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Positive Solutions of Second-order Difference Equation with Variable Coefficient on the Infinite Interval*

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Abstract In this paper, based on the one-signed Green's function and the compact results on the infinite interval, we obtain the existence and multiplicity of positive solutions for the boundary value problems

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) + f(n, x(n)) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim\limits_{n\to\infty} x(n) = 0 \end{cases}$$

by the fixed point theorem in cones. The main results extend some results in the previous literature.

Keywords Positive solution, Green's function, Compact, Infinite interval.

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1. Introduction

The continuous boundary value problem on the half-line occur in the mathematical modeling of various applied problems, for example, discussion on electrostatic probe measurements in solid-propellant rocket exhausts [11], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, heat transfer in the radial flow between parallel circular disks [13] and investigation of temperature distribution in the problem of phase change of solids with temperature-dependence thermal conductivity [13]. Hence, the existence of positive solutions to the infinite interval boundary value problem of second-order ordinary differential equations have been studied by many authors (see [3, 5–7, 9, 12, 15, 16] and their references). However, the existence and multiplicity of the positive solutions to second-order difference equations on the half-line have only few results such as [1, 2, 8, 14].

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, $N(a,\ b) = \{a, a+1, \cdots, b\}$, for $a < b$.

In 2001, Agarwal et al., [2] studied the positive solutions of the following bound-

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ary value problem on the infinite interval

$$\begin{cases} \Delta^2 x(n-1) + f(n, x(n)) = 0, \quad n \in \mathbb{N} \\ x(0) = 0, \qquad \lim_{n\to\infty} x(n) = \gamma \in \mathbb{R} \end{cases} \tag{1.1}$$

by employing upper and lower solution methods. In 2006, Tian and Ge [14] obtained the existence of multiple positive solutions for the problem

$$\begin{cases} \Delta^2 x(n-1) - p\Delta x(n-1) - qx(n-1) + f(n, x(n)) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta\Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases} \tag{1.2}$$

where $p, \ \alpha, \ \beta \geq 0, \ \alpha^2 + \beta^2 > 0, \ q > 0, \ 1 + p > q$, and $f : \mathbb{N} \times [0, \infty) \to [0, \infty)$ are continuous. The main proofs are based on the fixed point theorem in Fréchet space.

Definitely, the natural question is whether or not the positive solution of problem (1.2) on the infinite interval exists in the Banach space. The key points are the compact results on the infinite interval and the one-signed Green's function of (1.2) and its bounded properties. This is an interesting problem which is different from the properties of Green's function on finite interval.

Motivated by what has been mentioned above, we discuss the one-signed property of Green's function and its bounded properties, and obtain the existence and multiplicity of positive solutions of the following problem

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) + f(n, x(n)) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta\Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases} \tag{1.3}$$

where $p : \mathbb{N} \to [0, \ \infty)$, $q : \mathbb{N} \to (0, \ \infty)$ are bounded functions, $\alpha, \ \beta \geq 0, \ \alpha^2 + \beta^2 > 0$ and $f : \mathbb{N} \times [0, \infty) \to [0, \infty)$ are continuous.

Notice that (1.3) generalizes (1.2). It is worth pointing out that Green's function of the associated linear problem

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta\Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0 \end{cases} \tag{1.4}$$

cannot be explicitly expressed by elementary functions. These make our approach more difficult. Fortunately, we find Perron's theorem [8] and the compact theorem in the Banach space $l_\diamond = \{x(\cdot) \in l^\infty(\mathbb{N}_0) \,|\, \lim_{n\to\infty} x(n) = x(\infty)\}$ [10] to overcome these difficulties.

The rest of this paper is arranged as follows. In Section 2, we construct the Green's function of (1.4), and prove its one-signed and bounded properties. In Section 3, we state the compact theorem on the infinite interval and the transfer problem (1.3) to the compact summing operator in Banach space $l_\diamond$. In Section 4, we give the existence and multiplicity results of positive solutions for problem (1.3).

Throughout this paper, we denote the summation of $x(n)$ from $n = a$ to $n = b$ by $\sum_{n=a}^{b} x(n)$ with the understanding that $\sum_{n=a}^{b} x(n) = 0$ for all $a > b$, and the product of $x(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{b} x(n)$ with the understanding that $\prod_{n=a}^{b} x(n) = 1$ for all $a > b$.

2. The Green's function and its properties

To obtain the Green's function of (1.4), we need some restrictions on the functions $p(\cdot),\ q(\cdot)$ as follows.

(H1) $p:\mathbb{N}\to[0,\ \infty),\ q:\mathbb{N}\to(0,\ \infty)$ are bounded functions. Denote

$$p^*=\sup_{n\in\mathbb{N}}p(n),\quad p_*=\inf_{n\in\mathbb{N}}p(n),\quad q^*=\sup_{n\in\mathbb{N}}q(n),\quad q_*=\inf_{n\in\mathbb{N}}q(n).$$

(H2) $1+p(n)-q(n)>0,\ n\in\mathbb{N}$, $1+p_*-q_*>0$ and $1+p^*-q^*>0$.

(H3) $\lim\limits_{n\to\infty}p(n)=p_0\geq 0,\ \lim\limits_{n\to\infty}q(n)=q_0>0$ and $\prod\limits_{i=1}^{\infty}(1+p(i)-q(i))<\infty$.

Lemma 2.1. *Assume that (H1)-(H2) hold. Then, the initial value problem*

$$\begin{cases}\Delta^2x(n-1)-p(n)\Delta x(n-1)-q(n)x(n-1)=0, & n\in\mathbb{N},\\ \alpha x(0)-\beta\Delta x(0)=0, \qquad x(1)=1\end{cases}\tag{2.1}$$

has a unique solution $u(n)$ defined on $\mathbb{N}_0$. Moreover, $\Delta u(n)>0$ on $\mathbb{N}$, and u is increasing on $\mathbb{N}$.

Proof. By the existence and uniqueness of the solution to initial value problem [8], it follows that (2.1) has the unique solution $u(n)$ defined on $\mathbb{N}_0$.

Now, we prove the assertion by induction. First, from $\alpha u(0)-\beta\Delta u(0)=0$, we have $u(0)=\frac{\beta}{\alpha+\beta}u(1)=\frac{\beta}{\alpha+\beta}\geq 0$, $\Delta u(0)=u(1)-u(0)=\frac{\alpha}{\alpha+\beta}\geq 0$. Since $\alpha^2+\beta^2>0$, it follows that

$$\Delta u(1)=[1+p(1)]\Delta u(0)+q(1)u(0)>0,\ \Delta u(1)\geq\Delta u(0).$$

Secondly, we assume that if $k\leq n$, then

$$\Delta u(k)=[1+p(k)]\Delta u(k-1)+q(k)u(k-1)>0 \text{ and } \Delta u(k)\geq\Delta u(k-1).$$

Thus, we conclude

$$\begin{aligned}\Delta u(n+1)&=[1+p(n+1)]\Delta u(n)+q(n+1)u(n)\\ &\geq[1+p(n+1)]\Delta u(n-1)+q(n+1)u(n-1)>0\end{aligned}$$

and $\Delta u(n+1)\geq\Delta u(n)$.

Hence, $\Delta u(n)>0,\ n\in\mathbb{N}$. Together with (2.1), this yields $\Delta^2u(n-1)>0,\ n\in\mathbb{N}$. Therefore, $u(n)$ and $\Delta u(n)$ are increasing on $\mathbb{N}$. □

Lemma 2.2. *Suppose that (H1)-(H2) hold. Then, the problem*

$$\begin{cases}\Delta^2x(n-1)-p(n)\Delta x(n-1)-q(n)x(n-1)=0, & n\in\mathbb{N},\\ \alpha x(0)-\beta\Delta x(0)=1, \qquad \lim\limits_{n\to\infty}x(n)=0\end{cases}\tag{2.2}$$

has the unique solution $v(n)$ defined on $\mathbb{N}_0$. Moreover, $v(n)>0,\ \Delta v(n)<0$ on $\mathbb{N}$.

