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Flocking of Multi-particle Swarm with Group Coupling Structure and Measurement Delay*

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Abstract We investigate the flocking conditions of a group coupling system with time delays, in which the communication between particles includes inter-group and intra-group interactions, and the time delay comes from the theory of moving object observation. As an effective model, we introduce a system of nonlinear functional differential equations to describe its dynamic evolution mechanism. By constructing two differential inequalities on velocity and velocity fluctuation from a continuity argument, and using the Lyapunov functional approach, we present some sufficient conditions for the existence of asymptotic flocking solutions to the coupling system, in which an upper bound of the delay allowed by the system is quantitatively given to ensure the emergence of flocking behavior. All results are novel and can be illustrated by using some specific numerical simulations.

Keywords Time-asymptotic flocking, Group coupling, Intra-group and inter-group interaction, Measurement delay.

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1. Introduction

"Flocking" is a collective behavior that widely exists in biological populations and human society such as migration of birds, directional movement of fish, hunting by wolves and gathering of bacteria [15,33,34]. The research of biological flocking dates back to the simulation of bird behavior in 1987 [36]. Based on this, subsequent researchers proposed a series of group motion models such as Vicsek model [40], Couzin model [7], Cucker-Smale model [9, 10] and their variants. Among them, the pioneering progress was that Cucker and Smale had proposed a second-order nonlinear dynamic model (C-S model) with a Newtonian interaction function to analyze the flocking mechanism of a multi-particle swarm in [9, 10]. The flocking dynamics and the related topics based on the C-S model have been extensively studied from many perspectives such as collision avoidance [4, 8, 25, 30], random effects [1, 6, 11, 14, 16, 21, 32, 39], generalized flocking [31], fixed-time flocking [26, 41] and multi-cluster flocking [43]. It is worth noting that in the work mentioned above, the interaction weight function has a unified form, either the symmetric

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type proposed in [9, 10] that depends on the metric distance, or the asymmetric type proposed in [31] that depends on the relative distance scaling.

However, as the scale of system nodes increases, the interaction modes between nodes become complex and diverse. To solve the dynamic mechanism modeling, some scholars have adopted a direct method, using Vlasov-type equations to describe the infinite size system [17–20]. However, from a practical perspective, the number of nodes in a biological population or engineering system cannot be infinite such that a swarm of bees or micro drone systems, resulting in the model proposed in [17–20] are no longer applicable. In addition, we know that real systems are often affected by the things that are not fully understood such as the most common time delays first proposed in [27]. In fact, time delay has also been widely studied in other fields, accumulating rich theoretical results [13, 24, 37]. Therefore, these deterministic models need to be extended for more complex changes. Given this, from the perspective of mathematical mechanism, we use the idea of chunking to highlight the diversity of interaction functions in the system, and further preliminarily assume that the system interaction mode consists of inter-group interaction and intra-group interaction, as shown in Figure 1 in Section 4, and then consider the time delay derived from the theory of moving object observation proposed in [2].

Next, we briefly discuss our continuous time flocking model involving time delay. Let $(x_i, v_i) \in \mathbb{R}^{2d}$ be the position and velocity coordinates of the i-th particle, and be governed by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \ i \in \mathcal{N}, \\ \dot{v}_i(t) = \alpha \sum\limits_{j\in\mathcal{N}_1} \psi(r_{ji})\widetilde{v}_{ji}(t) + \kappa \sum\limits_{j\in\mathcal{N}_2} a_{ij}\widetilde{v}_{ji}(t), i \in \mathcal{N}_1, \\ \dot{v}_i(t) = \kappa \sum\limits_{j\in\mathcal{N}_1} a_{ij}\widetilde{v}_{ji}(t) + \beta \sum\limits_{j\in\mathcal{N}_2} \psi(r_{ji})\widetilde{v}_{ji}(t), i \in \mathcal{N}_2, \end{cases} \tag{1.1}$$

