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Dynamic Analysis of Stochastic Spruce Budworm Differential Model with Time Delay*

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Abstract In this paper, we consider a stochastic spruce budworm differential model with time delay. Based on the nonnegative initial conditions, the existence and uniqueness of the global positive solution are easily found. Then, we obtain the ultimate boundedness of solution in mean under the same conditions. Furthermore, we verify that the sample Lyapunov exponent of solution is less than a positive constant. Finally, numerical examples are presented to show the consistency of the theoretical results.

Keywords Spruce budworm model, Stochastic perturbation, Global solution.

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1. Introduction

The spruce budworm, found in spruce-fir forests in the United States and eastern Canada, is one of the most destructive insect species. On the basis of [3], its periodic outbreaks may result in the loss of large amounts of food and natural resources. According to records from the United States and Canada in [12], spruce budworm outbreaks have occurred approximately every 40 years since the 18th century, each lasting about 10 years and causing enormous damage to forest resources. In [2, 7], Canadian scholars Ludwing et al., and Dwyer et al., established the following classical spruce budworm model

$$\mathrm{d}u(t) = \left[ru(t)(1 - \frac{u(t)}{K}) - \frac{Bu^2(t)}{A + u^2(t)} \right] \mathrm{d}t, \tag{1.1}$$

where $u(t)$ represents the density of spruce budworm population at time t, $r > 0$ represents the population growth rate, and K represents the environmental carrying capacity. $B > 0$ is the predation rate of predators or parasites of u, and $A > 0$ means the saturate effect of the predators or parasites at the high density of u. In [17], Wang and Yeh investigated the bifurcation of model (1.1) with reaction diffusion.

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The spruce budworm generally completes its life-cycle within one year in [13], and can be divided into four stages: egg, larva, cocoon and adult, among which the larva can be divided into six juvenile stages. From an egg to the second juvenile stage, it accounts for about three quarters of the entire reproductive cycle. During this period, there is almost no activity of aphids, and birds mainly prey on the budworm after this period of time. In 2008, Vaidya and Wu in [15] treated the egg to the second instar larvae as immature stage and a time-delay differential equation with age structure, for the adult spruce budworm was established. The spruce budworm model with time delay can be described as follows

$$\mathrm{d}u(t) = [-ru(t) + \beta e^{-c\tau}u(t-\tau)e^{-\alpha u(t-\tau)} - \frac{Bu^2(t)}{A^2+u^2(t)}]\mathrm{d}t \tag{1.2}$$

with initial conditions

$$u(s) = \rho(s) \ for \ s \in [-\tau, 0], \quad \rho \in C([-\tau, 0], R_+). \tag{1.3}$$

Here, $R_+ = [0, +\infty)$, $u(t)$ represents the density of adult spruce budworm, $r > 0$ is the average mortality of adult budworm, $B > 0$ is the predation rate of birds, and $A > 0$ indicates the population density of spruce budworm when the predation rate reaches a half of the maximum. $\tau > 0$ is the time taken from birth to maturity. $c > 0$ is the average mortality of spruce budworm larvae. $b(u) = \beta ue^{-\alpha u}$ represents the birth function of spruce budworm and $\beta, \alpha > 0$.

Nevertheless, in the natural world, the spruce budworm model is inevitably more or less influenced by environment noises. In [10], May proposed that parameters in the system like environmental capacity and population growth rate had exhibited random fluctuations due to environmental noises. Many scholars have studied population behavior with random disturbances. For example, Peng and Zhang studied the stochastic predator-prey model with non-constant mortality rate in [11]. In [6], Liu and Zhu studied the stability of a budworm growth model with stochastic perturbation. Moreover, Song et al., [14] explored the dynamical behavior of stochastic Beddington-DeAngelis predator-prey model with distributed delay. Therefore, in [1,16], the delayed differential equations of stochastic spruce budworm are more suitable to model the data of (1.2). We assume that the average mortality r is disturbed with $r \to r - \sigma \mathrm{d}B(t)$, $B(t)$ is one-dimensional Brownian motion with $B(0) = 0$ defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$, and σ^2 is the intensity of the noise. In addition, consider $c = 0$, so that the mortality of spruce budworm larvae is 0. Then, we get the following stochastic function

$$\mathrm{d}u(t) = \left[-ru(t) + \beta u(t-\tau)e^{-\alpha u(t-\tau)} - \frac{Bu^2(t)}{A^2+u^2(t)}\right]\mathrm{d}t + \sigma u(t)\mathrm{d}B(t). \tag{1.4}$$

This article focuses on the following aspects. In Section 2, we provide preliminaries results. In Section 3, some properties of (1.4), such as the existence and uniqueness global positive solution of (1.4) with initial values (1.3), ultimate boundedness and the sample Lyapunov exponent of (1.4), are given. Examples and numerical simulations are carried out to support the results in Section 4. Finally, we briefly summarize the work of this paper.