Proof. To this end, we divide the proof into four steps.

Step 1. We show that (2.2) has a solution v with $v(n) > 0$ in $\mathbb{N}_0$. Let us consider the following problem

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) = 0, \quad n \in N(1,\ m) \\ \alpha x(0) - \beta \Delta x(0) = 1, \qquad x(m+1) = 0. \end{cases} \tag{2.3}$$

We claim that for each $m \in \mathbb{N}$, (2.3) has a positive solution $v := v_m$ with

$$v(n) > 0,\ \Delta v(n) < 0,\ \ \forall\ n \in N(1,\ m). \tag{2.4}$$

In fact, suppose on the contrary that there exists $n_1 \in N(1,\ m)$ such that

$$v(n_1) = 0,\ v(n) > 0 \text{ for } n \in N(1,\ n_1 - 1).$$

Then, $\Delta v(n_1 - 1) = v(n_1) - v(n_1 - 1) < 0$. Since the other case $\Delta v(n_1 - 1) = 0$ would imply that $v(n) = 0$ for $n \in N(1,\ m)$, which is a contradiction. Noticing that $\Delta v(n_1 - 1) < 0$ and $v(n_1) = 0$, and together with (H2), it implies

$$\begin{aligned} \Delta v(n_1) &= [1 + p(n_1)]\Delta v(n_1 - 1) + q(n_1)v(n_1 - 1) \\ &= [1 + p(n_1)]v(n_1) - [1 + p(n_1) - q(n_1)]v(n_1 - 1) \\ &= -[1 + p(n_1) - q(n_1)]v(n_1 - 1) < 0. \end{aligned}$$

That is, $v(n_1 + 1) < v(n_1) = 0$. Moreover, we have $\Delta v(n_1 + 1) = [1 + p(n_1 + 1)]\Delta v(n_1) + q(n_1 + 1)v(n_1) < 0$ and imply $v(n_1 + 2) < v(n_1 + 1) < 0$. The rest may be deduced by analogy, it follows that

$$\Delta v(m) < 0 \text{ and } v(m+1) < v(m) < 0.$$

This contradicts with the boundary condition $v(m+1) = 0$. Thus, we get

$$v(n) > 0 \text{ on } N(0,\ m) \ \text{ and } \Delta v(m) = v(m+1) - v(m) < 0.$$

On the other hand, since

$$\Delta v(m) = [1 + p(m)]\Delta v(m-1) + q(m)v(m-1) < 0,$$

we have $\Delta v(m-1) = -\frac{q(m)}{1+p(m)}v(m-1) < 0$. The rest may be deduced by analogy. Hence, we omit it. Therefore, $\Delta v(n) < 0,\ n \in N(1,\ m)$.

Step 2. For each $m \geq 1$, we show that $v_m(n) < v_{m+1}(n),\ n \in N(1,\ m)$. Let $w(n) := v_{m+1}(n) - v_m(n),\ n \in N(0,\ m+1)$. Then,

$$\begin{cases} \Delta^2 w(n-1) - p(n)\Delta w(n-1) - q(n)w(n-1) = 0, \quad n \in N(1,\ m) \\ \alpha w(0) - \beta \Delta w(0) = 1, \qquad w(m+1) = d, \end{cases} \tag{2.5}$$

where $d := v_{m+1}(m+1) > 0$. We claim

$$w(n) > 0,\ \Delta w(n) > 0, \quad \forall\ n \in N(1,\ m). \tag{2.6}$$

In fact, suppose on the contrary that there exists $n_2 \in N(1,\ m)$ such that

$$w(n_2) = 0 \text{ and } w(n) > 0 \text{ for } n \in N(n_2 + 1,\ m).$$

Applying the same method used in Step 1, we may deduce (2.6).

Step 3. Define the function $\bar{v}_m : [0,\ \infty) \to [0,\ \infty)$ by

$$\bar{v}_m(n) = \begin{cases} v_m(n), & n \in N(0,\ m+1), \\ 0, & n \in N(m+1,\ \infty). \end{cases}$$

Then, $\bar{v}_m \in l^\infty(0,\ \infty)$. Moreover, from Step 1 and Step 2, we have $0 \le \bar{v}_m(n) < 1,\ n \in \mathbb{N}_0$, and

$$\bar{v}_1(n) \le \bar{v}_2(n) \le \cdots \le \bar{v}_m(n) \le \cdots,\ n \in \mathbb{N}_0.$$

Let $v^*(n) := \lim\limits_{m\to\infty} \bar{v}_m(n),\ n \in \mathbb{N}_0$. Then, $v^* \in l^\infty(0,\ \infty)$. Hence, v^* is a solution of (2.2).

Step 4. We show that v^* is a unique solution of (2.2). On the contrary, suppose that (2.2) has two solutions $v_1,\ v_2$, set $\phi = v_1 - v_2$. Without loss of generality, suppose $\phi(n) \ge 0$ and $\phi(n_0 - 1) \ne 0$. Then,

$$\begin{cases} \Delta^2\phi(n-1) - p(n)\Delta\phi(n-1) - q(n)\phi(n-1) = 0, & n \in \mathbb{N} \\ \alpha\phi(0) - \beta\Delta\phi(0) = 0, & \phi(\infty) = 0. \end{cases}$$

From $\alpha\phi(0) - \beta\Delta\phi(0) = 0$, we know $\phi(0) = \frac{\beta}{\alpha+\beta}\phi(1) \le \phi(1)$. Thus,

$$\Delta\phi(0) \ge 0 \quad \text{and} \quad \Delta\phi(1) = [1 + p(1)]\Delta\phi(0) + q(1)\phi(0) > 0.$$

The rest may be deduced by analogy, we omit it, and we have

$$\Delta\phi(n_0) = [1 + p(n_0)]\Delta\phi(n_0 - 1) + q(n_0)\phi(n_0 - 1) > 0.$$

Therefore, $\phi(\infty) > 0$, and this contradicts with $\phi(\infty) = 0$. □

Definition 2.1 ([8]). A homogeneous linear equation

$$u(t+m) + p_{m-1}(t)u(t+m-1) + \cdots + p_0(t)u(t) = 0,\ t \in \mathbb{N} \tag{2.7}$$

is said to be of "Poincaré type", if $\lim\limits_{t\to\infty} p_k(t) = p_k$ for $k = 0,\ 1,\ \cdots,\ m-1$ (i.e., if the coefficient functions convergent to constant, as t goes to infinity). Here, m is a given integer.

Lemma 2.3 (Perron's theorem, [8]). *Assume that equation (2.7) is of "Poincaré type", and the roots of $\lambda_1,\ \cdots,\ \lambda_m$ of $\lambda^m + p_{m-1}\lambda^{m-1} + \cdots p_0 = 0$ satisfy $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|$. Moreover, suppose that $p_0(t) \ne 0$ for each t. Then, there are m independent solutions $u_1,\ u_2,\ \cdots,\ u_m$ of equation (2.7) that satisfy*

$$\lim_{t\to\infty} \frac{u_i(t+1)}{u_i(t)} = \lambda_i, \quad (i = 1,\ 2,\ \cdots,\ m).$$

Lemma 2.4. *Assume that (H1)-(H3) hold. Then, the unique solution of (2.2) satisfies*

$$\lim_{n\to\infty} \frac{v(n+1)}{v(n)} = \lambda_1, \tag{2.8}$$

where $0 < \lambda_1 = \frac{2+p_0-\sqrt{p_0^2+4q_0}}{2} < 1$.

