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Triple Positive Solutions of Boundary Value Problems for High-order Fractional Differential Equation at Resonance with Singularities*

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Abstract In this paper, we investigate the existence of triple positive solutions of boundary value problems for high-order fractional differential equation at resonance with singularities by using the fixed point index theory and the Leggett-Williams theorem. The spectral theory and some new height functions are also employed to establish the existence of triple positive solutions. The nonlinearity involved is arbitrary fractional derivative, and permits singularity.

Keywords Triple positive solution, Fractional differential equation, Resonance, Singularity.

MSC(2010) 34B18, 34A08, 34F15, 35A21.

1. Introduction

In this paper, we consider the following boundary value problem for high-order fractional differential equation (FBVP for short) at resonance

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t),D_{0+}^{\beta}x(t))=0,\ 0<t<1, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0,\ D_{0+}^{\beta}x(1)=\lambda\int_0^{\eta}m(t)D_{0+}^{\beta}x(t)dt, \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative, $m\in L[0,1]$ is nonnegative and may be singular at $t=0$ and $t=1$, $n-1<\alpha<n$, $n\geq 3$, $\alpha-(n-1)>\beta>0$, $0<\eta\leq 1$, $\lambda\int_0^{\eta}m(t)t^{\alpha-\beta-1}dt=1$, and the nonlinearity $f(t,x,y)$ permits singularities at $t=0,1$ and $x=y=0$.

We note that (1.1) happens to be at resonance in the case that the corresponding homogeneous boundary value problem

$$\begin{cases} D_{0+}^{\alpha}x(t)=0,\ 0<t<1, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0,\ D_{0+}^{\beta}x(1)=\lambda\int_0^{\eta}m(t)D_{0+}^{\beta}x(t)dt, \end{cases} \tag{1.2}$$

has a solution $ct^{\alpha-1}, c\in\mathbb{R}, c\neq 0$, as a nontrivial solution. That is, the derivative operator in the boundary value problem is not invertible.

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The research on the boundary value problem of fractional differential equation is mainly carried out to extend some effective methods in integer differential equation such as special function method, Laplace transform, Fourier transform, iterative method, operator calculus, combination method and fixed point theorem (see [3, 13, 18, 22]). The main results focus on the linear non-resonant boundary value problem (see [1, 21, 23–27]). The study on the resonant boundary value problem is not perfect.

The resonant boundary value problem is that the homogeneous problem corresponding to the equation has a nontrivial solution. That is to say, the derivative operator in the boundary value problem is not invertible. Boundary value problem at resonance, as a kind of special boundary value problem in differential equations, has a very wide application prospect in the fields of celestial mechanics, aerodynamics, material mechanics, fluid mechanics and so on (see [2, 6, 9, 12, 16]).

For integer-order boundary value problems at resonance, the methods used by researchers generally include topological degree theory, Mawhin's overlap degree theorem, fixed point theorem, critical point theorem, function analysis theory, phase plane analysis method and so on (see [4, 5, 7, 10, 17, 20]). Compared with the boundary value problem of integer-order differential equations at resonance, the study of fractional order started late. The first one was Kosmatov's application of Mawhin's continuity theorem to study the following fractional-order three-point boundary value problem at resonance (see [8])

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), u'(t)) = 0, \ 0 < t < 1, \\ D_{0+}^{\alpha-2}u(0) = 0, \ \eta u(\xi) = u(1). \end{cases}$$

In most cases of the real life, it is necessary to solve the positive solution of the differential equation under the boundary value conditions, which requires sufficient theoretical proof. For the boundary value problems of integer or fractional-order differential equations at resonance, it is not difficult to find that no matter using Mawhin's overlap degree theorem or generalized continuity theorem by Professor Ge, we can only obtain the existence of solutions, but cannot guarantee the existence of positive solutions. The study on the positive solution of boundary value problems for fractional-order differential equations at resonance has only been paid attention to by researchers in the recent years. There are only a few pieces of literature (see [5, 8, 14, 15, 17, 20]). Yang and Wang applied Leggett-Williams fixed point theorem to study the result that the fractional boundary value problem at resonance has at least one positive solution (see [19])

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)) = 0, \ 0 < t < 1, \\ u(0) = 0, \ \eta u'(0) = u'(1). \end{cases}$$

As far as we know, triple positive solutions of boundary value problems for high-order fractional differential equation at resonance with singularities has not been considered. Inspired by the work above, we aim to fill this gap. This paper is organized as follows. First, we reduce non-perturbed boundary value problems at resonance to the equivalent non-resonant perturbed problems with the same boundary conditions. Then, we derive the Green's function and corresponding properties. Finally, the existence of triple positive solutions is obtained by using the Leggett-Williams theorem and the fixed point index theory.

2. Illustration of proof

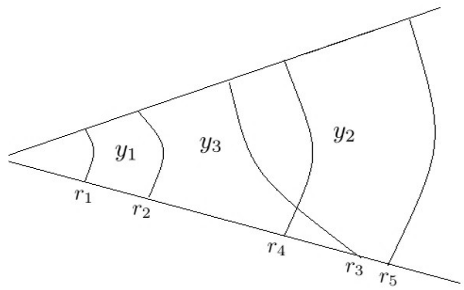

Figure 1. Illustration of proof

3. Preliminaries

In this section, we present some preliminaries and lemmas that will be used in this paper.

