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Poincaré Bifurcation from an Elliptic Hamiltonian of Degree Four with Two-saddle Cycle*

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Abstract In this paper, we consider Poincaré bifurcation from an elliptic Hamiltonian of degree four with two-saddle cycle. Based on the Chebyshev criterion, not only one case in the Liénard equations of type $(3, 2)$ is discussed again in a different way from the previous ones, but also its two extended cases are investigated, where the perturbations are given respectively by adding $\varepsilon y(d_0 + d_2 v^{2n})\frac{\partial}{\partial y}$ with $n \in \mathbb{N}^+$ and $\varepsilon y(d_0 + d_4 v^4 + d_2 v^{2n+4})\frac{\partial}{\partial y}$ with $n = -1$ or $n \in \mathbb{N}^+$, for small $\varepsilon > 0$. For the above cases, we obtain all the sharp upper bound of the number of zeros for Abelian integrals, from which the existence of limit cycles at most via the first-order Melnikov functions is determined. Finally, one example of double limit cycles for the latter case is given.

Keywords Perturbed Hamiltonian system, Poincaré bifurcation, Abelian integral, Chebyshev criterion.

MSC(2010) 34C05, 37C07.

1. Introduction

This paper deals with Liénard equations in the following form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = P(v) + yQ(v), \end{cases} \tag{1.1}$$

where P and Q are two polynomials in the variable v, and if deg $P = m$ and deg $Q = n$, equations (1.1) are called Liénard system of type (m, n). When it comes to the famous 16th problem of Hilbert, equations (1.1) have been studied extensively on asking for an upper bound on the number of limit cycles [18, 19], especially in respect to the weak 16th problem of Hilbert [1]. For studying the limit cycles in certain vector fields with specific degree for (1.1), we construct the perturbed Hamiltonian system by setting $Q(v) = \varepsilon g(v)$ with $\varepsilon > 0$, but it is small as follows

$$\frac{\mathrm{d}v}{\mathrm{d}t} = H_y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -H_v + \varepsilon g(v)y, \tag{1.2}$$

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where $H(v,y) = \frac{y^2}{2} + \int P(v)\mathrm{d}v$ is a Hamiltonian polynomial function of degree $m+1$. Thus, we can investigate Poincaré bifurcation of limit cycles via calculating the zeros of the elliptic integrals, i.e., Abelian integrals obtained by integrating the related 1-form $yq(v)\mathrm{d}v$ over the compact level curves of the Hamiltonian H.

There are some effective research methods and many good results which are obtained for the zeros of Abelian integrals and Poincaré bifurcation of system (1.2) (see the review literature [16], the recent papers [2,10,25] and references therein). As for the existence, uniqueness and number of limit cycles, it is worth noting that Jiang and Han [15] extended the investigation from the continuous nonlinear Liénard-type differential systems (1.1) to the discontinuous ones, and discussed the qualitative properties of the crossing limit cycles.

When $m+n \le 4$, type (m,n) of the Liénard equations (1.1), except for type $(1,3)$, has been given a complete study, and the corresponding Abel integrals were also proved to have one at most zero [3,16]. As for its type (m,n) with $m+n=5$, the comprehensive and detailed studies on type $(3,2)$ have been carried out by Dumortier and Li in a series of papers [4–7], and the corresponding supremum of the zero number of Abel integral was obtained respectively for all the five different cases.

We know that type $(3,2)$ corresponds to small perturbations of Hamiltonian vector fields with an elliptic Hamiltonian of degree four, and the perturbations are given by adding $\varepsilon y(x^2+\beta x+\alpha)\frac{\partial}{\partial y}$ for small $\varepsilon > 0$. It is worth mentioning that, for Two-saddle Cycle Case (A) and Saddle Loop Case (B) presented in [4], after linear rescaling their Hamiltonians are given by the following functions

$$H(v,y) = \frac{y^2}{2} - \frac{1}{4}v^4 - \frac{(\lambda-1)}{3}v^3 + \frac{\lambda}{2}v^2, \tag{1.3}$$

where $\lambda \ge 1$, and $\lambda = 1$ corresponds to Case (A), while $\lambda > 1$ corresponds to Case (B). Case (A) is rather simple as the limiting situation of Case (B), and its Abel integral was proved to have one at most zero in [14] and [4] respectively.

Here, we shall develop a utilized approach based on Chebyshev criterion [9,21] to prove the above conclusion once again, and we shall investigate the extended Case (A) more, namely for $\lambda = 1$ with the following more generic form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = (v+1)v(v-1) + \varepsilon g(v)y, \end{cases} \tag{1.4}$$

where $g(v) = \sum_{i=0}^{2m} d_i v^i$, $m \in \mathbb{N}^+, d_i \in \mathbb{R}$. Obviously, the $g(v)$ in the perturbed part of system (1.4) is no longer restricted to quadratic polynomial, and the following two classes of $g(v)$ will be discussed here

$$\begin{aligned} &g_a(v) = d_0 + d_{2n}v^{2n},\ n = 1,2,\cdots, \\ &g_b(v) = d_0 + d_4 v^4 + d_{2n+4}v^{2n+4},\ n = -1,1,2,\cdots. \end{aligned} \tag{1.5}$$

The Picard-Focus equations and Riccati equations in algebraic geometry were applied as the important research approaches in [4], which are still often used (see, for example, [13,17]). Here, by using the method of Chebyshev criterion presented in [9,21], and via strict proof of symbolic calculation, we determine the zero-point number of the Abelian integrals, i.e., the first-order Melnikov functions. Many applications have shown that these skills are uniquely effective in studying the existence of limit cycles at most via the Poincaré bifurcation for the first-order Melnikov functions (see, for example, [20,23,24]). In addition, as one of the techniques used here, the judgment criteria of sign invariant region for bivariate polynomial function, proposed in [8], is the key to the study.