where $\mathcal{N} := \{1, 2, \cdots, N\}$, $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ and $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$. In addition, $N_i = |\mathcal{N}_i| (i = 1, 2)$, $N = |\mathcal{N}|$ and $N_1 + N_2 = N$; $r_{ji} = \|x_j - x_i\|$, $\widetilde{v}_{ji}(t) = v_j(t-\tau) - v_i(t)$ for $j \in \mathcal{N}$, and $\tau > 0$ is the time delay derived from the theory of moving object observation. In (1.1), $\alpha > 0$ and $\beta > 0$ are intra-group coupling strengths; $\kappa > 0$ is inter-group coupling strength; ψ the intra-group interaction function, which is bounded, positive, non-increasing and Lipschitz continuous on $\mathbb{R}^+$ with $\psi(0) = 1$; a_{ij} represents the interaction strength between groups, and is a bounded positive constant, assuming that $0 < a_{ij} = a_{ji} \le 1$ for $i \in \mathcal{N}_k, \ j \in \mathcal{N}_l$ with $k \ne l \in \{1, 2\}$ is satisfied. Notice that the well-posedness of the time-delayed system (1.1) can be found in [22], which proves the existence and uniqueness of the classical solutions to system (1.1) for the given continuous initial conditions.

Specifically, system (1.1) can be degenerated into the C-S system proposed in [9, 10], and the threshold phenomena between global and local flocking has been observed. With our observations, if $\kappa = 0$ and $\tau = 0$ in (1.1), then there is no interaction between the two groups in the system, which can be regarded as two independent two groups that do not interfere with each other. In this case, some sharp conditions for flocking behavior can be obtained according to the classic flocking results in [9, 10, 19], as illustrated in Appendix A. Since then, these flocking estimates have been generalized for the generally non-increasing communication weight in [19], and the model considered is (1.1) with $\mathcal{N}_k = \emptyset (k = 1$ or $2)$ and $\tau = 0$. For another, notice that typical time delays include transmission delay [3, 5], processing delay [42] and their combination [27, 35], and the C-S model with time delay have been studied in [2,3,5,12,14,23,27,35,38,42]. Unfortunately, in most work, either the theoretical analysis is carried out under the premise that the time delay is small enough, or the flocking condition is not related to the delay. In addition, as far as we know, the flocking dynamics of coupling system with time delays has not been discussed from the perspective of moving object observations. The above two points are the main motivations for us to introduce a nonlinear second-order dynamic model with measurement delay and group coupling structure, and explore the flocking conditions. We emphasize that the flocking conditions proposed by us are closely related to the delay, and give a sufficient estimate of the upper bound of the delay tolerated by the system, which means the assumption that the delay sufficiently small in [5,35] can be removed from our framework.

Below we set the relevant initial conditions, $(x_i(\theta), v_i(\theta)) = (\varphi_i(\theta), \phi_i(\theta))$ for $\theta \in [-\tau, 0]$ for system (1.1) to specify a solution, where φ_i and ϕ_i are given continuous vector-valued functions. Subsequently, the time-asymptotic flocking defined in [19] can be extended to system (1.1), as stated below.

Definition 1.1. Let $\{x_i(t), v_i(t)\}(i \in \mathcal{N})$ be a solution to system (1.1) for a given initial data, a time-asymptotic flocking can be achieved, if and only if $\sup_{t\geq -\tau} D_x(t) < +\infty$ and $\lim_{t\to+\infty} D_v(t) = 0$ for $i, j \in \mathcal{N}$, where the spatial position diameter $D_x(t)$ and the velocity fluctuation $D_v(t)$ are defined by

$$D_x(t) = \max_{1\leq i,j\leq N}\{\|x_i(t) - x_j(t)\|\}, D_v(t) = \max_{1\leq i,j\leq N}\{\|v_i(t) - v_j(t)\|\}. \tag{1.2}$$

Since this work focuses on the asymptotic properties of the system state, we mainly consider the case where the time is enough. That is, $t > \tau$. In fact, when $t \in [0, \tau]$, the boundedness of the velocity of each particle and the velocity fluctuations between them can be obtained through the comparison principle (see Lemma 2.2 in [29]). Combined with Definition 1, we just need to prove $\sup_{t>\tau} D_x(t) < +\infty$ and $\lim_{t\to+\infty} D_v(t) = 0$ for $i, j \in \mathcal{N}$.

The rest of this work is organized as follows. Section 2 explores the delay-dependent flocking dynamics and quantifies an upper bound of the delay that allows flocking behavior to occur. The flocking dynamics of the coupling systems without time delay is discussed in Section 3. Section 4 presents numerical simulations of several examples.

2. Flocking conditions related to time delay

This section studies the flocking dynamic evolution mechanism of a two-group coupling system with measurement delay involving graph theory and Lyapunov stability theory.