Let $D^{\beta}_{0+}x(t) = y(t)$. By the method of [15], it is easy to see that (1.1) is equivalent to

$$\begin{cases} D^{\alpha-\beta}_{0+}y(t) + f(t, I^{\beta}_{0+}y(t), y(t)) = 0, \quad 0 < t < 1, \\ I^{\beta}_{0+}y(0) = D^{1-\beta}_{0+}y(0) = D^{2-\beta}_{0+}y(0) = \cdots = D^{n-2-\beta}_{0+}y(0) = 0, \ y(1) = \lambda \int_0^{\eta} m(t)y(t)dt. \end{cases} \tag{3.1}$$

Let

$$g(\tau) = \sum_{j=0}^{+\infty} \frac{[(j+1)(\alpha-\beta)-n+1][(j+1)(\alpha-\beta)-n]\tau^j}{\Gamma((j+1)(\alpha-\beta))}.$$

It is easy to get $g'(\tau) > 0$ on $(0, +\infty)$ and

$$g(0) = \frac{(\alpha-\beta-n+1)(\alpha-\beta-n)}{\Gamma(\alpha-\beta)} < 0, \quad \lim_{\tau\to+\infty} g(\tau) = +\infty.$$

Therefore, there exists a unique root $\tau^* > 0$ such that

$$g(\tau^*) = 0.$$

Obviously, the resonant FBVP (3.1) is equivalent to the FBVP

$$\begin{cases} -D^{\alpha-\beta}_{0+}y(t) + \tau y(t) = f(t, I^{\beta}_{0+}y(t), y(t)) + \tau y(t), \quad 0 < t < 1, \\ I^{\beta}_{0+}y(0) = D^{1-\beta}_{0+}y(0) = D^{2-\beta}_{0+}y(0) = \cdots = D^{n-2-\beta}_{0+}y(0) = 0, \ y(1) = \lambda \int_0^{\eta} m(t)y(t)dt. \end{cases} \tag{3.2}$$

In this paper, we list the following assumptions.

(H_1) $\tau \in (0, \tau^*]$ is a constant.

(H_2) f is continuous on $(0,1) \times (0,+\infty) \times (0,+\infty)$.

For the sake of convenience, we use the notations

$$G(t)=\sum_{j=0}^{+\infty}\frac{\tau^{j}t^{(j+1)(\alpha-\beta)-1}}{\Gamma((j+1)(\alpha-\beta))},$$

$$K(t,s)=\frac{1}{G(1)}\begin{cases}G(t)G(1-s), & 0\le t\le s\le 1,\\ G(t)G(1-s)-G(t-s)G(1), & 0\le s\le t\le 1,\end{cases}$$

$$H(t,s)=K(t,s)+\frac{\lambda G(t)\int_0^\eta m(t)K(t,s)dt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}.$$

Lemma 3.1. *Assume $h\in L[0,1]$. Then, the unique solution of the problem*

$$\begin{cases}-D_{0+}^{\alpha-\beta}y(t)+\tau y(t)=h(t), \quad 0<t<1,\\ I_{0+}^{\beta}y(0)=D_{0+}^{1-\beta}y(0)=D_{0+}^{2-\beta}y(0)=\cdots=D_{0+}^{n-2-\beta}y(0)=0,\ y(1)=\lambda\int_0^\eta m(t)y(t)dt,\end{cases}\tag{3.3}$$

can be expressed by

$$y(t)=\int_0^1 H(t,s)h(s)ds.$$

Proof. By [7,11], the solution of (3.3) can be expressed by

$$y(t)=-\int_0^t G(t-s)h(s)ds+c_1G(t)+c_2G'(t)+\cdots+c_nG^{(n-1)}(t).$$

By $I_{0+}^{\beta}y(0)=D_{0+}^{1-\beta}y(0)=D_{0+}^{2-\beta}y(0)=\cdots=D_{0+}^{n-2-\beta}y(0)=0$, we have $c_2=c_3=\cdots=c_n=0$.

Then, we get

$$y(t)=-\int_0^t G(t-s)h(s)ds+c_1G(t),$$

$$y(1)=-\int_0^1 G(1-s)h(s)ds+c_1G(1).$$

By $y(1)=\lambda\int_0^\eta m(t)y(t)dt$, we obtain

$$c_1=\frac{\int_0^1 G(1-s)h(s)ds-\lambda\int_0^\eta m(t)\int_0^t G(t-s)h(s)dsdt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}.$$

Therefore, the solution of (3.3) is

$$\begin{aligned}&y(t)\\ &=-\int_0^t G(t-s)h(s)ds+\frac{\int_0^1 G(1-s)h(s)ds-\lambda\int_0^\eta m(t)\int_0^t G(t-s)h(s)dsdt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}G(t)\\ &=\frac{\int_0^1 G(t)G(1-s)h(s)ds-\int_0^t G(t-s)G(1)h(s)ds}{G(1)}-\frac{\int_0^1 G(t)G(1-s)h(s)ds}{G(1)}\\ &+\frac{\int_0^1 G(1-s)h(s)ds}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}G(t)-\frac{\lambda\int_0^\eta m(t)\int_0^t G(t-s)h(s)dsdt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}G(t)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 K(t,s)h(s)ds + \frac{\lambda \int_0^\eta m(t) \int_0^1 G(t)G(1-s)h(s)dsdt}{G(1)[G(1) - \lambda \int_0^\eta m(t)G(t)dt]} G(t) \\
&- \frac{\lambda \int_0^\eta m(t) \int_0^t G(t-s)G(1)h(s)dsdt}{G(1)[G(1) - \lambda \int_0^\eta m(t)G(t)dt]} G(t) \\
&= \int_0^1 K(t,s)h(s)ds + \frac{\lambda \int_0^\eta m(t) \int_0^1 K(t,s)h(s)dsdt}{G(1) - \lambda \int_0^\eta m(t)G(t)dt} G(t) \\
&= \int_0^1 H(t,s)h(s)ds.
\end{aligned} \tag{3.4}
$$

This completes the proof.