The paper is organized as follows. In Section 2, we introduce some preparatory knowledge of determining the number of zeros for Abelian integral, and give our main conclusion. In Section 3, we provide the strict proof of the main conclusion through the symbolic computation in the three subsections. In the final section, the existence of global limit cycles at most via the first-order Melnikov functions is proved, and one specific example of two limit cycles is given to illustrate the conclusion obtained.

2. Preliminary methods and main result

From $(1.3)_{\lambda=1}$, system $(1.4)_{\varepsilon=0}$ has the Hamiltonian function

$$H(v,y)=\frac{y^2}{2}+\frac{1}{4}v^2(2-v^2). \tag{2.1}$$

Then, we have the phase portraits of system $(1.4)_{\varepsilon=0}$ in Figure 1.

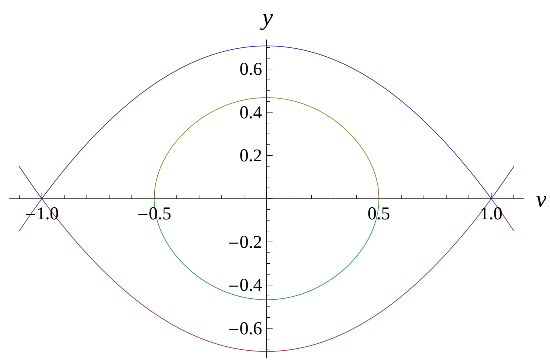


Figure 1. The phase portraits of system $(1.4)_{\varepsilon=0}$

We can find the closed orbits sketched in Figure 1 defined by the following function

$$\Gamma_h: H(v,y)=h,\ \ h\in(0,\frac{1}{4}). \tag{2.2}$$

The period annulus Γ_h of this center (i.e., the origin) is just inside the two heteroclinic orbits connecting two-saddle points $(1,0)$ and $(-1,0)$, whose projection is the open interval $(-1,1)$.

In addition, from the Hamiltonian function (2.1) of the unperturbed system $(1.4)_{\varepsilon=0}$, we have

$$H(v,y)=A(v)+B(v)y^2, \tag{2.3}$$

where $A(v)=\frac{1}{4}v^2(2-v^2)$ and $B(v)=\frac{1}{2}$. Additionally, the Melnikov function

(Abelian integral) corresponding to system (1.4) is

$$I(h) = \oint_{\Gamma_h} g(v)y\mathrm{d}v = \oint_{\Gamma_h} (d_0 + d_1 v + \cdots + d_{2m}v^{2m})y\mathrm{d}v \\ = d_0\tilde{I}_0(h) + d_1\tilde{I}_1(h) + \cdots + d_{2m}\tilde{I}_{2m}(h), \tag{2.4}$$

where $\tilde{I}_i(h) = \oint_{\Gamma_h} v^i y\mathrm{d}v$, $0 \le i \le 2m$, and the readers can note that the number of simple zeros of $I(h)$ corresponds to the number of limit cycles of system (1.4) by Poincaré bifurcation.

Obviously, according to the symmetry of the integral region and the integrand being an odd function, we can easily calculate and obtain the following result

$$\tilde{I}_i(h) = \oint_{\Gamma_h} v^{2k-1}y\mathrm{d}v = 0, k = 1, 2, \cdots . \tag{2.5}$$

Thus, we will not consider the odd terms of $g(v)$ in system (1.4).

In the following, let us recall the relevant definitions and conclusions on the Chebyshev criteria to determine the zero number of Abelian integrals, and also refer to the literature [9, 21] for details.

Definition 2.1. Assuming that $\{f_0(x), f_1(x), \cdots, f_{n-1}(x)\}$ is an ordered set of analytic functions on an open interval $\mathbb{L}$ of $\mathbb{R}$, this set is called an extended complete Chebyshev system (ECT-system), if, for all $i = 1, 2, \cdots, n$, any nontrivial linear combination

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{i-1} f_{i-1}(x)$$

has at most $i-1$ isolated zeros on $\mathbb{L}$ counted with multiplicities.

Lemma 2.1 (See [9]). *$\{f_0(x), f_1(x), \cdots, f_{n-1}(x)\}$ is an ECT-system on $\mathbb{L}$, if and only if, for each $i = 1, 2, \cdots, n$, the Wronskian $W[f_0(x), f_1(x), \cdots, f_{i-1}(x)] \neq 0$ for all $x \in \mathbb{L}$.*

To prove our conclusions, we will apply the following lemmas.

Lemma 2.2 (See [9]). *Let Γ_h be an oval inside the level curve $\{A(v) + \frac{1}{2}y^2 = h\}$ (such as (2.3)), and a function F such that F/A' is analytic at $v = 0$. Then, for any $k \in \mathbb{N}$,*

$$\oint_{\Gamma_h} F(v)y^{k-2}\mathrm{d}v = \oint_{\Gamma_h} G(v)y^k\mathrm{d}v$$

where $G(v) = \frac{1}{k}\left(\frac{F}{A'}\right)'(v)$.