Proof. The equation $\Delta^2 x(n-1)-p(n)\Delta x(n-1)-q(n)x(n-1)=0$ is equivalent to

$$x(n+1)-(2+p(n))x(n)+(1+p(n)-q(n))x(n-1)=0.$$

Since $\lim_{n\to\infty} p(n)=p_0$, $\lim_{n\to\infty} q(n)=q_0$, we can get

$$\lambda^2-(2+p_0)\lambda+(1+p_0-q_0)=0. \tag{2.9}$$

By simple calculation, it follows that (2.9) has two eigenvalues

$$\lambda_1=\frac{2+p_0-\sqrt{p_0^2+4q_0}}{2},\ \lambda_2=\frac{2+p_0+\sqrt{p_0^2+4q_0}}{2}.$$

It is easy to verify

$$0<\lambda_1<1, \qquad \lambda_2>1.$$

From Lemma 2.2, $\Delta v(n)<0$ on $\mathbb{N}$, that is, $v(n+1)<v(n)$. This together with Lemma 2.3, we obtain $\lim_{n\to\infty}\frac{v(n+1)}{v(n)}=\lambda_1$. □

Lemma 2.5. *Assume that (H1)-(H3) hold. Then, there exists $M>0$ such that*

$$\sup_{n\in\mathbb{N}_0} u(n)v(n)<M.$$

Proof. By Liouville formula [8], we have

$$u(n)=c_1v(n)+c_2v(n)\sum_{i=0}^{n-1}\frac{\prod_{k=1}^{i}[1+p(k)-q(k)]}{v(i)v(i+1)}$$

for some constants c_1 and c_2. Applying Stolz Theorem [8], it follows that

$$\begin{aligned}\lim_{n\to\infty} u(n)v(n) &= \lim_{n\to\infty} c_1v^2(n)+c_2v^2(n)\sum_{i=0}^{n-1}\frac{\prod_{k=1}^{i}[1+p(k)-q(k)]}{v(i)v(i+1)}\\ &= c_2\lim_{n\to\infty}\frac{\prod_{k=1}^{n}[1+p(k)-q(k)]v(n)v(n+1)}{v^2(n)-v^2(n+1)}\\ &= \lim_{n\to\infty}\frac{\prod_{k=1}^{n}[1+p(k)-q(k)]}{\frac{v(n)}{v(n+1)}-\frac{v(n+1)}{v(n)}}\\ &= \frac{\prod_{k=1}^{\infty}[1+p(k)-q(k)]\lambda_1}{1-\lambda_1^2}<\infty.\end{aligned}$$

Hence, there exists $M>0$ such that $\sup_{n\in\mathbb{N}_0} u(n)v(n)<M$. □

Let

$$G(n,\ i)=(\alpha+\beta)\begin{cases}\omega(i)u(i)v(n), & 1\le i\le n-1,\\ \omega(i)u(n)v(i), & i\ge n,\end{cases} \tag{2.10}$$

where $\omega(i)=\prod_{k=1}^{i}[1+p(k)-q(k)]^{-1}$.

Lemma 2.6. *Assume that (H1)-(H3) hold. For any $h \in l^1(\mathbb{N})$, then the problem*

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) + h(n) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0 \end{cases} \tag{2.11}$$

is equivalent to the sum equation

$$x(n) = \sum_{i=1}^{\infty} G(n,\ i)h(i), \quad n \in \mathbb{N}_0. \tag{2.12}$$

Proof. First, we show that the unique solution of (2.11) can be represented by (2.12). In fact, we know that the equation

$$\Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) = 0, \ n \in \mathbb{N}$$

has two linear independent solutions u and v, since $\begin{vmatrix} u(0) & v(0) \\ u(1) & v(1) \end{vmatrix} = -\frac{1}{\alpha+\beta} \neq 0$. Now, by the method variation of constant [8], we can obtain that the unique solution of (2.11) can be represented by (2.12).

Next, we check that the function defined by (2.12) is a solution of (2.11). From (2.12), we have

$$\begin{aligned} x(n+1) &= (\alpha+\beta)\Big[v(n+1)\sum_{i=1}^{n}\omega(i)u(i)h(i) + u(n+1)\sum_{i=n+1}^{\infty}\omega(i)v(i)h(i)\Big], \\ x(n) &= (\alpha+\beta)\Big[v(n)\sum_{i=1}^{n-1}\omega(i)u(i)h(i) + u(n)\sum_{i=n}^{\infty}\omega(i)v(i)h(i)\Big], \\ x(n-1) &= (\alpha+\beta)\Big[v(n-1)\sum_{i=1}^{n-2}\omega(i)u(i)h(i) + u(n-1)\sum_{i=n-1}^{\infty}\omega(i)v(i)h(i)\Big]. \end{aligned}$$

Hence,

$$\begin{aligned} &\Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) \\ =&[1+p(n)-q(n)](\alpha+\beta)\omega(n)h(n)[u(n-1)v(n) - u(n)v(n-1)] \\ =&[1+p(n)-q(n)](\alpha+\beta)\omega(n)h(n)\prod_{k=1}^{n-1}[1+p(k)-q(k))][u(0)v(1)-u(1)v(0)] \\ =&h(n)(\alpha+\beta)\omega(n)\prod_{k=1}^{n}[1+p(k)-q(k)]\frac{-1}{\alpha+\beta} = -h(n). \end{aligned}$$

It is easy to see that $\alpha G(0,\ i) - \beta\Delta G(0,\ i) = 0$ implies $\alpha x(0) - \beta \Delta x(0) = 0$. Applying the facts that $\sup_{n\in\mathbb{N}_0} u(n)v(n) < M,\ h \in l^1(0,\ \infty)$ and $\prod_{k=1}^{\infty}[1+p(k)-q(k)] < \infty$, it follows that for any $\epsilon > 0$, there exists $N_1 > 0$ such that

$$\sum_{i=n}^{\infty}\omega(i)u(i)v(i)|h(i)| \leq M\sum_{i=n}^{\infty}\omega(i)|h(i)| < \frac{\epsilon}{3(\alpha+\beta)},\ \forall\ n \geq N_1.$$

From the fact $\lim_{n\to\infty} v(n) = 0$, there exists $N_2 > 0$ such that

$$u(N_1)v(n)\sum_{i=1}^{\infty}\omega(i)|h(i)| < \frac{\epsilon}{3(\alpha+\beta)},\ \forall\ n \geq N_2.$$

Let $N = \max\{N_1,\ N_2\}$. Then, for $n > N$, we get

$$\begin{aligned}
|x(n)| =& \left|(\alpha+\beta)\Big[\sum_{i=1}^{n-1}\omega(i)u(i)v(n)h(i) + \sum_{i=n}^{\infty}\omega(i)u(n)v(i)h(i)\Big]\right| \\
\leq&(\alpha+\beta)\Big[\sum_{i=1}^{N_1-1}\omega(i)u(i)v(n)|h(i)| + \sum_{i=N_1}^{n-1}\omega(i)u(i)v(n)|h(i)| + \sum_{i=n}^{\infty}\omega(i)u(n)v(i)|h(i)|\Big] \\
\leq&(\alpha+\beta)u(N_1)v(n)\sum_{i=1}^{\infty}\omega(i)|h(i)| + (\alpha+\beta)2M\sum_{i=N_1}^{\infty}\omega(i)|h(i)| \\
<&\frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} x(n) = 0$. □