Lemma 3.2. *Let $s^* \in (0,1)$ such that $s^* = (1-s^*)^{\alpha-\beta-2}$. Then, $K(t,s)$ satisfies*

(1) $K(t,s) \geq \rho_1 s(1-s)^{\alpha-\beta-1}(1-t)t^{\alpha-\beta-1}, \quad \forall t,s \in [0,1]$;

(2) $K(t,s) \leq \rho_2 s(1-s)^{\alpha-\beta-1}, \quad \forall t,s \in [0,1]$, *where*

$$
\rho_1 = \frac{1}{G(1)[\Gamma(\alpha-\beta)]^2}, \quad \rho_2 = \frac{[G'(1)]^2}{G(1)s^*}.
$$

Proof. For $t \in [0,1]$, it is easy to check

$$
G(t) = \sum_{j=0}^{+\infty} \frac{\tau^j t^{(j+1)(\alpha-\beta)-1}}{\Gamma((j+1)(\alpha-\beta))} = \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \sum_{j=1}^{+\infty} \frac{\tau^j t^{(j+1)(\alpha-\beta)-1}}{\Gamma((j+1)(\alpha-\beta))} \geq \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \tag{3.5}
$$

and

$$
G(t) = \sum_{j=0}^{+\infty} \frac{\tau^j t^{(j+1)(\alpha-\beta)-1}}{\Gamma((j+1)(\alpha-\beta))} \leq t^{\alpha-\beta-1} \sum_{j=0}^{+\infty} \frac{\tau^j}{\Gamma((j+1)(\alpha-\beta))} = t^{\alpha-\beta-1}G(1). \tag{3.6}
$$

By direct calculation, we have

$$
G(1) < G'(1). \tag{3.7}
$$

Case (i): $0 \leq t \leq s \leq 1$.
By (3.5), we get

$$
K(t,s) = \frac{G(t)G(1-s)}{G(1)} \geq \frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{G(1)[\Gamma(\alpha-\beta)]^2} \geq \frac{s(1-s)^{\alpha-\beta-1}(1-t)t^{\alpha-\beta-1}}{G(1)[\Gamma(\alpha-\beta)]^2}.
$$

By (3.6) and (3.7), we obtain

$$
K(t,s) = \frac{G(t)G(1-s)}{G(1)} \leq t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}G(1) \leq \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-\beta-1}.
$$

Case (ii): $0 \leq s \leq t \leq 1$.
Let

$$
G(t) = t^{n-2}G_1(t), \quad G_1(t) = \sum_{j=0}^{+\infty} \frac{\tau^j t^{(j+1)(\alpha-\beta)-n+1}}{\Gamma((j+1)(\alpha-\beta))}.
$$

It is easy to get

$$G_1'(t)=\sum_{j=0}^{+\infty}\frac{[(j+1)(\alpha-\beta)-n+1]\tau^j t^{(j+1)(\alpha-\beta)-n}}{\Gamma((j+1)(\alpha-\beta))}>0,$$

$$G_1''(t)=t^{(j+1)(\alpha-\beta)-n-1}g(\tau)\le t^{(j+1)(\alpha-\beta)-n-1}g(\tau^*)=0.$$

Therefore,

$$\begin{aligned}&\frac{\partial}{\partial s}[G_1(t)G_1(1-s)-G_1(t-s)G_1(1)]\\&=-G_1(t)G_1'(1-s)+G_1'(t-s)G_1(1)\\&\ge G_1'(1-s)[G_1(1)-G_1(t)].\end{aligned}\tag{3.8}$$

Integrating (3.8) with respect to s, we get

$$\begin{aligned}&G_1(t)G_1(1-s)-G_1(t-s)G_1(1)\\&\ge\int_0^s G_1'(1-s)[G_1(1)-G_1(t)]\\&=[G_1(1)-G_1(1-s)][G_1(1)-G_1(t)]>0,\end{aligned}$$

which implies

$$G_1(t)G_1(1-s)\ge G_1(t-s)G_1(1).\tag{3.9}$$

Then, by (3.5) and (3.9), we have

$$\begin{aligned}K(t,s)&=\frac{G(t)G(1-s)-G(t-s)G(1)}{G(1)}\\&=\frac{t^{n-2}(1-s)^{n-2}G_1(t)G_1(1-s)-(t-s)^{n-2}G_1(t-s)G_1(1)}{G(1)}\\&\ge\frac{[t^{n-2}(1-s)^{n-2}-(t-s)^{n-2}]G_1(t)G_1(1-s)}{G(1)}\\&\ge\frac{[t(1-s)-(t-s)]t^{n-3}(1-s)^{n-3}G_1(t)G_1(1-s)}{G(1)}\\&\ge\frac{s(1-t)t^{n-3}(1-s)^{n-3}t^{\alpha-\beta-n+1}(1-s)^{\alpha-\beta-n+1}}{G(1)[\Gamma(\alpha-\beta)]^2}\\&\ge\frac{s(1-s)^{\alpha-\beta-1}(1-t)t^{\alpha-\beta-1}}{G(1)[\Gamma(\alpha-\beta)]^2}.\end{aligned}$$