Lemma 2.3 (See [9]). *Let us consider the Abelian integrals*

$$I_i(h) = \oint_{\Gamma_h} f_i(v)y^{2s-1}\mathrm{d}v, \quad i = 0, 1, \cdots, n-1$$

where, for each $h \in \mathbb{L}$, Γ_h is the oval surrounding the origin inside the level curve $\{A(v) + \frac{1}{2}y^2 = h\}$(such as (2.3)). Let σ be the involution associated to $A(v)$, and we define

$$\ell_i(v) := \frac{f_i(v)}{A'(v)} - \frac{f_i(\sigma(v))}{A'(\sigma(v))}.$$

Then, $\{I_0, I_1, I_2, \cdots, I_{n-1}\}$ is an ECT-system on $\mathbb{L}$, if $s > n-2$, and $\{\ell_0, \ell_1, \cdots, \ell_{n-1}\}$ is a CT-system on $(0, v_0^+)$.

We also introduce the following lemma given by [8], namely, the judgment criteria of sign invariant region for bivariate polynomial function, which will be used to prove our main conclusion.

Lemma 2.4 (See [8]). *Set $\Omega = \mathbb{R}$ and let*

$$G_\theta(x) = g_n(\theta)x^n + g_{n-1}(\theta)x^{n-1} + \cdots + g_1(\theta)x + g_0(\theta) \tag{2.6}$$

be a family of real polynomials also polynomially depending on a real parameter θ. Assume that there exists an open interval $L \subset \mathbb{R}$ such that: (i) there is some $\theta_0 \in L$ such that $G_{\theta_0}(x) > 0$ on Ω; (ii) for all $\theta \in L$, the discriminant of G_θ with respect to x is not equal to zero; (iii) for all $\theta \in L$, $g_n(\theta) \neq 0$. Then, for all $\theta \in L$, $G_\theta(x) > 0$ on Ω.

Next, our main conclusion is given in the following theorem, whose proof will be completed in the following two sections.

Theorem 2.1. *For the perturbed system (1.4), when $g(v) = g_a(v)$ in (1.5), the zero-point number of Abelian integral can reach one and one at most, taking into account the multiplicity; when $g(v) = g_b(v)$ in (1.5), the zero-point number of Abelian integral can reach two and two at most, taking into account the multiplicity.*

3. Proof of the main conclusion

We will apply symbolic computation to prove the main conclusion of Theorem 2.1 step by step strictly.

First, we quickly figure out

$$\tilde{I}_i(h) = \oint\limits_{\Gamma_h} v^i y \mathrm{d}v = \frac{1}{h}\oint\limits_{\Gamma_h} (A(v) + \frac{1}{2}y^2)v^i y \mathrm{d}v = \frac{1}{h}\oint\limits_{\Gamma_h} A(v)v^i y \mathrm{d}v + \frac{1}{2h}\oint\limits_{\Gamma_h} v^i y^3 \mathrm{d}v,$$

where $i = 0, 2k$, $k \in \mathbb{N}^+$. Then, applying Lemma 2.2, we obtain

$$\oint\limits_{\Gamma_h} A(v)v^i y \mathrm{d}v = \frac{1}{3}\oint\limits_{\Gamma_h} \left(\frac{Av^i}{A'}\right)' y^3 \mathrm{d}v.$$

If we write

$$\tilde{I}_0(h) = \frac{1}{h}I_0(h), \quad \tilde{I}_{2k}(h) = \frac{1}{h}I_{2k}(h),$$

where

$$I_0(h) = \oint\limits_{\Gamma_h} f_0(v)y^3 \mathrm{d}v, \quad I_{2k}(h) = \oint\limits_{\Gamma_h} f_{2k}(v)y^3 \mathrm{d}v,$$

then we have $f_0(v)$ and $f_{2k}(v)$ as follows

$$f_i(v) = -\frac{v^{i-1}\left(2vAA'' - 2iAA' - 5vA'^2\right)}{6A'^2}, \quad i = 0, 2k, \; k \in \mathbb{N}^+. \tag{3.1}$$

Furthermore, we apply Lemma 2.3, yielding

$$\ell_i(v) = \frac{f_i(v)}{A'(v)} - \frac{f_i(\sigma(v))}{A'(\sigma(v))}, \tag{3.2}$$

where $i = 0, 2k,\ k \in \mathbb{N}^+$, and $\sigma(v)$ is the involution associated to $A(v)$. Actually, $\sigma(v)$ and v together serve as the two abscissa of the intersections of the period annulus Γ_h with v-axis. Due to the symmetry with respect to y-axis for the phase diagram of $H(v, y) = h$, we have $\varphi = \sigma(v) = -v$.

Subsequently, we will finish the proof for the main conclusion of Theorem 2.1 in three steps, i.e., the following three subsections divided.