2. Preliminaries

Definition 2.1. System (1.4) is said to be ultimate bounded in mean, if there is a positive constant M independent of the initial conditions (1.3) such that

$$\limsup_{t\to+\infty} E|x(t)| \le M.$$

Lemma 2.1. *For any initial value $u(0)$, system (1.4) has a unique local positive solution solution $x(t)$, $t \in [-\tau, \tau_e)$, where τ_e is explosion time.*

Proof. Defining a function $f(t)$ such that $f(t) = \ln u(t)$, by Itô's formula, we consider the group of equations

$$\begin{cases} \mathrm{d}f(t) = \mathrm{d}\ln u(t) = \left[-re^{f(t)} - \dfrac{\sigma^2}{2} + \beta e^{f(t-\tau)}e^{-\alpha e^{f(t-\tau)}} - \dfrac{Be^{2f(t)}}{A^2+e^{2f(t)}}\right]\mathrm{d}t + \sigma \mathrm{d}B(t) \\ f(s) = \ln \rho(s) \ for \ s \in [-\tau, 0], \quad \rho \in C([-\tau,0], R_+). \end{cases} \tag{2.1}$$

Since the coefficients of (1.4) satisfy the locally Lipschitz condition, there is a unique max local solution $u(t)$ on $[-\tau, \tau_e)$ for any given initial value, and τ_e is called explosion time. We get $u(t) = e^{f(t)}$, which is the unique local positive solution to system (1.4) with initial value (1.3). □

3. Main results

3.1. Existence and uniqueness of the global positive solution

Theorem 3.1. *If $r > \frac{\sigma^2}{2}$, then for any initial value $u_0 \in R_+$, system (1.4) has a unique global positive solution $u(t)$ for $t \in [-\tau, +\infty)$.*

Proof. By Lemma 2.1, to show the solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficient large such that $u_0 \in \left[\frac{1}{k_0}, k_0\right]$, for each integer $k \ge k_0$, define the stoping time

$$\tau_k = \inf\left\{t \in [-\tau, \tau_e) : \min u(t) \le \frac{1}{k} \ or \ \max u(t) \ge k\right\},$$

and we set $\inf\emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$ and $\tau_\infty \le \tau_e$, a.s. If we can prove $\tau_\infty = \infty$ *a.s.*, then $\tau_e = \infty$ *a.s.*, for all $t \ge 0$.

If the statement is false, then there are constants $T \ge 0$ and $\varepsilon \in (0,1)$, and $\exists k_1 \ge k_0$, k_1 is an integer. Therefore,

$$P(\tau_k \le T) \ge \varepsilon \ for \ all \ k \ge k_1.$$

Define a C^2-function $V(u) = u - 1 - \ln u + u^2$. The positiveness of $V(u)$ is confirmed by $u - 1 - \ln u + u^2 > 0$, for all $u > 0$.

By Itô's formula, we have

$$\mathrm{d}V(u) = LV(u)\mathrm{d}t + (\sigma u(t) - \sigma + 2\sigma u^2(t))\mathrm{d}B(t),$$

where

$$\begin{aligned} LV(u) =& \left(1 - \frac{1}{u(t)} + 2u(t)\right)\left(-ru(t) - \frac{Bu^2(t)}{A^2+u^2(t)} + \beta u(t-\tau)e^{-\alpha u(t-\tau)}\right) \\ &+ \frac{1}{2}\left(\frac{1}{u^2(t)} + 2\right)\sigma^2 u^2(t) \\ \leq& \frac{\sigma^2}{2} + r + \frac{Bu(t)}{A^2+u^2(t)} - 2ru^2(t) + \frac{\beta}{e\alpha} + \frac{2\beta u(t)}{e\alpha} + \sigma^2 u^2(t) \\ \leq& \frac{\sigma^2}{2} + r + \frac{B}{2A} + \frac{\beta}{e\alpha} + \frac{2u(t)\beta}{e\alpha} - 2ru^2(t) + \sigma^2 u^2(t). \end{aligned} \tag{3.1}$$

