tau &\leq \int_0^1 \int_{\tau(\kappa(\alpha)+\kappa(\beta))}^{\kappa(\alpha)+\kappa(\beta)} L(u)du\kappa(\beta)d\tau \\ &= \int_0^{\kappa(\alpha)+\kappa(\beta)} L(u)\frac{u}{\kappa(\alpha)+\kappa(\beta)}\kappa(\beta)du, \\ \int_0^1 ||[T'(\alpha^*)]^{-1}(T'(\alpha^\tau) - T'(\alpha^*))||\kappa(\alpha)d\tau &\leq \int_0^1 \int_0^{2\tau\kappa(\alpha)} L_0(u)du\kappa(\alpha)d\tau \\ &= \int_0^{2\kappa(\alpha)} L_0(u)\left(\kappa(\alpha) - \frac{u}{2}\right)du,\end{aligned}$$

where $\alpha^\tau = \alpha^* + \tau(\alpha - \alpha^*)$ and $\beta^\tau = \alpha^* + \tau(\beta - \alpha^*)$. □

Lemma 2.2 ([11]). *Suppose that L is positive integrable. Assume that the function L is non-decreasing, and then $\frac{1}{t^2}\int_0^t L(u)udu$ is also non-decreasing.*

Lemma 2.3. *Suppose that L is positive integrable. Assume that the function L is non-decreasing, and then $\frac{1}{t}\int_0^t L(u)(t-u)du$ is also non-decreasing.*

Proof. Similar to the previous lemma, the proof may be proceeded. □

3. Local convergence of method (1.3)

Here, we prove the existence theorem for scheme (1.3) under radius Lipschitz condition first.

Theorem 3.1. *Let us suppose that $T(\alpha^*) = 0$, T has a continuous derivative in $B(\alpha^*, r)$, $[T'(\alpha^*)]^{-1}$ exists, $[T'(\alpha^*)]^{-1}T'$ fulfills hypothesis* (2.3) *and hypothesis* (2.4)*, and L is non-decreasing. Let r satisfy the expression*

$$\int_0^{2r} L_0(u)du \leq 1, \int_0^{2r} \{L(u)u + 2rL_0(u)\}du \leq 2r. \quad (3.1)$$

Then, the three-step method (1.3) *is convergent for all $\alpha_0 \in B(\alpha^*, r)$ and*

$$||\alpha_{n+1} - \alpha^*|| \leq \frac{q_1q_2q_3}{\kappa(\alpha_0)\kappa(\beta_0)\kappa(\gamma_0)}\kappa(\alpha_n)^4, \quad (3.2)$$

where

$$q_1 = \frac{\int_0^{2\kappa(\alpha_0)} L(u)udu}{2\kappa(\alpha_0)(1 - \int_0^{2\kappa(\alpha_0)} L_0(u)du)},\ q_2 = \frac{\int_0^{\kappa(\alpha_0)+\kappa(\beta_0)} L(u)udu}{(\kappa(\alpha_0)+\kappa(\beta_0))(1 - \int_0^{2\kappa(\alpha_0)} L_0(u)du)},$$
$$q_3 = \frac{\int_0^{\kappa(\alpha_0)+\kappa(\gamma_0)} L(u)udu}{(\kappa(\alpha_0)+\kappa(\gamma_0))(1 - \int_0^{2\kappa(\alpha_0)} L_0(u)du)} \quad (3.3)$$

are less than unity. Furthermore,

$$||\alpha_n - \alpha^*|| \leq D^{4^n-1}||\alpha_0 - \alpha^*||,\ n = 1, 2, \cdots, D = q_1q_2\frac{\kappa(\alpha_0)^2}{\kappa(\beta_0)\kappa(\gamma_0)}. \quad (3.4)$$

Proof. Let us choose $\alpha_0 \in B(\alpha^*, r)$ and let r fulfill inequality (3.1). Clearly, q_1, q_2 and q_3 mentioned above in expression (3.3) are less than unity, since

$$q_1 = \frac{\int_0^{2\kappa(\alpha_0)} L(u)udu}{2\kappa(\alpha_0)^2(1-\int_0^{2\kappa(\alpha_0)} L_0(u)du)}\kappa(\alpha_0) \leq \frac{\int_0^{2r} L(u)udu}{2r^2(1-\int_0^{2r} L_0(u)du)}\kappa(\alpha_0)$$
$$\leq \frac{||\alpha_0-\alpha^*||}{r} < 1,$$
$$q_2 = \frac{\int_0^{\kappa(\alpha_0)+\kappa(\beta_0)} L(u)udu}{(\kappa(\alpha_0)+\kappa(\beta_0))^2(1-\int_0^{2\kappa(\alpha_0)} L_0(u)du)}(\kappa(\alpha_0+\kappa(\beta_0))$$
$$\leq \frac{\int_0^{2r} L(u)udu}{2r^2(1-\int_0^{2r} L_0(u)du)}(\kappa(\alpha_0)+\kappa(\beta_0)) \leq \frac{||\alpha_0-\alpha^*||+||\beta_0-\alpha^*||}{2r} < 1,$$