Now, Lemma 2.6, we have from that for $h \in l^1(\mathbb{N})$, the boundary value problems

$$\begin{cases} \Delta^2 x(n-1) - p_*\Delta x(n-1) - q_* x(n-1) + h(n) = 0,\ n \in \mathbb{N}, \\ \alpha x(0) - \beta\Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \Delta^2 x(n-1) - p^*\Delta x(n-1) - q^* x(n-1) + h(n) = 0,\ n \in \mathbb{N}, \\ \alpha x(0) - \beta\Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0 \end{cases} \tag{2.14}$$

are equivalent to the sum equation

$$x_1(n) = \sum_{i=1}^{\infty} G_1(n,\ i)h(i), \quad n \in \mathbb{N}_0, \tag{2.15}$$

$$x_2(n) = \sum_{i=1}^{\infty} G_2(n,\ i)h(i), \quad n \in \mathbb{N}_0, \tag{2.16}$$

where

$$G_1(n,\ i) = \frac{1}{a_1 - b_1}\begin{cases} (1+b_1)^n(\frac{1}{(1+b_1)^i} - \frac{\alpha-a_1\beta}{\alpha-b_1\beta}\frac{1}{(1+a_1)^i}), & 1 \leq i \leq n-1, \\ \frac{1}{(1+a_1)^i}((1+a_1)^n - \frac{\alpha-a_1\beta}{\alpha-b_1\beta}(1+b_1)^n),\ i \geq n, \end{cases} \tag{2.17}$$

and

$$G_2(n,\ i) = \frac{1}{a_2 - b_2}\begin{cases} (1+b_2)^n(\frac{1}{(1+b_2)^i} - \frac{\alpha-a_2\beta}{\alpha-b_2\beta}\frac{1}{(1+a_2)^i}), & 1 \leq i \leq n-1, \\ \frac{1}{(1+a_2)^i}((1+a_2)^n - \frac{\alpha-a_2\beta}{\alpha-b_2\beta}(1+b_2)^n),\ i \geq n, \end{cases} \tag{2.18}$$

respectively. Here,

$$a_1 = \frac{p_* + \sqrt{p_*^2 + 4q_*}}{2}, \quad b_1 = \frac{p_* - \sqrt{p_*^2 + 4q_*}}{2},$$

$$a_2 = \frac{p^* + \sqrt{p^{*2} + 4q^*}}{2}, \quad b_2 = \frac{p^* - \sqrt{p^{*2} + 4q^*}}{2}.$$

Lemma 2.7. *For all* $(n,\ i) \in \mathbb{N}_0 \times \mathbb{N}$,

$$G_2(n,\ i) \leq G(n,\ i) \leq G_1(n,\ i) < B,$$

where $B = \max\{1,\ \frac{\beta}{\alpha - b_1\beta},\ \frac{1}{a_1 - b_1} - \frac{(\alpha - a_1\beta)(1+a_1)}{(\alpha - b_1\beta)(a_1 - b_1)}\}$.

Proof. From (2.17), we can easily deduce $G_1(n,\ i) < B, (n,\ i) \in \mathbb{N}_0 \times \mathbb{N}$, where $B = \max\{1,\ \frac{\beta}{\alpha - b_1\beta},\ \frac{1}{a_1 - b_1} - \frac{(\alpha - a_1\beta)(1+a_1)}{(\alpha - b_1\beta)(a_1 - b_1)}\}$. Next, we only show $G(n,\ i) \leq G_1(n,\ i)$, and the other case can be treated by the same way.

On the contrary, suppose that there exists $(n_0, i_0) \in \mathbb{N} \times \mathbb{N}$, such that $G(n_0, i_0) > G_1(n_0, i_0)$. Let

$$\hat{h}(n) = \begin{cases} 0, & 1 \leq n \leq i_0 - 1, \\ n - i_0 + 1, & i_0 - 1 \leq n \leq i_0, \\ i_0 + 1 - n, & i_0 \leq n \leq i_0 + 1, \\ 0, & i_0 + 1 \leq n < \infty. \end{cases}$$

Then, $\hat{h}(n) \geq 0, \quad n \in \mathbb{N}$.

Let $x_1(n)$, $x_2(n)$ be the solutions of

$$\begin{cases} \Delta^2 x(n-1) - p_* \Delta x(n-1) - q_* x(n-1) + \hat{h}(n) = 0,\ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases}$$

and

$$\begin{cases} \Delta^2 x(n-1) - p^* \Delta x(n-1) - q^* x(n-1) + \hat{h}(n) = 0,\ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases}$$

respectively. Let $\hat{x}(n)$ be the solution of

$$\begin{cases} \Delta^2 x(n-1) + F(n, x(n-1), \Delta x(n-1)) = 0,\ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases} \tag{2.19}$$

where $F(n, x(n-1), \Delta x(n-1)) = -p(n)\Delta x(n-1) - q(n)x(n-1) + \hat{h}(n), \quad n \in \mathbb{N}$.

We claim

$$\Delta^2 x_1(n-1) + F(n, x_1(n-1), \Delta x_1(n-1)) \leq 0, \qquad n \in \mathbb{N}$$

and

$$\Delta^2 x_2(n-1) + F(n, x_2(n-1), \Delta x_2(n-1)) \geq 0, \qquad n \in \mathbb{N}.$$

In fact, if $1 \leq i_0 \leq n-2$, then

$$\Delta x_1(n-1) = (1+b_1)^{n-1} b_1 \left(\frac{1}{(1+b_1)^{i_0}} - \frac{\alpha - a_1\beta}{\alpha - b_1\beta}\frac{1}{(1+a_1)^{i_0}}\right) > 0.$$

If $i_0 \geq n$, then

$$\Delta x_1(n-1) = \frac{1}{(1+a_1)^{i_0}}[a_1(1+a_1)^{n-1} - \frac{\alpha - a_1\beta}{\alpha - b_1\beta} b_1(1+b_1)^{n-1}] > 0.$$

If $i_0 = n-1$, then

$$\Delta x_1(n-1) = \frac{b_1}{a_1 - b_1}[1 - \frac{\alpha - a_1\beta}{\alpha - b_1\beta} b_1 \frac{(1+b_1)^{n-1}}{(1+a_1)^{n-1}}] > 0.$$

Hence,

$$\begin{aligned} &\Delta^2 x_1(n) + F(n, x_1(n-1), \Delta x_1(n-1)) \\ &= (p_* - p(n))x_1(n) + (p(n) - p_* + q_* - q(n))x_1(n-1) \\ &\leq (p_* - p(n))x_1(n) + (p(n) - p_*)x_1(n) + (q_* - q(n))x_1(n) \\ &= (q_* - q(n))x_1(n) \leq 0. \end{aligned}$$

On the other hand, $\alpha\hat{x}(0) - \beta\Delta\hat{x}(0) = 0 = \alpha x_1(0) - \beta\Delta x_1(0)$, $\lim_{n\to\infty} \hat{x}(n) = 0 = \lim_{n\to\infty} x_1(n)$.