Noticing that $G'(t)$ is nondecreasing on $[0,1]$, then we can get

$$\frac{\partial}{\partial s}K(t,s)=\frac{-G(t)G'(1-s)+G'(t-s)G(1)}{G(1)}\le\frac{G'(1-s)[G(1)-G(t)]}{G(1)}.\tag{3.10}$$

Integrating (3.10) with respect to s, we have

$$\begin{aligned}K(t,s)&\le\int_0^s\frac{G'(1-s)[G(1)-G(t)]}{G(1)}ds\\&=\frac{[G(1)-G(1-s)][G(1)-G(t)]}{G(1)}\\&\le\frac{[G'(1)]^2s(1-s)}{G(1)}.\end{aligned}\tag{3.11}$$

By (3.6) and (3.7), we obtain

$$K(t,s) \le \frac{G(t)G(1-s)}{G(1)} \le G(1)t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} < G_1'(1)(1-s)^{\alpha-\beta-1}. \quad (3.12)$$

Let s^* be the unique root of the equation $s = (1-s)^{\alpha-\beta-2}$ on $[0,1]$. Then,

$$\min\{s, (1-s)^{\alpha-\beta-2}\} \le \frac{s(1-s)^{\alpha-\beta-2}}{s^*}. \quad (3.13)$$

Therefore, by (3.7) and (3.11)-(3.13), we get

$$\begin{aligned} K(t,s) &\le \min\{\frac{[G'(1)]^2 s(1-s)}{G(1)}, G'(1)(1-s)^{\alpha-\beta-1}\} \\ &\le \min\{s, (1-s)^{\alpha-\beta-2}\} \times \max\{\frac{G'(1)}{G(1)}, 1\} \times G'(1)(1-s) \\ &\le \frac{s(1-s)^{\alpha-\beta-2}}{s^*} \times \frac{G'(1)}{G(1)} \times G'(1)(1-s) \\ &= \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-\beta-1}. \end{aligned}$$

This completes the proof.

Lemma 3.3. *The function $H(t,s)$ satisfies*
(1) $H(t,s) \ge \omega_1 s(1-s)^{\alpha-\beta-1} t^{\alpha-\beta-1}$, $\forall t, s \in [0,1]$;
(2) $H(t,s) \le \omega_2 s(1-s)^{\alpha-\beta-1}$, $\forall t, s \in [0,1]$, *where*

$$\omega_1 = \frac{\rho_1(1-\lambda\int_0^\eta m(t)t^{\alpha-\beta}dt)}{\Gamma(\alpha-\beta)[G(1)-\lambda\int_0^\eta m(t)G(t)dt]}, \quad \omega_2 = \rho_2[1+\frac{\lambda G(1)\int_0^\eta m(t)dt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}].$$

Proof. Combining Lemma 3.2 with

$$\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \le G(t) \le t^{\alpha-\beta-1}G(1) \le G(1),$$

we have

$$\begin{aligned} H(t,s) &= K(t,s) + \frac{\lambda G(t)\int_0^\eta m(t)K(t,s)dt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt} \\ &\ge \frac{\lambda t^{\alpha-\beta-1}\int_0^\eta m(t)\rho_1 s(1-s)^{\alpha-\beta-1}(1-t)t^{\alpha-\beta-1}dt}{\Gamma(\alpha-\beta)[G(1)-\lambda\int_0^\eta m(t)G(t)dt]} \\ &= \frac{\rho_1(1-\lambda\int_0^\eta m(t)t^{\alpha-\beta}dt)}{\Gamma(\alpha-\beta)[G(1)-\lambda\int_0^\eta m(t)G(t)dt]} s(1-s)^{\alpha-\beta-1}t^{\alpha-\beta-1} \end{aligned}$$

and

$$\begin{aligned} H(t,s) &= K(t,s) + \frac{\lambda G(t)\int_0^\eta m(t)K(t,s)dt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt} \\ &\le \rho_2 s(1-s)^{\alpha-\beta-1} + \frac{\lambda G(t)\int_0^\eta m(t)\rho_2 s(1-s)^{\alpha-\beta-1}dt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt} \\ &\le \rho_2[1+\frac{\lambda G(1)\int_0^\eta m(t)dt}{G(1)-\lambda\int_0^\eta m(t)G(t)dt}]s(1-s)^{\alpha-\beta-1}. \end{aligned}$$

This completes the proof.