Taking $-(2r-\sigma^2)u^2(t) + \frac{2\beta}{\alpha e}u(t) \leq \frac{\beta^2}{\alpha^2 e^2(2r-\sigma^2)}$ leads to

$$LV(u) \leq \frac{\sigma^2}{2} + r + \frac{B}{2\sqrt{A}} + \frac{\beta}{\alpha e} + \frac{\beta^2}{\alpha^2 e^2(2r-\sigma^2)} \leq M. \tag{3.2}$$

Combining (3.1) and (3.2), we can obtain

$$\mathrm{d}V(u) \leq M\mathrm{d}t + (\sigma u(t) - \sigma + 2\sigma u^2(t))\mathrm{d}B(t). \tag{3.3}$$

Integrating both sides of (3.1) from 0 to $\tau_k \wedge T$ and then taking the expectations yield

$$EV(t) \leq E\int_0^{\tau_k \wedge T} M\mathrm{d}t + EV(u(0)) \leq MT + EV(u(0)).$$

Let $\Omega_k = \tau_k \leq T$, we have $P(\Omega_k) \geq \varepsilon$. Therefore, for every $\omega \in \Omega_k$, there is at least one of the $x(\tau_k, \omega)$ equaling either k or $\frac{1}{k}$. Note that $V(x(\tau_k))$ is no less than $(k-1-\ln k) \wedge (\frac{1}{k} - 1 - \ln\frac{1}{k})$. Consequently,

$$\infty > MT + E(V(0)) \geq E(V(u)) \geq E(1_{\Omega_k(\omega)}, V(u_k)) \geq \varepsilon(k-1-\ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right),$$

where $1_{\Omega_k(\omega)}$ is the indicator function of ω_k. Taking $k \to \infty$, we can induce $\varepsilon(k-1-\ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right) \to +\infty$, which is a contradiction. Hence, we can get $\tau_\infty = \infty$.

Now, the conclusion has been confirmed.

□

3.2. Ultimately bounded in mean

Theorem 3.2. *Let $r > \frac{\sigma^2}{2}$ hold, and the global solution of $x(t)$ has the properties*

$$\lim_{t\to+\infty} Eu(t) \leq \frac{\beta}{re\alpha} \quad \textit{and} \quad \limsup_{t\to+\infty} \frac{1}{t}\int_0^t Eu^2(s)\mathrm{d}s \leq \frac{4\beta^2}{\alpha^2 e^2(2r-\sigma^2)^2}.$$

From Definition 2.1, system (1.4) is ultimately bounded in mean.

Proof. From the inequality $\sup_{u\in R} ue^{-u} = \frac{1}{e}$, we can derive

$$\mathrm{d}u(t) \leq \left[-ru(t) - \frac{Bu^2(t)}{A^2+u^2(t)} + \frac{\beta}{e\alpha}\right]\mathrm{d}t + u(t)\sigma\mathrm{d}B(t).$$

Making use of the Itô's formula, we have

$$\begin{aligned}\mathrm{d}(e^{rt}u(t)) &\leq re^{rt}u(t)\mathrm{d}t - re^{rt}u(t)\mathrm{d}t - e^{rt}\frac{Bu^2(t)}{A^2+u^2(t)}\mathrm{d}t + e^{rt}\frac{\beta}{e\alpha}\mathrm{d}t + e^{rt}\sigma u(t)\mathrm{d}B(t)\\ &\leq \frac{\beta}{e\alpha}e^{rt}\mathrm{d}t + e^{rt}\sigma u(t)\mathrm{d}B(t).\end{aligned} \tag{3.4}$$

Integrating it from 0 to t on both sides of (3.4) and then taking the expectations, it follows from that

$$e^{rt}Eu(t) \leq u(0) + \int_0^t \frac{\beta}{e\alpha}e^{rs}\mathrm{d}s = u(0) + \frac{\beta}{re\alpha}(e^{rt}-1).$$

This implies $\limsup_{t\to\infty} Eu(t) \leq \frac{\beta}{re\alpha}$.