By using the similar method, we can prove

$$\Delta^2 x_2(n) + F(n, x_2(n-1), \Delta x_2(n-1)) \geq 0, \qquad n \in \mathbb{N}.$$

Next, we will show

$$x_2(n) \leq \hat{x}(n) \leq x_1(n), \quad n \in \mathbb{N}_0. \tag{2.20}$$

Define $F^*(n, x(n-1), \Delta x(n-1))$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$, by

$$F^*(n, x(n-1), \Delta x(n-1)) = \begin{cases} F(n, x_1(n-1), \Delta x_1(n-1)) + \frac{x - x_1(n)}{1+|x_1(n)|}, & x(n) \geq x_1(n), \\ F(n, x(n-1), \Delta x(n-1)), & x_2(n) \leq x(n) \leq x_1(n), \\ F(n, x_2(n-1), \Delta x_2(n-1)) + \frac{x - x_2(n)}{1+|x_2(n)|}, & x(n) \leq x_2(n). \end{cases}$$

Note that $F^*(n, x(n-1), \Delta x(n-1))$ is continuous as a function of x and Δx for each n. Furthermore, F^* is bounded, and agrees with F when $x_2(n) \leq x(n) \leq x_1(n)$. Let $\Lambda := \{u : \mathbb{N}_0 \to \mathbb{R} \mid \lim_{n\to\infty} u(n) = 0, \max_{n\in\mathbb{N}_0} |u_n| \leq \max\{\max_{n\in\mathbb{N}_0} |x_1|, \max_{n\in\mathbb{N}_0} |x_2|\}$. Then, by Brouwer's fixed point theorem [4], the boundary value problem

$$\begin{cases} \Delta^2 x(n-1) + F^*(n, x(n-1), \Delta x(n-1)) = 0, & n \in \mathbb{N}, \\ \alpha x(0) - \beta\Delta x(0) = 0, & \lim_{n\to\infty} x(n) = 0 \end{cases}$$

has a solution $x(n)$ on Λ.

We claim $x(n) \le x_1(n), \quad n \in \mathbb{N}_0$. On the contrary, suppose that $x(n) - x_1(n)$ has a positive maximum at some n_1 in $\mathbb{N}$. Consequently, set $w(n) = x(n) - x_1(n)$, then $w(n_1) > 0$, and we must have

$$\begin{aligned}\Delta^2 w(n_1 - 1) &\ge F^*(n_1, \Delta x(n_1 - 1), x(n_1 - 1)) - F(n_0, \Delta x_1(n_1 - 1), x_1(n_1 - 1))\\ &= \frac{x(n_1) - x_1(n_1)}{1 + |x_1(n_1)|} > 0.\end{aligned}$$

By Anti-Maximum principle [8], we have $w(n_1) < 0$, which contradicts with $w(n_1) > 0$. It follows that $x(n) \le x_1(n), \quad n \in \mathbb{N}_0$.

Similarly, $x_2(n) \le x(n), \quad n \in \mathbb{N}_0$. Thus, $x(n)$ is a solution of problem (2.19). That is,

$$x_2(n) \le \hat{x}(n) \le x_1(n), \quad n \in \mathbb{N}_0,$$

and accordingly,

$$\sum_{i=1}^{\infty} G_2(n,i)\hat{h}(i) \le \sum_{i=1}^{\infty} G(n,i)\hat{h}(i) \le \sum_{i=1}^{\infty} G_1(n,i)\hat{h}(i).$$

Therefore,

$$\begin{aligned}x_1(n_0) - \hat{x}(n_0) &= \sum_{i=1}^{\infty}[G_1(n_0,i) - G(n_0,i)]\hat{h}(i)\\ &= \sum_{i=1}^{i_0-1}[G_1(n_0,i) - G(n_0,i)]\hat{h}(i) + \sum_{i=i_0+1}^{\infty}[G_1(n_0,i) - G(n_0,i)]\hat{h}(i)\\ &\quad + [G_1(n_0,i_0) - G(n_0,i_0)]\hat{h}(i_0)\\ &< 0,\end{aligned}$$

which contradicts with the second inequality in (2.20). □

Lemma 2.8. *For any given $\theta \in (1, +\infty)$, we have*

$$G(n,\ i)v^{\theta}(n) \le G(i,\ i)v(i).$$

Proof. If $1 \le i \le n-1$, then

$$G(n,\ i)v^{\theta}(n) = (\alpha+\beta)\omega(i)u(i)v(n)v^{\theta}(n) \le (\alpha+\beta)\omega(i)u(i)v(i)v(n) \le G(i,\ i)v(i).$$

If $i \ge n$, then

$$\begin{aligned}G(n,\ i)v^{\theta}(n) &= (\alpha+\beta)\omega(i)u(n)v(i)v^{\theta}(n) \le (\alpha+\beta)\omega(i)u(n)v(i)v(n)\\ &\le \omega(i)u(i)v(i)v(i) = G(i,\ i)v(i).\end{aligned}$$

Hence, $G(n,\ i)v^{\theta}(n) \le G(i,\ i)v(i),\ (n,\ i) \in \mathbb{N}_0 \times \mathbb{N}$. □

Lemma 2.9. *For any subinterval $N(l_1,\ l_2) \subseteq \mathbb{N}$ with $0 < l_1 < l_2$, $n \in N(l_1,\ l_2)$ and $i \in \mathbb{N}$,*

$$G(n,\ i) \ge \delta G(i,\ i)v(i),$$

where $\delta := \min\{q(n) \,|\, n \in N(l_1,\ l_2)\}$*,* $l_1, l_2 \in \mathbb{N}$ *is a constant and*

$$q(n) = \min\{v(n),\ \frac{u(n)}{B}\},\ n \in \mathbb{N}_0.$$

Proof. If $1 \le i \le n-1$, then

$$\frac{G(n,\ i)}{G(i,\ i)v(i)} = \frac{\omega(i)u(i)v(n)}{\omega(i)u(i)v(i)v(i)} = \frac{v(n)}{v(i)v(i)} \ge v(n).$$

If $i \ge n$, then

$$\frac{G(n,\ i)}{G(i,\ i)v(i)} = \frac{\omega(i)u(n)v(i)}{\omega(i)u(i)v(i)v(i)} = \frac{u(n)}{u(i)v(i)} \ge \frac{u(n)}{B}.$$

Let $q(n) = \min\{v(n),\ \frac{u(n)}{B}\}$. Then,

$$G(n,\ i) \ge q(n)G(i,\ i)v(i),\ (n,\ i) \in \mathbb{N}_0 \times \mathbb{N}.$$

□

Remark 2.1. Note that u is increasing on $\mathbb{N}_0$, so u may be unbounded on $\mathbb{N}_0$, and there is no positive constant c, such that if $1 \le n \le i \le \infty$,

$$\frac{G(n,\ i)}{G(i,\ i)} = \frac{u(n)}{u(i)} > cu(n), \forall\ n \in \mathbb{N}.$$

Hence, it is impossible to prove

$$G(n,\ i) \ge q(n)G(i,\ i),\ (n,\ i) \in \mathbb{N}_0 \times \mathbb{N}.$$

3. Compactness results of the sum operator in Banach space

Let

$$l_\diamond = \{x(\cdot) \in l^\infty(\mathbb{N}_0) \,|\, \lim_{n\to\infty} x(n) = x(\infty)\}$$

with the norm $\|x\|_l = \sup_{n\in\mathbb{N}} |x(n)|$. Then, $l_\diamond$ is a Banach space.

Let

$$X = \{x(\cdot) \in l^\infty(\mathbb{N}_0) \,|\, \lim_{n\to\infty} |x(n)|v^\theta(n) = r \text{ for some } r \in \mathbb{R}\}$$

be endowed with the norm $\|x\| = \sup_{n\in\mathbb{N}_0}\{|x(n)|v^\theta(n)\}$, where $\theta > 1$ is a constant. Then, X is a Banach space. Here, $v(n)$ is the unique solution of (2.2).

Lemma 3.1 (Theorem 2.1, [10]). *Let* $\mathcal{F} \subset l_\diamond$ *be a set satisfying the following conditions:*

(A1) $\mathcal{F}$ *is bounded in* $l_\diamond$*;*

(A2) the function from $\mathcal{F}$ *are equiconvergent, i.e., given* $\epsilon > 0$*, it corresponds to* $N(\epsilon) > 0$ *such that*

$$\|f(n) - f(\infty)\| < \epsilon \quad \textit{for any } n \ge N(\epsilon) \textit{ and } f \in \mathcal{F}.$$

Then, $\mathcal{F}$ *is compact in* $l_\diamond$*.*

Lemma 3.2 (Theorem 2.2, [10]). *Let $\mathcal{F} \subset l_\diamond$ be compact in $l_\diamond$. Then,*

(i) $\mathcal{F}$ is bounded in $l_\diamond$;

(ii) the function from $\mathcal{F}$ are equiconvergent.