Let $E = C[0,1]$ be the Banach space with the maximum norm $\|x\| = \max\limits_{0\le t\le 1} |x(t)|$. Let

$$\chi_1(t) = \frac{\omega_1\Gamma(\alpha-\beta)}{\omega_2\Gamma(\alpha)}t^{\alpha-1}, \quad \chi_2(t) = \frac{\omega_1}{\omega_2}t^{\alpha-\beta-1}.$$

Define a cone

$$P = \{x \in E : x(t) \ge \chi_2(t)\|x\|, \quad t \in [0,1]\}.$$

Letting $0 < a < 1$, we denote

$$\chi^* = \min_{t\in[a,1]} \chi_2(t), \quad \vartheta(x) = \min_{t\in[a,1]} x(t), \quad x \in P.$$

Letting $\forall R \ge r > 0$, we set

$$\begin{aligned} &P_r = \{x \in P : \|x\| < r\},\\ &P(\vartheta, r, R) = \{x \in P : r \le \vartheta(x), \|x\| \le R\},\\ &\mathring{P}(\vartheta, r, R) = \{x \in P : r < \vartheta(x), \|x\| \le R\}. \end{aligned}$$

Define some height functions as follows

$$\begin{aligned} &\Phi(t,r,R) = \max\{f(t,x,y) + \tau y : r\chi_1(t) \le x \le \frac{Rt^\beta}{\Gamma(\beta+1)}, r\chi_2(t) \le y \le R\},\\ &\phi_1(t,r) = \min\{f(t,x,y) : r\chi_1(t) \le x \le \frac{rt^\beta}{\Gamma(\beta+1)}, r\chi_2(t) \le y \le r\},\\ &\phi_2(t,r,R) = \min\{f(t,x,y) + \tau y : \frac{rt^\beta}{\Gamma(\beta+1)} \le x \le \frac{Rt^\beta}{\Gamma(\beta+1)}, r \le y \le R\}. \end{aligned}$$

Lemma 3.4 ([4]). *Letting P be a cone in Banach space E, Ω be a bounded open set in E and $\theta \in \Omega$, $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator.*

(1) *If $\exists x_0 \in P \setminus \{\theta\}$ such that $x - Ax \ne \lambda x_0$, $\forall \lambda \ge 0$, $x \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 0$;*

(2) *If $Ax \ne \lambda x$, $\forall \lambda \ge 1$, $x \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 1$.*

Lemma 3.5 ([10]). *Let $A : \overline{K}_{r_3} \to P$ be a completely continuous operator. If there exist a concave positive functional ϑ with $\vartheta(x) \le \|x\|$ ($x \in P$) and the numbers $r_3 \ge r_2 > r_1 > 0$ satisfying the following conditions*

(1) *$\mathring{P}(\vartheta, r_1, r_2) \ne \emptyset$ and $\vartheta(Ax) > r_1$, if $x \in P(\vartheta, r_1, r_2)$;*

(2) *$Ax \in \overline{K}_{r_3}$, if $x \in P(\vartheta, r_1, r_3)$;*

(3) *$\vartheta(Ax) > r_1$ for all $x \in P(\vartheta, r_1, r_3)$ with $\|Ax\| > r_2$. Then, $i(A, \mathring{P}(\vartheta, r_1, r_3), \overline{K}_{r_3}) = 1$.*

4. Main results

Theorem 4.1. *Assume that there exist the positive numbers $r_1 < r_2 < r_3 < r_4 \le r_5$ with $r_3 \le r_4\chi^*$ such that*

(A_1) *$f \in C[(0,1) \times (0, \frac{r_5}{\Gamma(\beta+1)}) \times (0, r_5)]$ with $f(t,x,y) \ge -\tau y$;*

(A_2) *$\Phi(t, r_1, r_5) \in L[0,1]$;*

(A_3) $\phi_1(t, r_1) \geq 0$;
(A_4) $\int_0^1 \Phi(s, r_2, r_2)s(1-s)^{\alpha-\beta-1}ds < r_2\omega_2^{-1}$;
(A_5) $\int_a^1 \phi_2(s, r_3, r_4)s(1-s)^{\alpha-\beta-1}ds > r_3[\chi^*\omega_2]^{-1}$, $a \in (0,1)$;
(A_6) $\int_0^1 \Phi(s, r_3, r_5)s(1-s)^{\alpha-\beta-1}ds \leq r_5\omega_2^{-1}$.
Then, the resonant $FBVP$ (1.1) has at least three positive solutions.

Proof. Denote two operators

$$Ay(t) = \int_0^1 H(t,s)[f(s, I_{0+}^{\beta}y(s), y(s)) + \tau y(s)]ds,$$
$$Ly(t) = \int_0^1 H(t,s)y(s)ds.$$

Therefore, $L : P \to P$ is a completely continuous linear operator. Then, by means of the Krein-Rutmann theorem and Lemma 3.2, we know that the spectral radius of L is $r(L) = \tau^{-1}$, and $\psi(t) = t^{\alpha-\beta-1}$ is the corresponding eigenfunction. That is, $L\psi = \tau^{-1}\psi$. For any $y \in \overline{P}_{r_5} \setminus P_{r_1}$, we have $r_1\chi_2(t) \leq y(t) \leq r_5$ and $r_1\chi_1(t) \leq I_{0+}^{\beta}y(t) \leq \frac{r_5 t^{\beta}}{\Gamma(\beta+1)}$.

Next, we will prove $A : \overline{P}_{r_5} \setminus P_{r_1} \to P$ is completely continuous.