2.1. Preliminaries

Based on graph theory, a weighted undirected graph $\mathcal{G}(C)$ is the graph $\mathcal{G}$ with a nonnegative adjacency matrix $C = (c_{ij})_{N\times N}$, where c_{ij} represents the interaction weight between the particles i and j. If $i, j \in \mathcal{N}_1$, then $c_{ij} = \alpha\psi(r_{ji})$. If $i \in \mathcal{N}_1, j \in \mathcal{N}_2$ or $i \in \mathcal{N}_2$, $j \in \mathcal{N}_1$, then $c_{ij} = \kappa a_{ij}$. If $i, j \in \mathcal{N}_2$, then $c_{ij} = \beta\psi(r_{ji})$. The Laplacian matrix $L = (l_{ij})_{N\times N}$ of $\mathcal{G}(C)$ is defined by $l_{ij} = -c_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{j=1}^{N} c_{ij}$. Moreover, let ξ be the smallest positive eigenvalue of L.

As in [14, 29, 35], a structural assumption is required on the matrix of communication rates, which can be stated as follows: there exists a constant $\nu > 0$ such that $\xi \geq \nu > 0$. This is a technical but instrumental and standardly used assumption that we use as well to perform our convergence analysis. In fact, the above assumption has been briefly mentioned in [10] and applied to construct the flocking conditions of a second-order dynamic model without delay. It allows both conditional flocking and unconditional flocking to be implemented under some other requirements on initial configurations.

Under the aforementioned setup, we have

$$\begin{aligned}\langle Lv, v\rangle = \frac{\alpha}{2}\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_1}\psi(r_{ji})\left\|v_{ji}(t)\right\|^2 + \frac{\beta}{2}\sum_{i\in\mathcal{N}_2}\sum_{j\in\mathcal{N}_2}\psi(r_{ji})\left\|v_{ji}(t)\right\|^2 \\ + \kappa\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2}a_{ij}\left\|v_{ji}(t)\right\|^2,\end{aligned} \tag{2.1}$$

where $r_{ji} = \|x_j - x_i\|$ and $v_{ji} = v_j - v_i$. In addition, denote $v_c = \frac{1}{N}\sum_{i=1}^N v_i \in \mathbb{R}^d$ and define $w = (w_1, \cdots, w_N)$ by $w_i = v_i - v_c$ for $i \in \mathcal{N}$. Thus, we have $\sum_{i=1}^N w_i = 0$ and

$$\|w(t)\|^2 = \sum_{i\in\mathcal{N}}\|w_i(t)\|^2 = \frac{1}{2N}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\|w_{ij}(t)\|^2, \tag{2.2}$$

where $w_{ij} = w_i - w_j$. For simplicity, we introduce several notations,

$$\begin{aligned}&C_1 = \alpha N_1 + \kappa N_2;\ C_2 = \kappa N_1 + \beta N_2;\ P_\tau(t) = \sum_{j\in\mathcal{N}}\int_{t-\tau}^t \|\dot{v}_j(s)\|^2 ds;\\ &C_3 = (C_1^2 + C_2^2)N(2\nu)^{-1};\ C_4 = 2N(\alpha^2+\kappa^2)N_1 + 2N(\kappa^2+\beta^2)N_2;\\ &C_5 = (\alpha^2 N_1 + \kappa^2 N_2)N_1 + (\kappa^2 N_1 + \beta^2 N_2)N_2,\end{aligned} \tag{2.3}$$

where $\nu > 0$ is a constant.

2.2. Main results

The conditions under which the flocking behavior of system (1.1) occurs are summarized as follows.

Theorem 2.1. *Let $\{x_i(t), v_i(t)\}(i \in \mathcal{N})$ be a solution to system (1.1) for a given initial data. If the time delay τ satisfies $\tau \leq \tau_c = \min\{\ln M_0, (8C_5M_0)^{-1}\}$, then the system converges asymptotically, and flocking behavior occurs, where $M_0 = 1 + \nu(16C_3C_4)^{-1}$, $C_i (i = 3, 4, 5)$ are indicated in (2.3), and ν is a positive constant determined by the interactive topology.*

Remark 2.1. There are two explanations about the above theorem.