Note: Follows from Lemmas 3.1 and 3.2, it concludes a compact theorem in the Banach space X.

Lemma 3.3. *Let $M \subset X$ and M satisfy the following conditions:*

(i) M is bounded in X;

(ii) the functions belonging to $\{y \,|\, y(n) = x(n)v^\theta(n),\ x \in M\}$ are equiconvergent, i.e., given $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that

$$\|y(n) - y(\infty)\| < \epsilon \quad \text{for any } n \geq N(\epsilon).$$

Then, M is compact in X.

To prove our main results, we give the following assumptions.

(H4) $f : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and satisfies

$$\exists\ s > 0 :\ \ s \neq 1,\ 0 \leq f(n,\ x) \leq k_1(n) + k_2(n)x^s,\ \forall\ (n,\ x) \in \mathbb{N} \times \mathbb{R}^+,$$

where $k_1,\ k_2 \in l^\infty(0,\ \infty)$.

(H5) Assume the summation

$$M_1 = \sum_{i=1}^{\infty} G(i,\ i)v(i)k_1(i) < \infty, \quad M_2 = \sum_{i=1}^{\infty} G(i,\ i)[v(i)]^{1-\theta s}k_2(i) < \infty,$$

where $\theta > 1$ is a constant, and there exists $r > 0$, such that $M_1 + M_2 r^s < r$.

For some constants $l_1,\ l_2 \in \mathbb{N}$ with $l_1 < l_2$, we denote

$$m := \sum_{i=l_1}^{l_2} G(i,\ i)v(i).$$

Define a cone of X

$$P = \{x \in X \,|\, x(n) \geq 0,\ n \in \mathbb{N},\ \text{and}\ x(n) \geq q(n)\|x\|\},$$

and the operator $A : X \to X$

$$Ax(n) = \sum_{i=1}^{\infty} G(n,\ i)f(i,\ x(i)), \quad n \in \mathbb{N}_0.$$

Lemma 3.4. *Assume that (H1)-(H5) hold. Then, $A(P) \subset P$ and $A : P \to P$ are completely continuous.*

Proof. We divide the proof into the following steps.

Step 1. $A(P) \subseteq P$.

For any $x \in P$, from Lemma 2.8, we have

$$v^\theta(n)Ax(n) = \sum_{i=1}^{\infty} v^\theta(n)G(n,\ i)f(i,\ x(i)) \tag{3.1}$$

$$\leq \sum_{i=1}^{\infty} v(i)G(i,\ i)[k_1(i) + k_2(i)|x(i)|^s] \tag{3.2}$$

$$\leq \sum_{i=1}^{\infty} v(i)G(i,\ i)k_1(i) + \sum_{i=1}^{\infty}[v(i)]^{1-\theta s}G(i,\ i)k_2(i)\|x\|^s \tag{3.3}$$

$$\leq M_1 + M_2\|x\|^s. \tag{3.4}$$

Therefore, $\sup_{n\in\mathbb{N}_0}\{|Ax(n)|v^{\theta}(n)\} \leq M_1 + M_2\|x\|^s < \infty$. That is, $Ax \in X, \forall\ x \in P$. By Lemma 2.8 and Lemma 2.9, we have

$$\begin{aligned} Ax(n) &= \sum_{i=1}^{\infty} G(n,\ i)f(i,\ x(i)) \\ &\geq \sum_{i=1}^{\infty} q(n)G(i,\ i)v(i)f(i,\ x(i)) \\ &\geq \sum_{i=1}^{\infty} q(n)v^{\theta}(\xi)G(\xi,\ i)f(i,\ x(i)) \\ &= q(n)v^{\theta}(\xi)Ax(\xi), \quad \forall\ \xi \in \mathbb{N}_0. \end{aligned}$$

Setting ξ with $v^{\theta}(\xi)Ax(\xi) = \|Ax\|$, we deduce

$$Ax(n) \geq q(n)\|Ax\|,\ \forall\ x \in P.$$

Therefore, $A(P) \subseteq P$.

Step 2. $A : P \to P$ is continuous.

Assume that $\{x_k\}_{k=1}^{\infty} \subseteq P$, $x_0 \in P$ and $\lim_{k\to\infty} x_k = x_0$. Then, there exists a constant $M > 0$, such that $\|x_k\| \leq M,\ k \in \mathbb{N}_0$. Thus,

$$\begin{aligned} &\sum_{i=1}^{\infty} G(i,\ i)v(i)|f(i,\ x_k(i)) - f(i,\ x_0(i))| \\ \leq& 2\sum_{i=1}^{\infty} G(i,\ i)v(i)[k_1(i) + k_2(i)|x|^s] \\ \leq& 2M_1 + 2M^sM_2 < \infty. \end{aligned}$$

Hence, according to the continuous of f and Lemma 2.8, we have

$$\begin{aligned} &\lim_{k\to\infty} \|Ax_k - Ax_0\| \\ =& \lim_{k\to\infty}\sup_{n\in\mathbb{N}_0} |\sum_{i=1}^{\infty} v^{\theta}(n)G(n,\ i)f(i,\ x_k(i)) - \sum_{i=1}^{\infty} v^{\theta}(n)G(n,\ i)f(i,\ x_0(i))| \\ \leq& \lim_{k\to\infty}\sup_{n\in\mathbb{N}_0} \sum_{i=1}^{\infty} G(i,\ i)v(i)|f(i,\ x_k(i)) - f(i,\ x_0(i))| = 0. \end{aligned}$$

Thus, $A : P \to P$ is continuous.

Step 3. We show that $A : P \to P$ is compact.

Let $D \subseteq P$ be bounded. Then, there exists $M > 0$ such that $\|x\| < M,\ x \in D$. First, we show that $A(D)$ is bounded set in X. For any $x \in D$, from (H4)-(H5) and by applying the method to prove (3.4), it follows that

$$\|Ax\| \leq M_1 + M_2 M^s,$$

which implies that $A(D)$ is bounded in X.

Second, we show that the functions belonging to $\{(Ax)(\cdot)v^{\theta}(\cdot) \mid x \in D\}$ are equiconvergent. Let $\sigma := \frac{\theta-1}{2}$, then $\sigma > 0$. Since $\lim_{n\to\infty} v(n) = 0$, we have that for all $\epsilon > 0$, there exists $N > 0$ such that

$$|v(n) - 0| < \left(\frac{\epsilon}{M_1 + M_2 M^s}\right)^{\frac{1}{\sigma}}, \quad \forall\ n \geq N.$$

Thus, from Lemma 2.8, it follows that for above $\epsilon > 0$, there exists $N > 0$ such that $x \in D$ and $n \geq N$, implying that

$$\begin{aligned} 0 \leq v^{\theta}(n)Ax(n) &= \sum_{i=1}^{\infty} v^{\theta}(n)G(n,\ i)f(i,\ x(i)) \\ &\leq v^{\sigma}(n)\sum_{i=1}^{\infty} v^{1+\sigma}(n)G(n,\ i)[k_1(i) + k_2(i)|x(i)|^s] \\ &\leq v^{\sigma}(n)[\sum_{i=1}^{\infty} v(i)G(i,\ i)k_1(i) + \sum_{i=1}^{\infty}[v(i)]^{1-\theta s}G(i,\ i)k_2(i)\|x\|^s] \\ &\leq v^{\sigma}(n)[M_1 + M_2\|x\|^s] < \epsilon. \end{aligned}$$

Hence, the functions belonging to $\{y \mid y(n) = x(n)v^{\theta}(n),\ x \in M\}$ are equiconvergent.