(i) $\forall y \in \overline{P}_{r_5} \setminus P_{r_1}$, by Lemma 3.3, we obtain

$$\begin{aligned}
Ay(t) &= \int_0^1 H(t,s)[f(s, I_{0+}^{\beta}y(s), y(s)) + \tau y(s)]ds \\
&\geq \int_0^1 \omega_1 s(1-s)^{\alpha-\beta-1}t^{\alpha-\beta-1}[f(s, I_{0+}^{\beta}y(s), y(s)) + \tau y(s)]ds \\
&= \chi_2(t)\int_0^1 \omega_2 s(1-s)^{\alpha-\beta-1}[f(s, I_{0+}^{\beta}y(s), y(s)) + \tau y(s)]ds \\
&\geq \chi_2(t)\max_{0\leq t\leq 1}\int_0^1 H(t,s)[f(s, I_{0+}^{\beta}y(s), y(s)) + \tau y(s)]ds \\
&= \chi_2(t) \parallel Ay \parallel .
\end{aligned}$$

Then, $A : \overline{P}_{r_5} \setminus P_{r_1} \to P$ is well defined.

(ii) Assume $\{y_n\} \subset \overline{P}_{r_5} \setminus P_{r_1}$ and $\parallel y_n - y_0 \parallel \to 0 (n \to +\infty)$. Then, $r_1 \leq \parallel y_n \parallel \leq r_5$, $n = 0, 1, 2, \ldots$.

For $\forall \varepsilon > 0$, by virtue of the absolute continuity of integral, $\exists \delta \in (0, \frac{1}{2})$ is such that

$$\int_0^{\delta} \omega_2 s(1-s)^{\alpha-\beta-1}\Phi(t, r_1, r_5)ds < \frac{\varepsilon}{6},$$

$$\int_{1-\delta}^{1} \omega_2 s(1-s)^{\alpha-\beta-1}\Phi(t, r_1, r_5)ds < \frac{\varepsilon}{6}.$$

Since f is uniformly continuous on $[\delta, 1-\delta] \times [r_1\chi_1(t), \frac{r_5 t^{\beta}}{\Gamma(\beta+1)}] \times [r_1\chi_2(t), r_5]$ and $\parallel y_n - y_0 \parallel \to 0$, there exists $N > 0$ such that, for $\forall n > N$, we get

$$|f(t, I_{0+}^{\beta}y_n(t), y_n(t)) - f(t, I_{0+}^{\beta}y_0(t), y_0(t))| < \frac{\varepsilon}{3\int_0^1 \Phi(t, r_1, r_5)ds}, \quad t \in [\delta, 1-\delta].$$

Therefore,

$$\begin{aligned}
&\|Ay_n - Ay_0\| \\
&\le \max_{0\le t\le 1}\int_0^1 H(t,s)|f(t,I_{0+}^{\beta}y_n(t),y_n(t)) - f(t,I_{0+}^{\beta}y_0(t),y_0(t))|ds \\
&\le \int_0^1 \omega_2 s(1-s)^{\alpha-\beta-1}|f(t,I_{0+}^{\beta}y_n(t),y_n(t)) - f(t,I_{0+}^{\beta}y_0(t),y_0(t))|ds \\
&\le 2\int_0^{\delta}\omega_2 s(1-s)^{\alpha-\beta-1}\Phi(t,r_1,r_5)ds + 2\int_{1-\delta}^1 \omega_2 s(1-s)^{\alpha-\beta-1}\Phi(t,r_1,r_5)ds \\
&\quad + \int_{\delta}^{1-\delta}\omega_2 s(1-s)^{\alpha-\beta-1}|f(t,I_{0+}^{\beta}y_n(t),y_n(t)) - f(t,I_{0+}^{\beta}y_0(t),y_0(t))|ds \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

Then, A is continuous.

(iii) For $\forall D\subset \overline{P}_{r_5}\setminus P_{r_1}$ is a bounded set. Then, $r_1\chi_2(t)\le\| y \|\le r_5, \forall y\in D$. We obtain

$$\begin{aligned}
|Ay(t)| &= |\int_0^1 H(t,s)[f(s,I_{0+}^{\beta}y(s),y(s))+\tau y(s)]ds| \\
&\le \int_0^1 H(t,s)|f(s,I_{0+}^{\beta}y(s),y(s))+\tau y(s)|ds.
\end{aligned}$$

By $s, I_{0+}^{\beta}y(s), y(s)$ is continuous and bounded on $(0,1)\times[r_1\chi_1(t),\frac{r_5t^{\beta}}{\Gamma(\beta+1)}]\times[r_1\chi_2(t),r_5]$, and we get that $f(s,I_{0+}^{\beta}y(s),y(s))$ is also bounded. It is clear that $H(t,s)$ is bounded. Then, $A(D)$ is uniformly bounded.

It is obvious that $H(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. For $\forall\varepsilon>0$, there exists $\delta>0$ such that, for $\forall t_1,t_2\in[0,1], |t_1-t_2|<\delta$, we obtain

$$|H(t_1,s)-H(t_2,s)|<\frac{\varepsilon}{\int_0^1\Phi(t,r_1,r_5)ds}.$$

Therefore,

$$\begin{aligned}
&|Ay(t_1)-Ay(t_2)| \\
&\le \int_0^1 |H(t_1,s)-H(t_2,s)|[f(s,I_{0+}^{\beta}y(s),y(s))+\tau y(s)]ds \\
&\le \int_0^1 \frac{\varepsilon}{\int_0^1\Phi(t,r_1,r_5)ds}\Phi(t,r_1,r_5)ds=\varepsilon.
\end{aligned}$$

Then, $A(D)$ is equicontinuous.