Taking Itô's formula again results in

$$\begin{aligned}\mathrm{d}u^2(t) &= \left[2u(t)(-ru(t) - \frac{Bu^2(t)}{A^2+u^2(t)} + \beta u(t-\tau)e^{-u(t-\tau)\alpha})\right]\mathrm{d}t + 2u^2(t)\sigma\mathrm{d}B(t)\\ &\leq \left[-(2r-\sigma^2)u^2(t) + \frac{2u(t)\beta}{e\alpha}\right]\mathrm{d}t + 2u^2(t)\sigma\mathrm{d}B(t).\end{aligned} \tag{3.5}$$

Integrating from 0 to t on both sides of (3.5) and then taking the expectations, it follows from that

$$0 \leq Eu^2(t) \leq u^2(0) + \int_0^t E\left[-2r - \sigma^2u^2(t) + \frac{2\beta u(t)}{e\alpha}\right]\mathrm{d}t. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$(r - \frac{\sigma^2}{2})u^2(t) + \frac{2u(t)\beta}{e\alpha} \leq \frac{2\beta^2}{(2r-\sigma^2)}. \tag{3.7}$$

Obviously, we have

$$(r - \frac{\sigma^2}{2})\int_0^t Eu^2(s)\mathrm{d}s \leq u^2(0) + \frac{2\beta^2}{(2r-\sigma^2)}.$$

That is,

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t Eu^2(s)\mathrm{d}s \leq \frac{4\beta^2}{\alpha^2e^2(2r-\sigma^2)^2}.$$

Now, the result has been confirmed.

□

3.3. The sample Lyapunov exponent

Theorem 3.3. *Let $u(t)$ be the positive solution of system (1.4) with the initial value $u(0) \in R_+$, then the sample Lyapunov exponent of the solution of (1.4) should not be greater than $\frac{N}{2}$ with the condition $r > \frac{\sigma^2}{2}$. That is,*

$$\limsup_{t\to\infty}\frac{\ln u(t)}{t} \leq \frac{N}{2},\ a.s.$$

Proof. Using Itô's formula, we can obtain the following inequality

$$\begin{aligned}\ln(1+u^2(t)) \leq & \ln(1+u^2(0)) + \int_0^t \frac{1}{1+u^2(s)}\left[-2ru^2(s) + 2\frac{\beta}{\alpha e}u(s)\right]\mathrm{d}s \\ & + \int_0^t \sigma^2 u^2(s)\frac{1-u^2(s)}{(1+u^2(s))^2}\mathrm{d} + 2\int_0^t \frac{\sigma^2 u^2(s)}{1+u^2(s)}\mathrm{d}B(s) \\ = & \ln(1+u^2(0)) + \int_0^t \frac{1}{1+u^2(s)}\left[-(2r-\sigma^2)u^2(s) + 2\frac{\beta}{\alpha e}u(s)\right]\mathrm{d}s \\ & - \int_0^t \sigma^2 u^2(s)\frac{u^2(s)}{(1+u^2(s))^2}\mathrm{d}s + 2\int_0^t \frac{\sigma u^2(s)}{1+u^2(s)}\mathrm{d}B(s).\end{aligned}$$

On the one hand, using the exponential martingale inequality in [9], we have

$$P\left\{\sup_{0\leq t\leq n}\left[2\int_0^t \frac{\sigma u^2(s)}{1+u^2(s)}\mathrm{d}B(s) - \int_0^t \frac{\sigma^2 u^4(s)}{(1+u^2(s))^2}\mathrm{d}s\right] > 2\ln n\right\} \leq \frac{1}{n^2}, \quad n \geq 0.$$

According to the Borel-Cantelli lemma, it yields that for almost all $\omega \in \Omega$, and there is a random integer $n_0 = n_0(\omega) \geq 1$ such that

$$\sup_{0\leq t\leq n}\left[2\int_0^t \frac{\sigma^2 u^2(s)}{1+u^2(s)}\mathrm{d}B(s) - \int_0^t \frac{\sigma^2 u^4(s)}{(1+u^2(s))^2}\mathrm{d}s\right] \leq 2\ln n, \quad n \geq n_0.$$

That is, $2\int_0^t \frac{\sigma^2 u^2(s)}{1+u^2(s)}\mathrm{d}B(s) \leq \int_0^t \frac{\sigma^2 u^4(s)}{(1+u^2(s))^2}\mathrm{d}s + 2\ln n$, for all $0 \leq t \leq n$, $n \geq n_0$ is almost for sure.

On the other hand, $-(2r-\sigma^2)u^2(s) + 2\frac{\beta}{\alpha e}u(s) \leq N(1+u^2(s))$ and $-(2r-\sigma^2)u^2(s) + 2\frac{\beta}{\alpha e}u(s) \leq \frac{\beta^2}{\alpha^2 e^2(2r-\sigma^2)}$, where $N = \min\{\frac{\beta^2}{\alpha^2 e^2(2r-\sigma^2)}, \frac{\beta}{\alpha e}\}$.