(i) Assume that $\alpha = \beta$ in (1.1), $\sum_{j\in\mathcal{N}_1}\psi(r_{ji}) = 1$ for $i \in \mathcal{N}_1$ and $\sum_{j\in\mathcal{N}_2} a_{ij} = 1$ for $i \in \mathcal{N}_1$. Let ξ^1 be a left eigenvector to the simple eigenvalue 1 of the (non-normalized) Laplacian matrix corresponding to the adjacency matrix $C = (c_{ij})_{N\times N}$. Thus, proceeding as Theorem 3 in [28], the asymptotic velocity v_∞ satisfies $v_\infty = (1+\gamma\tau)^{-1}\left\langle \xi^1, \phi(0) + \gamma\int_{-\tau}^0 \phi(\theta)d\theta\right\rangle$, where $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$ and $\gamma = \alpha + \kappa$; $\langle\cdot,\cdot\rangle$ are the inner products of vectors.

(ii) Although the flocking conditions proposed in Theorem 2.1 are sufficient, and the upper bound of delay tolerated by the system is not optimal, it provides an important reference for the delay control of multi-agent systems in engineering applications.

To prove Theorem 2.1, we present some auxiliary lemmas.

Lemma 2.1. *Letting* $\{x_i(t), v_i(t)\}(i \in \mathcal{N})$ *be a global solution to (1.1) for a given initial data, then we have*

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 \leq -\frac{\nu}{2}\|w(t)\|^2 + C_3\tau P_\tau(t), \tag{2.4}$$

where $\nu > 0$ *is a constant, and* C_3 *and* $P_\tau(t)$ *are as indicated in (2.3).*

Proof. Since $w_i = v_i - v_c$, for $i \in \mathcal{N}$ and $\sum_{i\in\mathcal{N}} w_i = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 = \sum_{i\in\mathcal{N}} w_i(t)\cdot(\dot{v}_i(t) - \dot{v}_c) = \sum_{i\in\mathcal{N}} \dot{v}_i(t)\cdot w_i(t). \tag{2.5}$$

Substituting the second equation and third equation of (1.1) into (2.5), and using (2.1) and the following fact

$$\widetilde{v}_{ji}(t) = v_j(t-\tau) - v_j(t) + v_j(t) - v_i(t), \text{ for } i,\ j \in \mathcal{N}, \tag{2.6}$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 = {}& \alpha\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_1}\psi(r_{ji})(v_j(t-\tau) - v_j(t))\cdot w_i(t)\\ &+ \beta\sum_{i\in\mathcal{N}_2}\sum_{j\in\mathcal{N}_2}\psi(r_{ji})(v_j(t-\tau) - v_j(t))\cdot w_i(t)\\ &+ \kappa\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2}a_{ij}(v_j(t-\tau) - v_j(t))\cdot w_i(t)\\ &+ \kappa\sum_{i\in\mathcal{N}_2}\sum_{j\in\mathcal{N}_1}a_{ij}(v_j(t-\tau) - v_j(t))\cdot w_i(t)\\ &- \langle Lw, w\rangle,\end{aligned} \tag{2.7}$$

where we have used $\sum_{i=1}^N w_i = 0$. For simplicity, we denote

$$\begin{aligned}&U := (u_1, \cdots, u_{N_1}),\ u_i(t) := \sum_{j\in\mathcal{N}_1}\psi(r_{ji})(v_j(t-\tau) - v_j(t)),\\ &p_j(t) := \int_{t-\tau}^t \|\dot{v}_j(s)\|ds,\ \mathbf{I} := \left|\alpha\sum_{i,j\in\mathcal{N}_1}\psi(r_{ji})(v_j(t-\tau) - v_j(t))\cdot w_i(t)\right|.\end{aligned} \tag{2.8}$$

Noting that $u_i(t) = \sum_{j\in\mathcal{N}_1}\psi(r_{ji})\int_{t-\tau}^t \dot{v}_j(s)ds$, $a_{ji} = a_{ij} \leq 1$ and $\psi(\cdot) \leq 1$, we have

$$\|U(t)\| \leq \sum_{i\in\mathcal{N}_1}\|u_i\| \leq \sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_1}\psi(r_{ji})\int_{t-\tau}^t \|\dot{v}_j(s)\|ds = N_1\sum_{j\in\mathcal{N}_1}p_j(t). \tag{2.9}$$