By virtue of the Arzela-Ascoli theorem, we know that A is compact.

Then, $A:\overline{P}_{r_5}\setminus P_{r_1}\to P$ is completely continuous. By the extension theorem of the completely continuous operator, A can be extended to a completely continuous $\widetilde{A}:P\to P$. For the sake of convenience, we still denote it as A.

For any $y\in\partial P_{r_2}$, we have $r_2\chi_2(t)\le y(t)\le r_2$ and $r_2\chi_1(t)\le I_{0+}^{\beta}y(t)\le\frac{r_2t^{\beta}}{\Gamma(\beta+1)}$. By Lemma 3.3 and (A_4), we obtain

$$Ay(t)\le\omega_2\int_0^1 s(1-s)^{\alpha-\beta-1}\Phi(s,r_2,r_2)ds<r_2,$$

which implies $Ay \neq \lambda y$, $\forall \lambda \geq 1$. By Lemma 3.4, we have

$$i(A, P_{r_2}, P) = 1. \tag{4.1}$$

For any $y \in \partial P_{r_5}$, we have $r_5\chi_2(t) \leq y(t) \leq r_5$ and $r_5\chi_1(t) \leq I_{0+}^{\beta} y(t) \leq \frac{r_5 t^{\beta}}{\Gamma(\beta+1)}$. By Lemma 3.3 and (A_6), we obtain

$$Ay(t) \leq \omega_2 \int_0^1 s(1-s)^{\alpha-\beta-1}\Phi(s, r_3, r_5)ds < r_5.$$

Similarly, we get

$$i(A, P_{r_5}, P) = 1. \tag{4.2}$$

Next, we will prove A has a positive fixed point on P_{r_2}. Suppose that A has no fixed point on ∂P_{r_1}. Then, we will prove if $\exists \psi \in P \setminus \{\theta\}$ such that

$$y - Ay \neq \lambda\psi, \quad \forall \lambda \geq 0, \quad y \in \partial P_{r_1}. \tag{4.3}$$

Otherwise, suppose that there exist $\lambda_0 > 0$ and $y_1 \in \partial P_{r_1}$ such that

$$y_1 - Ay_1 = \lambda_0 \psi.$$

Thus, $y_1 \geq \lambda_0\psi$. Set $\lambda^* = \sup\{\lambda : y_1 \geq \lambda\psi\}$. Then, $y_1 \geq \lambda^*\psi$. By (A_3), we get

$$Ay_1(t) = \int_0^1 H(t,s)[f(s, I_{0+}^{\beta} y_1(s), y_1(s)) + \tau y_1(s)]ds \geq \tau \int_0^1 H(t,s) y_1(s) ds = \tau L y_1.$$

Therefore,

$$y_1 = Ay_1 + \lambda_0\psi \geq \tau L y_1 + \lambda_0 \psi \geq \tau L(\lambda^*\psi) + \lambda_0\psi = (\lambda^* + \lambda_0)\psi,$$

which contradicts with the definition of λ^*, and then (4.3) holds. By Lemma 3.4, we have

$$i(A, P_{r_1} \cap P, P) = 0. \tag{4.4}$$

Therefore, (4.1) and (4.4) yield that A has a fixed point $y_1 \in P_{r_2} \setminus P_{r_1}$.

Then, we will prove $i(A, \mathring{P}(\vartheta, r_3, r_5), \overline{P}_{r_5}) = 1$.

(i) It is obvious that $\mathring{P}(\vartheta, r_3, r_4) \neq \emptyset$. For any $y \in P(\vartheta, r_3, r_4)$, we have $r_3 \leq y(t) \leq r_4$ and $\frac{r_3 t^{\beta}}{\Gamma(\beta+1)} \leq I_{0+}^{\beta} y(t) \leq \frac{r_4 t^{\beta}}{\Gamma(\beta+1)}$, for $t \in [a, 1]$. By virtue of Lemma 3.3 and (A_5), we obtain

$$\begin{aligned}
\vartheta(Ay) = \min_{t\in[a,1]} Ay(t) &= \min_{t\in[a,1]} \int_0^1 H(t,s)[f(s, I_{0+}^{\beta} y(s), y(s)) + \tau y(s)]ds \\
&\geq \min_{t\in[a,1]} \omega_1 t^{\alpha-\beta-1} \int_0^1 s(1-s)^{\alpha-\beta-1}[f(s, I_{0+}^{\beta} y(s), y(s)) + \tau y(s)]ds \\
&= \min_{t\in[a,1]} \omega_2 \chi_2(t) \int_0^1 s(1-s)^{\alpha-\beta-1}[f(s, I_{0+}^{\beta} y(s), y(s)) + \tau y(s)]ds \\
&\geq \omega_2 \chi^* \int_a^1 s(1-s)^{\alpha-\beta-1}\phi_2(s, r_3, r_4)ds > r_3.
\end{aligned}$$

(ii) For any $y \in P(\vartheta, r_3, r_5)$, we have $r_3\chi_2(t) \le y(t) \le r_5$ and $r_3\chi_1(t) \le I^{\beta}_{0+}y(t) \le \frac{r_5 t^{\beta}}{\Gamma(\beta+1)}$, for $t \in [0,1]$. By virtue of Lemma 3.3 and (A_6), we get

$$\begin{aligned} Ay(t) &= \int_0^1 H(t,s)[f(s, I^{\beta}_{0+}y(s), y(s)) + \tau y(s)]ds \\ &\le \omega_2 \int_0^1 s(1-s)^{\alpha-\beta-1}[f(s, I^{\beta}_{0+}y(s), y(s)) + \tau y(s)]ds \\ &\le \omega_2 \int_0^1 s(1-s)^{\alpha-\beta-1}\Phi(s, r_3, r_5)ds \le r_5. \end{aligned}$$

Therefore, $Ay \in \overline{P}_{r_5}$.