and the similar explanation for q_3. Now, if $\alpha_n \in B(\alpha^*, r)$, then by virtue of hypothesis (2.4), relation (3.1) and Banach Lemma, we can write

$$||[T'(\alpha_n)]^{-1}T'(\alpha^*)|| \leq \frac{1}{1-\int_0^{2\kappa(\alpha_n)} L_0(u)du}. \tag{3.5}$$

By the Taylor's series expansion of $T(\alpha_n)$, for the first sub-step of scheme (1.3), we can write

$$||\beta_n-\alpha^*|| \leq ||[T'(\alpha_n)]^{-1}T'(\alpha^*)||.||\int_0^1 [T'(\alpha^*)]^{-1}[T'(\alpha_n)-T'(\alpha_n^\tau)]d\tau||.||(\alpha_n-\alpha^*)||$$
$$\leq \frac{1}{1-\int_0^{2\kappa(\alpha_n)} L_0(u)du}\int_0^1\int_{2\tau\kappa(\alpha_n)}^{2\kappa(\alpha_n)} L(u)du\kappa(\alpha_n)d\tau. \tag{3.6}$$

By similar analogy for the the second and last sub-steps of scheme (1.3), we can write

$$||\gamma_n-\alpha^*|| \leq ||[T'(\alpha_n)]^{-1}T'(\alpha^*)||.||\int_0^1 [T'(\alpha^*)]^{-1}[T'(\alpha_n)-T'(\beta_n^\tau)]d\tau||.||(\beta_n-\alpha^*)||$$
$$\leq \frac{1}{1-\int_0^{2\kappa(\alpha_n)} L_0(u)du}\int_0^1\int_{\tau(\kappa(\alpha_n)+\kappa(\beta_n))}^{\kappa(\alpha_n)+\kappa(\beta_n)} L(u)du\kappa(\beta_n)d\tau \tag{3.7}$$

and

$$||\alpha_{n+1}-\alpha^*|| \leq ||[T'(\alpha_n)]^{-1}T'(\alpha^*)||.||\int_0^1 [T'(\alpha^*)]^{-1}[T'(\alpha_n)-T'(\gamma_n^\tau)]d\tau||.||(\gamma_n-\alpha^*)||$$
$$\leq \frac{1}{1-\int_0^{2\kappa(\alpha_n)} L_0(u)du}\int_0^1\int_{\tau(\kappa(\alpha_n)+\kappa(\gamma_n))}^{\kappa(\alpha_n)+\kappa(\gamma_n)} L(u)du\kappa(\gamma_n)d\tau. \tag{3.8}$$

Since $\kappa(\alpha_n)$, $\kappa(\beta_n)$ and $\kappa(\gamma_n)$ are monotonically decreasing, for all $n = 0, 1, ...$, we have

$$||\beta_n-\alpha^*|| \leq \frac{q_1}{\kappa(\alpha_0)}\kappa(\alpha_n)^2,$$

$$||\gamma_n-\alpha^*|| \leq \frac{q_1q_2}{\kappa(\alpha_0)\kappa(\beta_0)}\kappa(\alpha_n)^3$$

and

$$||\alpha_{n+1}-\alpha^*|| \leq \frac{q_1q_2q_3}{\kappa(\alpha_0)\kappa(\beta_0)\kappa(\gamma_0)}\kappa(\alpha_n)^4,$$

where q_1, q_2 and q_3 are defined in equation (3.3). Also, it can be seen that inequality (3.4) may be easily derived from expression (3.2). □

4. Uniqueness of the solution

In this section, we discuss about the uniqueness of the solution for method (1.3).

Theorem 4.1. *Let us suppose that $T(\alpha^*) = 0$, T has a continuous derivative in $B(\alpha^*, r)$, $[T'(\alpha^*)]^{-1}$ exists, and $[T'(\alpha^*)]^{-1}T'$ fulfills hypothesis* (2.4)*. Let r satisfy the expression*

$$\int_0^{2r} L_0(u)(2r-u)du \leq 2r. \tag{4.1}$$

Then, the equation $T(\alpha) = 0$ has a unique solution α^* in $B(\alpha^*, r)$.