3.1. Zeros of Φ_0 and Φ_2

According to (3.1) and (3.2), we easily obtain that

$$\ell_0 = -\tfrac{8-13v^2+7v^4}{6v(v^2-1)^3}, \qquad \ell_{2\mathrm{n}} = -\tfrac{\left(2nv^4-6nv^2+4n+7v^4-13v^2+8\right)\left(v^{2n}+(-v)^{2n}\right)}{12(v-1)^3v(v+1)^3},$$

and directly calculate the Wronskian $W[\ell_0(v)]$ and $W[\ell_0(v), \ell_{2n}(v)]$, yielding

$$\begin{aligned}
\Phi_0(v) &:= W[\ell_0(v)] = \ell_0(v),\\
\Phi_{2n}(v) &:= W[\ell_0(v), \ell_{2n}(v)]\\
&= \tfrac{n\left(14nv^6-54nv^4+68nv^2-32n+49v^6-117v^4+140v^2-64\right)\left(v^{2n}+(-v)^{2n}\right)}{36(v-1)^5v^3(v+1)^5}.
\end{aligned} \tag{3.3}$$

Naturally, Φ_2 has the following form

$$\Phi_2(v) = W[\ell_0(v), \ell_2(v)] = \tfrac{63v^6-171v^4+208v^2-96}{18v(v^2-1)^5}, \tag{3.4}$$

and more it is found to have only one zero-point v_* in the interval $(0, 1)$, namely,

$$v_* \in (\underbrace{0.96225935214\cdots 8382, 0.96225935214\cdots 8383}_{10^{-50}}).$$

In fact, applying Sturm's Theory, we can verify this result strictly. Therefore, $\{\ell_0(v), \ell_2(v)\}$ is not a CT-system on $(0, v_0^+)$ with $v_0^+ = 1$. Then, we cannot utilize Lemma 2.3 to prove that $\{I_0, I_2\}$ is an ECT-system on $(0, 1/4)$.

Thus, we shall try to calculate $\tilde{I}_0$ and $\tilde{I}_2$ directly, yielding

$$\begin{aligned}
\tilde{I}_0(h) = \oint_{\Gamma_h} y\mathrm{d}v = \tfrac{1}{3(1-4h)^{1/4}}\sqrt{\tfrac{c_1}{h}}\{(1+\sqrt{(1-4h)}\mathrm{EllipticE}[\tfrac{1-\sqrt{1-4h}}{1+\sqrt{1-4h}}]\\
-(1-4h+\sqrt{(1-4h)}\mathrm{Elliptick}[\tfrac{1-\sqrt{1-4h}}{1+\sqrt{1-4h}}]\}
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\tilde{I}_2(h) = \oint_{\Gamma_h} v^2y\mathrm{d}v = -\tfrac{4}{15(1-4h)^{1/4}}\sqrt{\tfrac{c_1}{h}}\{(-1+3h)(1+\sqrt{(1-4h)})\\
\cdot\,\mathrm{EllipticE}[\tfrac{1-\sqrt{1-4h}}{1+\sqrt{1-4h}}] + (1-4h+(1-3h)\sqrt{(1-4h)})\mathrm{Elliptick}[\tfrac{1-\sqrt{1-4h}}{1+\sqrt{1-4h}}]\},
\end{aligned} \tag{3.6}$$

where $c_1 = 2(\sqrt{1-4h} - 1 + 4h)$. Let $\tau = \sqrt{1-4h}$, then $h = \frac{1}{4}\sqrt{1-\tau^2}$, $0 < \tau < 1$, yielding

$$\begin{aligned}
\tilde{I}_0'(h) &= 2\sqrt{\tfrac{2}{1+\tau}}\mathrm{EllipticK}\left[\tfrac{1-\tau}{1+\tau}\right],\\
\tilde{I}_2'(h) &= -2\sqrt{2}\sqrt{1+\tau}\left(\mathrm{EllipticE}\left[\tfrac{1-\tau}{1+\tau}\right] - \mathrm{EllipticK}\left[\tfrac{1-\tau}{1+\tau}\right]\right).
\end{aligned} \tag{3.7}$$

In addition, by computing, we obtain

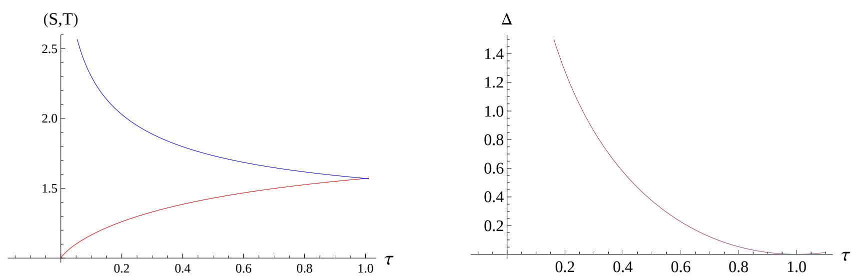
$$\left[\frac{\tilde{I}_2(h)}{\tilde{I}_0(h)}\right]' = \frac{\Delta}{\tilde{I}_0^2},$$

where

$$\Delta = \tilde{I}_0\tilde{I}_2' - \tilde{I}_0'\tilde{I}_2 = \tfrac{8}{15}\{5(1+\tau)\mathrm{EllipticE}\left[\tfrac{1-\tau}{1+\tau}\right]^2 \\ - 2(2+5\tau+\tau^2)\mathrm{EllipticE}\left[\tfrac{1-\tau}{1+\tau}\right]\mathrm{EllipticK}\left[\tfrac{1-\tau}{1+\tau}\right] + 2\tau(2+\tau)\mathrm{EllipticK}\left[\tfrac{1-\tau}{1+\tau}\right]^2\}.$$