(iii) For any $y \in P(\vartheta, r_3, r_5)$ with $\|Ay\| > r_4$, noticing $r_3 \le r_4\chi^*$, we have

$$\vartheta(Ay) = \min_{t\in[a,1]} (Ay)(t) \ge \chi^*\|Ay\| > \chi^* r_4 \ge r_3.$$

By Lemma 3.5, we obtain

$$i(A, \mathring{P}(\vartheta, r_3, r_5), \overline{P}_{r_5}) = 1. \tag{4.5}$$

By (4.1), (4.2) and (4.5), we get

$$i(A, P_{r_5} \setminus (\mathring{P}(\vartheta, r_3, r_5) \cup P_{r_2}), \overline{P}_{r_5}) = -1. \tag{4.6}$$

(4.5) and (4.6) yield that A has two fixed points $y_2 \in \mathring{P}(\vartheta, r_3, r_5)$ and $y_3 \in P_{r_5} \setminus (\mathring{P}(\vartheta, r_3, r_5) \cup P_{r_2})$.

It is clear that these fixed points are three positive solutions for FBVP (1.1). For a better comprehension, see Figure 1.

5. Example

Example 5.1. Consider the following the resonant FBVP

$$\begin{cases} D^{2.2}_{0+}x(t) + f(t, x(t), D^{0.1}_{0+}x(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \quad D^{0.1}_{0+}x(1) = 0.5\int_0^{0.8} t^{-0.2} D^{0.1}_{0+}x(t)dt, \end{cases} \tag{5.1}$$

with

$$f(t,x,y) =$$

$$\begin{cases} \dfrac{t^9 x^{-\frac{1}{4}}}{60\sqrt{1-t}} + \dfrac{y^{-\frac{1}{4}}}{150} + \dfrac{(1-t)^9 y^{-\frac{1}{4}}}{60\sqrt{t}} - \dfrac{y}{5}, & (t,x,y) \in (0,1)\times(0,+\infty)\times(0,1], \\ \dfrac{(1-t)^9}{60\sqrt{t}} + \dfrac{t^9 x^{-\frac{1}{4}}}{60\sqrt{1-t}} + \dfrac{y^5}{150} - \dfrac{y}{5}, & (t,x,y) \in (0,1)\times(0,+\infty)\times(1,6], \\ \dfrac{(1-t)^9}{60\sqrt{t}} + \dfrac{t^9 x^{-\frac{1}{4}}}{60\sqrt{1-t}} + \dfrac{y^{\frac{1}{2}} + 6^5 - \sqrt{6}}{150} - \dfrac{y}{5}, & (t,x,y) \in (0,1)\times(0,+\infty)\times(6,+\infty). \end{cases}$$

Equation (5.1) is obtained by taking the following values from equation (1.1), where $\alpha = 2.2$, $\beta = 0.1$, $n = 3$, $\lambda = 0.5$, $\eta = 0.8$, $m(t) = t^{-0.2}$. By $\frac{34.56\tau^3}{\Gamma(8.4)} \le g(\tau) - [-\frac{0.09}{\Gamma(2.1)} + \frac{2.64\tau}{\Gamma(4.2)} + \frac{14.19\tau^2}{\Gamma(6.3)}] \le \sum_{j=3}^{+\infty} \frac{34.56\tau^j}{\Gamma(8.4)}$, we obtain that a unique solution to $g(\tau)$ is $\tau^* \in (0.24, 0.25)$. Let $\tau = 0.2$ and $a = 0.8$. Simple computation shows $G(1) = 0.9814$, $G'(1) = 1.1412$, $s^* = 0.8351$, $\rho_1 = 0.9303$, $\rho_2 = 1.5891$, $\omega_1 = 0.9915$, $\omega_2 = 2.5888$, $\chi_1(t) = 0.3638t^{1.2}$, $\chi_2(t) = 0.383t^{1.1}$, $\chi^* = 0.2996$, $\chi^*\omega_2 = 0.7756$.

Let $r_1 = 0.0625$, $r_2 = 1$, $r_3 = 6$, $r_4 = 90$, $r_5 = 1764$. Obviously, (A_1), (A_2) and (A_3) hold. By direct calculation, we can get

$$\omega_2 \int_0^1 \Phi(s, r_2, r_2) s(1-s)^{1.1} ds \approx 0.0115 < r_2,$$

$$\chi^*\omega_2 \int_{0.8}^1 \phi_2(s, r_3, r_4) s(1-s)^{1.1} ds \approx 8.0818 > r_3,$$

$$\omega_2 \int_0^1 \Phi(s, r_3, r_5) s(1-s)^{1.1} ds \approx 1647.4027 \le r_5,$$

implying that (A_4), (A_5) and (A_6) hold. It follows from Theorem 4.1 that high-order FBVP at resonance has at least three positive solutions.

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