Therefore, $A : P \to P$ is compact. □

Finally, we give the fixed point theorem in cones.

Lemma 3.5 ([4]). *Let E be a Banach space and $K \subset E$ be a cone. Assume that $\Omega_1,\ \Omega_2$ are bounded open sunsets of E with $0 \in \Omega_1,\ \bar{\Omega}_1 \subset \Omega_2$, and let $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous such that either*

(i) $\|Au\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|,\ u \in K \cap \partial\Omega_2$; or

(ii) $\|Au\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|,\ u \in K \cap \partial\Omega_2$.

Then, A has a fixed point.

4. Existence and multiplicity of positive solutions

We will discuss the existence of positive solutions for (1.3) between the cases $s > 1$ (super-linear case) and $s < 1$ (sub-linear case).

Theorem 4.1. *Let $s > 1$ and assume that (H1)-(H5) and* $(H6)\ \liminf_{x\to+\infty} \min_{n\in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} \geq \frac{1}{v^{\theta}(l_2)\delta^2 m}$ *hold. Then, problem* (1.3) *has at least one nontrivial positive solution.*

Proof. Let r be defined by (H5). By the inequality of (H6), there exists a constant $T > \delta r > 0$ such that

$$\min_{n\in N(l_1,\ l_2)} f(n,\ x) \geq \frac{1}{v^{\theta}(l_2)\delta^2 m}x, \quad \forall\ x \geq T.$$

Hence, $f(n,\ x) \geq \frac{1}{v^{\theta}(l_2)\delta^2 m} x$, for any $x \geq T$ and $n \in N(l_1,\ l_2)$.

Let $R = \max\{2r,\ \frac{T}{\delta}\}$, and define the open sets

$$\Omega_1 = \{x \in X\ :\ \|x\| < r\}, \quad \Omega_2 = \{x \in X\ :\ \|x\| < R\}.$$

For any $x \in \partial\Omega_1 \cap P$, from (H4) and (H5), we obtain the following estimates

$$v^{\theta}(n)|Ax(n)| \leq M_1 + M_2\|x\|^s = M_1 + M_2 r^s < r.$$

Passing to the supremum over n, we infer

$$\|Ax\| \leq \|x\|,\ \forall\ x \in \partial\Omega_1 \cap P. \tag{4.1}$$

Since $0 < \delta < 1$, it follows that for any $x \in \partial\Omega_2 \cap P$, $\min\limits_{n \in N(l_1,\ l_2)} x(n) \geq \delta\|x\| = \delta R$ $\geq T$. From (H6), for $n \in N(l_1,\ l_2)$, we can get

$$\begin{aligned} v^{\theta}(n)Ax(n) &= \sum_{i=1}^{\infty} v^{\theta}(n)G(n,\ i)f(i,\ x(i)) \\ &\geq v^{\theta}(n)\sum_{i=l_1}^{l_2} G(i,\ i)v(i)\delta^2 \frac{1}{mv^{\theta}(l_2)\delta^2}\|x\| \\ &\geq \frac{1}{mv^{\theta}(l_2)} v^{\theta}(l_2)\sum_{i=l_1}^{l_2} G(i,\ i)v(i)\|x\| \\ &= \|x\|. \end{aligned}$$

Therefore, $\|Ax\| \geq \|x\|,\quad \forall\ x \in \partial\Omega_2 \cap P$. Hence, the operator A has a fixed point $x \in (\overline{\Omega}_2 \setminus \Omega_1) \cap P$. That is, problem (1.3) has a positive solution x satisfying

$$r \leq \|x\| \leq R.$$

□

The following result deals with the sublinear polynomial growth case, and can be proved in a similar argument. We omit the proof.

Theorem 4.2. *Let $s < 1$ and assume that (H1)-(H5) and $(H7)$ $\liminf\limits_{x \to 0} \min\limits_{n \in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} \geq \frac{1}{v^{\theta}(l_2)\delta^2 m}$ hold. Then, problem* (1.3) *has at least one nontrivial positive solution.*

Remark 4.1. In the case $p(\cdot) \equiv p_0, q(\cdot) \equiv q_0$, problem (1.3) was discussed in [14] by the fixed-point theorem in Fréchet space. Theorem 4.1 and Theorem 4.2 give a new results of the existence of positive solutions for (1.3) by using the fixed-point theorem in Banach space.

Finally, we prove the existence of two nontrivial positive solutions for problem (1.3) in the superlinear case.

Theorem 4.3. *Assume that $s > 1$, (H1)-(H5) and $(H8)$ $\liminf\limits_{x \to 0} \min\limits_{n \in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} = \liminf\limits_{x \to +\infty} \min\limits_{n \in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} = +\infty$ hold. Then, problem* (1.3) *has at least two positive solutions x_1 and x_2 in P such that $0 < \|x_1\| < r < \|x_2\|$.*

Proof. Consider the open set $\Omega_1 = \{x \in X : \|x\| < r\}$, where r is as introduced in (H5). As in the proof of Theorem 4.1, we can check

$$\|Ax\| \le \|x\|, \ \forall\, x \in \partial\Omega_1 \cap P. \tag{4.2}$$

Let the constant

$$M_3 := \frac{1}{\delta^2 v^\theta(n_0) m} \quad \text{for some} \quad n_0 \in N(l_1,\ l_2). \tag{4.3}$$

(a) The condition $\liminf_{x\to+\infty} \min_{n\in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} = +\infty$ implies that there exists some $r_0 > 0$ such that

$$f(n,\ x) \ge M_3 x, \quad \text{for} \quad n \in N(l_1,\ l_2) \quad \text{and} \quad x \ge r_0.$$

Consider the open set $\Omega_2 = \{x \in X : \|x\| < R\}$, where $R = \max\{2r,\ \frac{r_0}{\delta}\}$. Then, for any $x \in \partial\Omega_2 \cap P,\ x(n) \ge \delta\|x\| = \delta R \ge r_0,\ n \in N(l_1,\ l_2)$. Let $n_0 \in N(l_1,\ l_2)$, it follows that

$$\begin{aligned} v^\theta(n_0)Ax(n_0) &= \sum_{i=1}^{\infty} v^\theta(n_0) G(n_0,\ i) f(i,\ x(i)) \\ &\ge v^\theta(n_0) \sum_{i=l_1}^{l_2} G(i,\ i) v(i) \delta^2 M_3 \|x\| \\ &\ge M_3 \delta^2 v^\theta(n_0) \sum_{i=l_1}^{l_2} G(i,\ i) v(i) \|x\| \\ &\ge \|x\|. \end{aligned}$$

Hence,

$$\|Ax\| \ge \|x\|, \quad \forall\, x \in \partial\Omega_2 \cap P. \tag{4.4}$$

(b) From $\liminf_{x\to 0} \min_{n\in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} = \infty$ in (H8), we infer that, for the constant M_3 in (4.3), there exists $r_1 > 0$ such that

$$f(n,\ x) \ge M_3 x, \quad \text{for} \quad n \in N(l_1,\ l_2) \quad \text{and} \quad 0 \le x \le r_1.$$

Let $L = \min\{r_1 v^\theta(l_2), \frac{r}{2}\}$ and $\Omega_3 = \{x \in X : \|x\| < L\}$. For any $x_2 \in \partial\Omega_3 \cap P$, we have $x(n)v^\theta(n) \le L,\ \forall\, n \in \mathbb{N}_0$. Hence, $x(n)v^\theta(n) \le L,\ \forall\, n \in N(l_1,\ l_2)$. Therefore, $x(n) \le v^{-\theta}(l_2)L \le r_1,\ \forall\, n \in N(l_1,\ l_2)$. Proceeding as in part (a), we can prove

$$\|Ax\| \ge \|x\|, \quad \forall\, x \in \partial\Omega_3 \cap P. \tag{4.5}$$

By (4.2),(4.4) and (4.5), together with the fact $L < r < R$, Lemma 3.5 implies that the operator A has two fixed points in the cone P, $x_1 \in \bar{\Omega}_1 \backslash \Omega_3$ and $x_2 \in \bar{\Omega}_2 \backslash \Omega_1$ such that $0 < L \le \|x_1\| < r < \|x_2\| \le R$. Clearly, x_1 and x_2 are nontrivial positive solutions of problem (1.3). □

Finally, we give some examples to illustrate the main results.