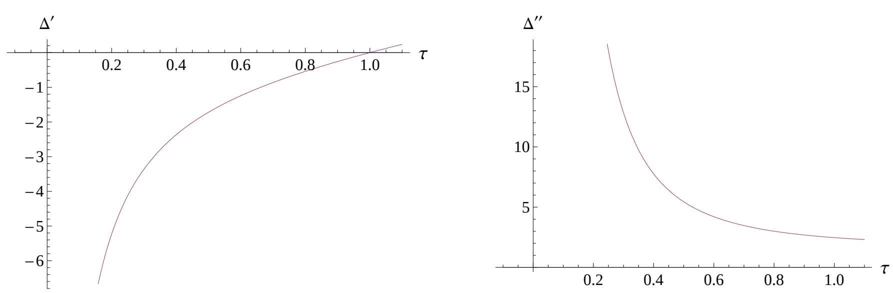
If we set $S = S(\tau) = \mathrm{EllipticE}\left[\frac{1-\tau}{1+\tau}\right]$, $T = T(\tau) = \mathrm{EllipticK}\left[\frac{1-\tau}{1+\tau}\right]$, then it is easily verified that on the interval $0 < \tau < 1$, $S(\tau)$ and $T(\tau)$ are monotone increasing and decreasing respectively, and $S(1) = T(1) = \frac{\pi}{2}, S(1) = 1, T(0) = +\infty$, as shown in Figure 2(i). If we let the ratio $\frac{T}{S} = K$, then we easily get $K \geq 1$ on the interval $0 < \tau < 1$.

For the above Δ, we can apply the method of numerical simulation to obtain the preliminary judgment: $\Delta > 0$ on the interval $0 < \tau < 1$, as shown in Figure 2(ii). However, unfortunately, we cannot find a quick way to prove this conclusion. Here, a tedious one has to be utilized.



(i) (ii)

Figure 2. The function images of S, T and Δ with respect to $\tau \in (0,1)$ in (i) and (ii) respectively



(i) (ii)

Figure 3. The function images of Δ' and Δ'' with respect to $\tau \in (0,1)$ in (i) and (ii) respectively

Next, we calculate the first and second derivatives of $\Delta(\tau)$ as follows

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\tau} = \frac{16}{15\tau(\tau-1)}[(1-2\tau^2)S^2 + 2\tau(\tau^2+\tau-1)ST + \tau^2(1-2\tau)T^2] := \Delta' \tag{3.8}$$

and

$$\tfrac{\mathrm{d}\Delta'}{\mathrm{d}\tau} = \tfrac{16S^2}{15(\tau+1)\tau^2(\tau-1)^2} f_\Delta := \Delta'', \tag{3.9}$$

where $f_\Delta = (\tau^4+1) + \tau(2\tau^4 - 4\tau^3 - \tau^2 + \tau - 2)K - \tau^2(2\tau^3 - 4\tau^2 + \tau - 1)K^2$. Then, we try to strictly prove that under the condition of $\tau \in (0,1)$, $\Delta'' > 0$ holds, which further proves that Δ' monotone increasing, as shown in Figure 3(i) and Figure 3(ii).

In fact, from $K \in [1, +\infty)$, let $K = 1 + \beta^2$, where $-\infty < \beta < +\infty$. Then, we obtain

$$\begin{aligned} f_\Delta = &-2\left(\beta^4+\beta^2\right)\tau^5 + \left(2\beta^2+1\right)^2\tau^4 \\ &- \left(\beta^4+3\beta^2+2\right)\tau^3 + \left(\beta^4+3\beta^2+2\right)\tau^2 - 2\left(\beta^2+1\right)\tau + 1. \end{aligned}$$

Furthermore, we compute the discriminant of f_Δ with respect to β,

$$\begin{aligned} &\delta_1(\tau): \\ &= \mathrm{Discriminant}[f_\Delta, \beta] \\ &= -16S^{12}(\tau-1)^6\tau^{10}\left(\tau^2+1\right)\left(2\tau^3-4\tau^2+\tau-1\right)\left(4\tau^4-8\tau^2-11\right)^2, \end{aligned} \tag{3.10}$$

and discover that $\delta_1(\tau)$ has never one zero point on the interval $(0,1)$. Thus, we take any value on the interval $(0,1)$, for example, $\tau = 1/2$, to substitute it into f_Δ, and yield

$$f_\Delta = \frac{5\beta^4}{16} - \frac{7\beta^2}{16} + \frac{5}{16} > 0, \ \ \forall\beta \in (-\infty, +\infty).$$

According to Lemma 2.4, we conclude as follows.

Lemma 3.1. *When $\tau \in (0,1)$, $\Delta'' > 0$ always holds, That is, Δ' is a monotone increasing function in τ.*

Moreover, when $\tau = 1$, $\Delta'(1) = 0$, we have $\Delta' < 0$, $\forall\tau \in (0,1)$. From this, $\Delta(\tau)$ is a monotone decreasing function, and when $\tau = 1$, $\Delta(1) = 0$. Thus, we obtain that $\Delta > 0$ always holds for all $\tau \in (0,1)$, which indicates $\left[\frac{I_2(h)}{I_0(h)}\right]' > 0$ for all $h \in (0, 1/4)$. Therefore, we have the following.