Proof. Let us suppose if possible $\beta^* \in B(\alpha^*, r)$ is another solution such that $\beta^* \neq \alpha^*$. By using Taylor's series expansion, we can write

$$\begin{aligned} ||\beta^* - \alpha^*|| &\leq ||[T'(\alpha^*)]^{-1}T'(\alpha^*)||.||\int_0^1 [T'(\alpha^*)]^{-1}[T'(\beta^{*\tau}) - T'(\alpha^*)]d\tau||.||(\beta^* - \alpha^*)|| \\ &\leq \int_0^1 \int_0^{2\tau\kappa(\beta^*)} L_0(u)du\kappa(\beta^*)d\tau. \end{aligned} \tag{4.2}$$

Using Lemma 2.1 in the above expression, it can be written as

$$\begin{aligned} ||\beta^* - \alpha^*|| &\leq \frac{1}{2\kappa(\beta^*)} \int_0^{2\kappa(\beta^*)} L_0(u)[2\kappa(\beta^*) - u]du(\beta^* - \alpha^*) \\ &\leq \frac{\int_0^{2r} L_0(u)(2r-u)du}{2r}\kappa(\beta^*) \leq ||\beta^* - \alpha^*||, \end{aligned} \tag{4.3}$$

which is contradictory. Thus, the result is obtained. □

Now, if we assume that L and L_0 are constants, then we obtain the following corollaries from the above mentioned theorems.

Corollary 4.1. *Let us suppose that α^* fulfills $T(\alpha^*) = 0$, T has a continuous derivative in $B(\alpha^*, r)$, $[T'(\alpha^*)]^{-1}$ exists, and $[T'(\alpha^*)]^{-1}T'$ fulfills hypotheses* (2.1) *and* (2.2)*. Let r satisfy the expression*

$$r(2L_0 + L) = 1. \tag{4.4}$$

Then, the three-step method (1.3) *is convergent for all $\alpha_0 \in B(\alpha^*, r)$, and inequalities* (3.2) *and* (3.4) *hold, where*

$$q_1 = \frac{L\kappa(\alpha_0)}{1 - 2L_0\kappa(\alpha_0)},\ q_2 = \frac{L(\kappa(\alpha_0) + \kappa(\beta_0))}{2(1 - 2L_0\kappa(\alpha_0))},\ q_3 = \frac{L(\kappa(\alpha_0) + \kappa(\gamma_0))}{2(1 - 2L_0\kappa(\alpha_0))} \tag{4.5}$$

are less than unity.

Corollary 4.2. *Let us suppose that α^* fulfills $T(\alpha^*) = 0$, T has a continuous derivative in $B(\alpha^*, r)$, $[T'(\alpha^*)]^{-1}$ exists and $[T'(\alpha^*)]^{-1}T'$ fulfills hypothesis* (2.2)*. Let r satisfy the equation*

$$rL_0 = 1. \tag{4.6}$$

Then, the equation $T(\alpha) = 0$ has a unique solution α^* in $B(\alpha^*, r)$. Moreover, the radius r of ball depends only on L_0.

5. Numerical example

Example 5.1. Choose $X = Y = C[0,1], \Omega = \overline{V}(0,1)$ and $x^* = 0$. Then, define t on Ω as

$$t(h)(x) = h(x) - \int_0^1 x\tau h(\tau)^3 d\tau.$$

Therefore,

$$t'(h(p))(x) = p(x) - 3\int_0^1 x\tau h(\tau)^2 p(\tau) d\tau \ for\ all\ p \in \Omega.$$

Then, we get

$$L_0(u) = 1.5u < L(u) = 3u.$$

Using (3.1) and $t'(x^*) = I$, we have the following cases.
The old case $L_0(u) = L(u) = 3u$ gives

$$r_0 = 0.245253.$$

Cases $L_0(u) = \frac{3}{2}u\ and\ L(u) = 3u$ give

$$r_1 = 0.324947.$$

Notice that $r_0 < r_1$.

6. Conclusions

In this study, the local convergence of a three-step Newton-type method of order four is applied under generalized Lipschitz conditions, in which instead of Lipschitz constants, some non-decreasing integrable functions are being used. It turns out that although the conditions are more general, they are also more flexible, leading to some advantages without any additional computational efforts.

Acknowledgements

The author would like to extend his sincere gratitude to the reviewers and editors for their valuable suggestions that have helped improve this paper.

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