Example 4.1. Let us consider the nonlinear boundary value problem

$$\begin{cases} \Delta^2 x(n-1) - \sin\frac{\pi}{(n+1)^2}\Delta x(n-1) - (1-\frac{1}{(n+1)^2})x(n-1) + f(n,x) = 0, \\ x(0) - \frac{1}{2}\Delta x(0) = 0, \qquad \lim\limits_{n\to\infty} x(n) = 0, \end{cases} \tag{4.6}$$

where $p(n) = \sin\frac{\pi}{(n+1)^2} \in (0, \frac{\sqrt{2}}{2}]$, $q(n) = 1 - \frac{1}{(n+1)^2} \in [\frac{3}{4}, 1)$ satisfy (H1)-(H3), $f(n,x(n)) = \frac{1}{n^2} + \frac{2}{n^3+3}x^5$ is continuous about x. It is easy to verify that f satisfies (H4)-(H5), and $\liminf\limits_{x\to+\infty} \min\limits_{n\in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} = \liminf\limits_{x\to+\infty} \min\limits_{n\in N(l_1,\ l_2)} [\frac{1}{xn^2} + \frac{2}{n^3+3}x^4] = \infty$ implies that (H6) holds. Thus, from Theorem 4.1, problem (4.6) has at least one positive solution.

Example 4.2. Let us consider the nonlinear boundary value problem

$$\begin{cases} \Delta^2 x(n-1) - (1-\cos\frac{\pi}{n+1})\Delta x(n-1) - \frac{n^3+3n^2+3}{(n+1)^3}x(n-1) + \frac{1}{n^2+4n+4} + \frac{1}{n^3}x^{\frac{1}{2}} = 0, \\ \frac{1}{4}x(0) - 3\Delta x(0) = 0, \qquad \lim\limits_{n\to\infty} x(n) = 0, \end{cases}$$

where $p(n) = 1-\cos\frac{\pi}{n+1} \in (0, 1]$, $q(n) = \frac{n^3+3n^2+3}{(n+1)^3} = 1-\frac{1}{(n+1)^3} \in [\frac{7}{8}, 1)$ satisfy (H1)-(H3). Notice that $f(n,x) = \frac{1}{n^2+4n+4} + \frac{1}{n^3}x^{\frac{1}{2}}$ is continuous on $\mathbb{N}\times\mathbb{R}^+$, and (H4)-(H5) are true. Since $\liminf\limits_{x\to 0} \min\limits_{n\in N(l_1,\ l_2)} \frac{f(n,\ x)}{x} = \liminf\limits_{x\to 0} \min\limits_{n\in N(l_1,\ l_2)} [\frac{1}{(n^2+4n+4)x} + \frac{1}{n^3}x^{-\frac{1}{2}}] = \infty$, it follows that (H7) is true. Thus, from Theorem 4.2, this problem has at least one positive solution.

Example 4.3. The nonlinear boundary value problem

$$\begin{cases} \Delta^2 x(n-1) - \sin\frac{\pi}{(n+1)^2}\Delta x(n-1) - (1-\frac{1}{(n+1)^2})x(n-1) + \frac{1}{n^2} + \frac{2}{n^3+3}x^4 = 0, \\ x(0) - \frac{1}{2}\Delta x(0) = 0, \qquad \lim\limits_{n\to\infty} x(n) = 0, \end{cases}$$

where $p(n) = \sin\frac{\pi}{(n+1)^2} \in (0, \frac{\sqrt{2}}{2}]$, $q(n) = 1 - \frac{1}{(n+1)^2} \in [\frac{3}{4}, 1)$ satisfy (H1)-(H3), and $f(n,x(n)) = \frac{1}{n^2} + \frac{2}{n^3+3}x^4$ is continuous about x and (H4)-(H5) hold. It is easy to see $\liminf\limits_{x\to 0} \min\limits_{n\in N(l_1,\ l_2)} \frac{1}{xn^2} + \frac{2}{n^3+3}x^3 = \liminf\limits_{x\to+\infty} \min\limits_{n\in N(l_1,\ l_2)} \frac{1}{xn^2} + \frac{2}{n^3+3}x^3 = +\infty$, which means that (H8) holds. Hence, from Theorem 4.3, this problem has at least two positive solutions.

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References

[1] R. P. Agarwal and D. O'Regan, *Existence and Approximation of Solutions of Non-linear Discrete Systems on Infinite Intervals*, Mathematical Methods in the Applied Sciences, 1999, 22, 91–99.

[2] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.

[3] P. B. Baily, L. F. Shampine and P. E. Waltman, *Nonlinear Two-point Boundary Value Problem*, Academic Press, New York, 1968.

[4] K. Deimling, *Nonlinear Functional Analysis*, Heidemic Press, Springer-Verlag, Berlin, 1988.

[5] S. Djebali and K. Mebarki, *Multiple positive solutions for singular BVPs on the positive half line*, Computers & Mathematics with Applications, 2008, 55(12), 2940–2952.

[6] M. Galewski, T. Moussaoui and I. Soufi, *On the existence of solutions for a boundary value problem on the half-line*, Electronic Journal of Qualitative Theory of Differential Equations, 2018, 12, 1–12.

[7] M. Jia, H. Zhang and Q. Chen, *Existence of positive solutions for fractional differential equation with integral boundary conditions on the half-line*, Boundary Value Problems, 2016, 104, 16 pages.

[8] W. G. Kelley and A. C. Peterson, *Difference Equations–An Introduction with Applications (Second Edition)*, Academic Press, New York, 2001.

[9] H. Li, G. Du and C. Yue, *On Two-point Boundary Value Problems for Second-order Difference Equation*, Journal of Nonlinear Modeling and Analysis, 2022, 4(3), 605–614.

[10] Y. Lu and R. Ma, *Discrete Arzelà-Ascoli Theorem on the Half Line and Its Application*, Communications in Mathematical Research, 2018, 34(3), 221–229.

[11] G. Maise and A. J. Sabadell, *Electrostatic Probe Measurements in Solid-propellant Rocket Exhausts*, AIAA Journal, 1970, 8(5), 895–901.

[12] R. Ma and B. Zhu, *Existence of positive solutions for a semipositone boundary value problem on the half-line*, Computers & Mathematics with Applications, 2009, 58(8), 1672–1686.

[13] T. Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York, 1979.

[14] Y. Tian and W. Ge, *Multiple positive solutions of boundary value problems for second-order discrete equations on the half-line*, Journal of Difference Equations and Applications, 2006, 12(2), 191–208.

[15] B. Yan, D. O'Regan and R. P. Agarwal, *Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity*, Journal of Computational and Applied Mathematics, 2006, 197(2), 365–386.

[16] M. Zima, *On positive solutions of boundary value problems on the half-line*, Journal of Mathematical Analysis and Applications, 2001, 259(1), 127–136.