Lemma 3.2. *For the perturbed system (1.4), when $g(v) = g_a(v)$ with $n = 1$ in (1.5), the maximum number of zeros of Abelian integral is one at most on $(0, 1/4)$.*

3.2. Zeros of Φ_0 and Φ_{2n} for the cases of $n \geq 2$

Now, we investigate zeros of Φ_0 and Φ_{2n} in (3.3) with $n \geq 2$ at the same time. Namely, we check whether the conditions of Lemma 2.3 are satisfied. If its conditions hold, then $\{I_0, I_{2n}\}$ is an ECT-system on $(0, 1/4)$.

First, from the expression of $\Phi_0(v)$ in (3.3) as follows

$$\Phi_0(v) = \ell_0(v) = -\frac{8 - 13v^2 + 7v^4}{6v(v^2-1)^3},$$

we easily verify that for Φ_0, and there is not a zero on the interval $0 < v < 1$.

Next, we try to find the zeros of Φ_{2n} in (3.3). In order to study whether Φ_{2n} has zeros, we only need to consider whether the following factor $q(v,n)$ of Φ_{2n} has zeros,

$$q(v,n) = 14nv^6 - 54nv^4 + 68nv^2 - 32n + 49v^6 - 117v^4 + 140v^2 - 64.$$

If we make the variable substitution: $v = \frac{\beta^2}{\beta^2+1}$ where $\beta \in (-\infty, +\infty)$, the reader can note that the value range of v is unchanged and still $(0,1)$. Then, we have

$$q(v,n) = \frac{d_{1n}}{(\beta^2+1)^6},$$

where $d_{1n} = -4n\beta^{12} - 28n\beta^{10} - 126n\beta^8 - 368n\beta^6 - 412n\beta^4 - 192n\beta^2 - 32n + 8\beta^{12} - 58\beta^{10} - 237\beta^8 - 720\beta^6 - 820\beta^4 - 384\beta^2 - 64$. By computing the discriminant of d_{1n} with respect to β,

$$\text{Discriminant}[d_{1n}, \beta] = 44135276348230860800000\,\delta(n), \tag{3.11}$$

where $\delta(n) = (n-2)(n+2)^3(2n+7)^2(1100n^4 - 7700n^3 + 20655n^2 + 208670n + 255571)^4$, we try to find out certain intervals of n such that $\delta(n) \neq 0$. In fact, there exists only one positive real root for $\delta(n) = 0$, i.e., $n = 2$.

Randomly choosing a value of $n > 2$, for example, $n = 3$, we have

$$d_{1n}\big|_{n=3} = -(4\beta^{12} + 142\beta^{10} + 615\beta^8 + 1824\beta^6 + 2056\beta^4 + 960\beta^2 + 160) < 0,$$

where $\beta \in (-\infty, +\infty)$. From this specific example and Lemma 2.4, we obtain that when $n > 2$, $d_{1n} < 0$ always holds. In fact, when $n = 2$, we have

$$d_{1n} = -\left(2\beta^2 + 1\right)\left(57\beta^8 + 216\beta^6 + 620\beta^4 + 512\beta^2 + 128\right) < 0,$$

which always holds for all $\beta \in (-\infty, +\infty)$. From this, when $n \geq 2$, $q(v,n) < 0$ always holds for all $v \in (0,1)$, yielding Φ_{2n} always being negative. Therefore, $\{I_0, I_{2n}\}$ is an ECT-system on $(0, 1/4)$, according to Lemma 2.3, we conclude as follows.

Lemma 3.3. *For the perturbed system (1.4), when $g(v) = g_a(v)$ with $n \geq 2$ in (1.5), the maximum number of zeros of Abelian integral is one at most on* $(0, 1/4)$.

From the above Lemmas 3.2 and 3.3, we can prove the first part of Theorem 2.1. As for the fact that the number can reach one, it will be proved in the following section.

3.3. Zeros of Φ_0, Φ_4 and Φ_{2n+4}

In this subsection, we will investigate the zeros of Φ_0, Φ_4 and Φ_{2n+4} at the same time, where $n = -1, 1, 2, \cdots$. Similarly, from (3.1) and (3.2), it is easy to obtain

$$\begin{aligned}
\ell_0 &= -\tfrac{8-13v^2+7v^4}{6v(v^2-1)^3}, \\
\ell_4 &= -\tfrac{v(12-19v^2+9v^4)}{6(v^2-1)^3}, \\
\ell_{2\text{n}+4} &= \tfrac{v^{2n+4}\left(2nv^4-6nv^2+4n+5v^4-13v^2+10\right)}{12(v-1)^2(v+1)^2}.
\end{aligned}$$

Next, we directly calculate the following Wronskian

$$
\begin{aligned}
&W[\ell_0(v)] = \ell_0(v) := \Phi_0(v),\\
&W[\ell_0(v), \ell_4(v)] = \tfrac{v(128-148v^2+77v^4)}{9(v^2-1)^4} := \Phi_4(v),\\
&W[\ell_0(v), \ell_4(v), \ell_{2n+4}(v)] = \tfrac{(n-2)n\tilde{q}\left(v^{2n}+(-v)^{2n}\right)}{27(v-1)^7v^2(v+1)^7} := \Phi_{2n+4}(v),
\end{aligned}
\tag{3.12}
$$

where $\tilde{q}(v,n) = 154nv^8 - 758nv^6 + 1452nv^4 - 1360nv^2 + 512n + 539v^8 - 1717v^6 + 3036v^4 - 2768v^2 + 1024$. According to the analysis in the last subsection, we know $\Phi_0(v) \neq 0$ and $\Phi_4(v) \neq 0$. Thus, we just whether the third Wronskian determinant Φ_{2n+4} has zeros needs consideration.