Combining the above conditions, we can obtain the following conclusion

$$\ln(1+u^2(s)) \leq \ln(1+u^2(0)) + Nt + 2\ln n,\ 0 \leq t \leq n,\ n \geq n_0\ a.s.$$

This implies

$$\frac{1}{t}\ln(1+u^2(t)) \leq \frac{1}{n-1}\left[\ln(1+u^2(0)) + Nn + 2\ln n\right].$$

Therefore, we get the sample Lyapunov exponent of system (1.4),

$$\limsup_{t\to\infty}\frac{1}{t}\ln u(t) \leq \limsup_{t\to\infty}\frac{1}{2t} \leq \limsup_{t\to\infty}\frac{1}{2(n-1)}\left[\ln(1+u^2(0)) + Nn + 2\ln n\right] = \frac{N}{2}. \tag{3.8}$$

Now, the conclusion has been confirmed. □

4. Numerical simulation

In this section, we will carry out some numerical simulations for system to illustrate our main result.

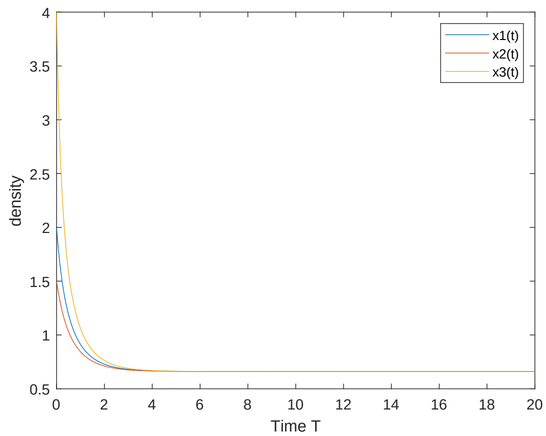

Figure 1. The trajectories of (1.4) for initial value 1.5, 2, 4 without white noise

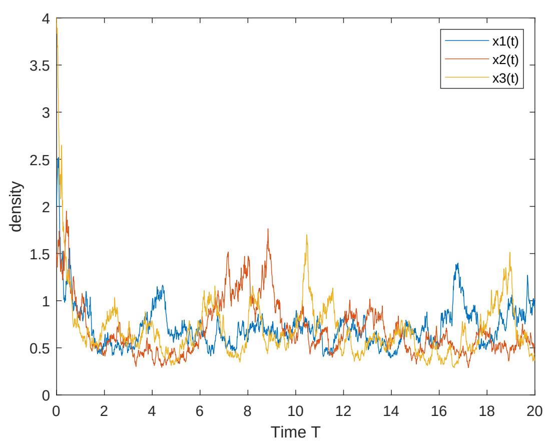

Figure 2. The trajectories of (1.4) for initial value 1.5, 2, 4 with white noise

Example 4.1. Combining the data in [15], letting $r = 0.19$, $A = 139496$, $B = 105700$, $\beta = 194594$, $\tau = 1$, $\alpha = 11.857$, $\sigma = 0.5$ and $r > \frac{\sigma^2}{2}$ hold, we can obtain the following stochastic spruce budworm delayed equation: $\mathrm{d}u(t) = [-0.19u(t) - \frac{105700u^2(t)}{139496+u^2(t)} + u(t-1)e^{-11.857u(t-1)}]\mathrm{d}t + 0.5u(t)\mathrm{d}B(t)$, with initial values $u_1(0) = 1.5$, $u_2(0) = 2$, $u_3(0) = 4$. According to Milstein's numerical method [4], the overcome is illustrated by the numerical simulation in Figure 2. We can clearly see that the images of the solutions with three different initial values gradually approach a fixed range and are always positive. Figure 1 shows the change in population size without white noise. In contrast to Figure 1, the change in spruce budworm population is no longer smooth after the addition of random perturbation factors.

5. Conclusion

In this paper, we have studied and explored the stochastic spruce budworm growth model with time delay. We show that when $r > \frac{\sigma^2}{2}$, the equation has a unique and global positive solution with ultimate boundedness in mean. In addition, we have estimated the sample Lyapunov exponent of (1.4). Throughout the paper, it can be seen that white noise affects many properties of the spruce budworm model.

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