Similar to the proof of Lemma 3.3, by the means of the substitution: $v = \frac{\beta^2}{\beta^2+1}$ where $\beta \in (-\infty, +\infty)$, then we have

$$\tilde{q}(v,n) = \frac{d_{4n}}{(\beta^2+1)^8},$$

where $d_{4n} = 228n\beta^{14} + 1890n\beta^{12} + 7280n\beta^{10} + 16892n\beta^8 + 20512n\beta^6 + 12976n\beta^4 + 4096n\beta^2 + 512n + 114\beta^{16} + 294\beta^{14} + 3651\beta^{12} + 14128\beta^{10} + 33196\beta^8 + 40736\beta^6 + 25904\beta^4 + 8192\beta^2 + 1024$.

By computing the discriminant of d_{4n} with respect to β, we have

$$\text{Discriminant}[d_{4n}, \beta] = 102860358224 \cdots 1122252800000000\, \tilde{\delta}(n), \tag{3.13}$$

where $\tilde{\delta}(n) = (n+2)^3(2n+7)^2(99614340n^6 - 1408831380n^5 + 6985984401n^4 - 6257138760n^3 - 336335118210n^2 - 915007828140n - 697962737115)^4$.

Then, by checking $\tilde{\delta}(n) = 0$, we obtain that there exist only two real roots $n_1 = -1.96\cdots$ and $n_2 = 11.35\cdots$. Additionally, by choosing two specific examples at random, namely when $n = 3$ and $n = 12$, we respectively have,

$$
\begin{aligned}
d_{4n}|_{n=3} &= 114\beta^{16} + 978\beta^{14} + 9321\beta^{12} + 35968\beta^{10} + 83872\beta^8 + 102272\beta^6\\
&\quad + 64832\beta^4 + 20480\beta^2 + 2560 > 0,\\
d_{4n}|_{n=12} &= 114\beta^{16} + 3030\beta^{14} + 26331\beta^{12} + 101488\beta^{10} + 235900\beta^8 + 286880\beta^6\\
&\quad + 181616\beta^4 + 57344\beta^2 + 7168 > 0
\end{aligned}
$$

which always hold for all $\beta \in (-\infty, +\infty)$. According to Lemma 2.4, whether $n_1 < n < n_2$ or $n > n_2$, $d_{4n} > 0$ always holds. This indicates that when $n = -1$ or $n \in \mathbb{N}$, $\tilde{q}(v,n) > 0$ always holds for all $v \in (0,1)$, yielding that Φ_{2n+4} is always positive. Therefore, $\{I_0, I_4, I_{2n+4}\}$ is an ECT-system on $(0, 1/4)$, according to Lemma 2.3, we conclude as follows.

Lemma 3.4. *For the perturbed system (1.4), when $g(v) = g_g(v)$ in (1.5), the maximum number of zeros of Abelian integral is two at most on* $(0, 1/4)$.

From the above Lemma 3.4, we prove the second part of Theorem 2.1. As for the fact that the number can reach two, it will be proved in the following section.

4. Limit cycles bifurcations and simulations

We know that the number of simple zeros of the Abelian integral can be translated to the number of limit cycles via Poincaré bifurcations. Further, according to Theorem

2.1, we know the maximum number of possible global limit cycles surrounding the origin of system (1.4) under the two perturbed cases in (1.5). Then, we give the following conclusion.

Proposition 4.1. *For Poincaré bifurcation around the origin of the perturbed system (1.4) via the first-order Melnikov functions, there exists one and at most one limit cycle for $g(v) = g_a(v)$, and there exist two and at most two limit cycles for $g(v) = g_b(v)$ in (1.5).*

Proof. By applying the methods of asymptotic expansions of Abelian integrals $\tilde{I}(h)$ (also called the first Melnikov function), presented in [11, 12, 22], we can find its zeros via the asymptotic expansion at $h = 0$ to correspond to the Hopf bifurcation values, and via the asymptotic expansion at $h = h_0 = \frac{1}{4}$ for the heteroclinic bifurcation.

In fact, we can take the other approach to find its zeros. Considering the case of $g_b(v)$ first, from expressions (2.4) and (3.1), we have

$$I(h) = d_0\tilde{I}_0(h) + d_4\tilde{I}_4(h) + d_{2n+4}\tilde{I}_{2n+4}(h) = \tfrac{1}{h}(d_0I_0 + d_4I_4 + d_{2n+4}I_{2n+4}), \quad (4.1)$$

where $I_i = I_i(h) = \oint_{\Gamma_h} f_i(v)y\mathrm{d}v, i = 0, 4, 2n+4$ with $n = -1, 1, 2, \cdots$. Additionally, setting $d_{2n+4} = 1$ with no harm and choosing arbitrarily two different $h_1^*, h_2^* \in (0, \frac{1}{4})$, then we have a group of equations as follows

$$\begin{cases} d_0I_0(h_1^*) + d_4I_4(h_1^*) = -I_{2n+4}(h_1^*), \\ d_0I_0(h_2^*) + d_4I_4(h_2^*) = -I_{2n+4}(h_2^*). \end{cases} \quad (4.2)$$

According to Lemma 3.3, $\{I_0, I_4\}$ is an extended complete Chebyshev (ECT) system on $(0, \frac{1}{4})$. Then, applying Theorem B and Lemma 2.3 in [9], we know that its all discrete Wronskian determinants of $\{I_0, I_4\}$ are not equal to zero. This implies that for the left coefficient matrix of equations (4.2), its determinant is not zero at (h_1^*, h_2^*). Thus, equations (4.2) have only one solution $\{d_0^*, d_4^*\}$. Letting $\mathbf{d} = (d_0^*, d_4^*, 1)$, then h_1^* and h_2^* are naturally two different zeros of Abelian integral $I(h, \mathbf{d})$ in (2.4), and more it follows from the proved conclusion in Theorem 2.1 that its maximum number of zeros is two at most, then there are no more zeros. Therefore, there necessarily exist some parameter values such that system (1.4) with $g(v) = g_b(v)$ has two big-amplitude limit cycles which bifurcate from the period annulus surrounding the origin.

With the same principle, the conclusion to the case of $g_(v) = g_a(v)$ can also be proved. Thus, we complete the proofs of this proposition and the remaining part of Theorem 2.1, and namely, the maximum number of zeros of Abelian integral (2.4) can be reached. □

Here, we give one numerical example of two big-amplitude limit cycles bifurcating around the origin, which straightforwardly verify the result of Proposition 4.1. Choosing $g_b(v) = d_0 + d_4v^4 + d_2v^2$ in system (1.4), i.e., case $n = -1$, and setting $h_1 = 1/5$ and $h_2 = 1/20$, then we have

$$\begin{aligned} &I_0(h_1) \approx 0.1602648, \quad I_2(h_1) \approx 0.0041999, \quad I_4(h_1) \approx 0.0002209; \\ &I_0(h_2) \approx 0.7014380, \quad I_2(h_2) \approx 0.0916512, \quad I_4(h_2) \approx 0.0245802. \end{aligned} \quad (4.3)$$

From equations (4.2), by letting $d_2 = 1$, we obtain

$$d_0 \approx -0.0219291, \quad d_4 \approx -3.1028822. \tag{4.4}$$

At this time, freely setting $\varepsilon = 0.05$ yields two limit cycles, as shown in Figure 4. From the three time histories in Figure 4(a), (b) and (c) respectively, we know that one limit cycle is stable, and the other is unstable.

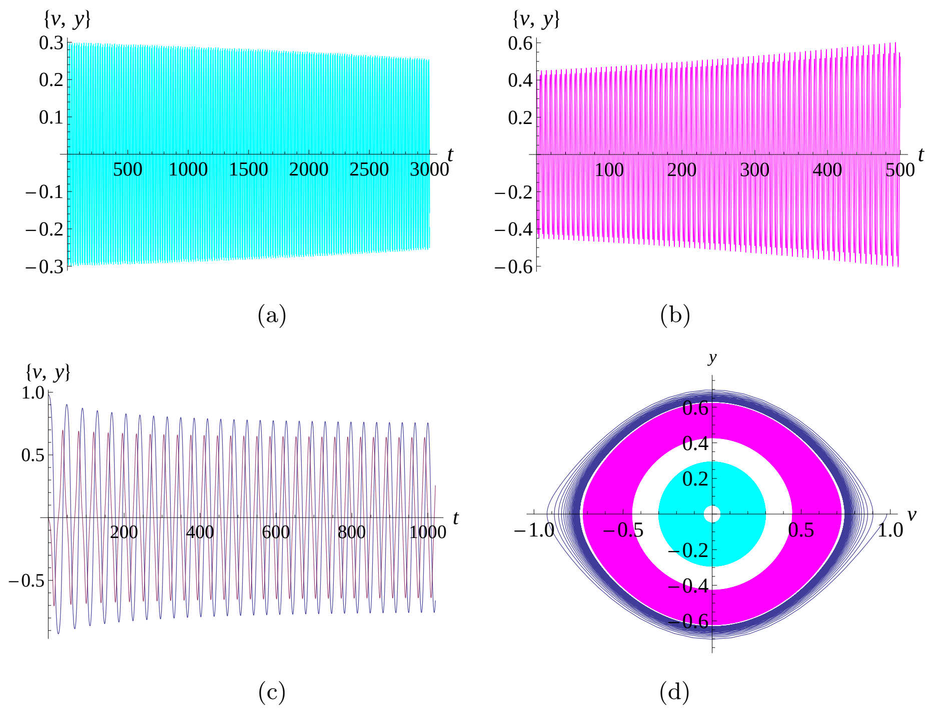


Figure 4. Time histories of (1.4) for $v(t)$ and $y(t)$ with the initial points: (a) $(v, y) = (0.3, 0)$; (b) $(v, y) = (0.45, 0)$; and (c) $(v, y) = (0.98, 0)$; (d)the phase portrait showing two limit cycles with the inner one unstable and the outer one stable

Conclusion

This paper focuses on Poincaré bifurcation from an elliptic Hamiltonian of degree four with two-saddle cycle. Based on the Chebyshev criterion and via symbolic calculation, once again, we prove the previous conclusion for the Liénard equations of type $(3, 2)$ in a different way. Moreover, for its two extended cases, the maximum number of Abelian integral zeros is determined respectively. We believe that the means used in this paper may be applied to other problems.

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