Thus, we further get

$$\mathbf{I} \leq \alpha \left| \sum_{i \in \mathcal{N}_1} u_i \cdot w_i \right| \leq \alpha \|U(t)\| \cdot \|w(t)\| \leq \alpha N_1 \|w(t)\| \sum_{j \in \mathcal{N}_1} p_j(t), \tag{2.10}$$

where $p_j(t)$ is shown in (2.8). Thus, repeating the derivations in (2.9) and (2.10) to estimate the 2nd, 3rd and 4th terms at the right end of (2.7), we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 \leq \ -\langle Lw, w\rangle + \Delta(t), \tag{2.11}$$

where $\Delta(t) = \sum_{k=1}^{2} C_k \|w(t)\| \sum_{j \in \mathcal{N}_k} p_j(t)$, $C_i (i = 1, 2)$ is shown in (2.3). According to Young's inequality, for $\delta_1 > 0$ and $\delta_2 > 0$, we have

$$\Delta(t) \leq \frac{C_1}{2\delta_1} \left(\sum_{j \in \mathcal{N}_1} p_j(t) \right)^2 + \frac{C_2}{2\delta_2} \left(\sum_{j \in \mathcal{N}_2} p_j(t) \right)^2 + \frac{1}{2} \left(C_1\delta_1 + C_2\delta_2 \right) \|w(t)\|^2.$$

Furthermore, according to Cauchy inequality, we can get

$$\Delta(t) \leq \frac{\tau}{2} \left(\frac{C_1 N_1}{\delta_1} + \frac{C_2 N_2}{\delta_2} \right) P_\tau(t) + \frac{1}{2} \left(C_1\delta_1 + C_2\delta_2 \right) \|w(t)\|^2. \tag{2.12}$$

Let $\delta_1 = \nu C_1 (C_1^2 + C_2^2)^{-1}$ and $\delta_2 = \nu C_2 (C_1^2 + C_2^2)^{-1}$, which means that (2.11) can be further abbreviated as (2.4). The proof is completed. □

Lemma 2.2. *Letting $\{x_i(t), v_i(t)\} (i \in \mathcal{N})$ be a solution to (1.1) for a given initial data, then we have*

$$\sum_{i=1}^{N} \|\dot{v}_i(t)\|^2 \leq 4C_4 \|w(t)\|^2 + 4C_5 \tau P_\tau(t), \tag{2.13}$$

where $C_i (i = 4, 5)$ and $P_\tau(t)$ are defined in (2.3).

Proof. Noting that $\sum_{i=1}^{N} \|\dot{v}_i(t)\|^2 = \sum_{i \in \mathcal{N}_1} \|\dot{v}_i(t)\|^2 + \sum_{i \in \mathcal{N}_2} \|\dot{v}_i(t)\|^2$, we estimate $\sum_{i \in \mathcal{N}_1} \|\dot{v}_i(t)\|^2$ and $\sum_{i \in \mathcal{N}_2} \|\dot{v}_i(t)\|^2$ respectively.

For $i \in \mathcal{N}_1$, we use the second equations of (1.1) and (2.6) to get

$$\begin{aligned} \dot{v}_i(t) &= \alpha \sum_{j \in \mathcal{N}_1} \psi(r_{ji}) \widetilde{v}_{ji}(t) + \kappa \sum_{j \in \mathcal{N}_2} a_{ij} \widetilde{v}_{ji}(t) \\ &= \alpha \sum_{j \in \mathcal{N}_1} \psi(r_{ji})(v_j(t-\tau) - v_j(t)) + \alpha \sum_{j \in \mathcal{N}_1} \psi(r_{ji}) w_{ji}(t) \\ &\quad + \kappa \sum_{j \in \mathcal{N}_2} a_{ij}(v_j(t-\tau) - v_j(t)) + \kappa \sum_{j \in \mathcal{N}_2} a_{ij} w_{ji}(t), \end{aligned} \tag{2.14}$$

and it further follows from $\psi(\cdot) \leq 1$ and $a_{ij} \leq 1$ that

$$\|\dot{v}_i(t)\| \leq \alpha \sum_{j \in \mathcal{N}_1} \left(p_j(t) + \|w_{ji}(t)\| \right) + \kappa \sum_{j \in \mathcal{N}_2} \left(p_j(t) + \|w_{ji}(t)\| \right),$$

where $p_j(t)=\int_{t-\tau}^{t}\|\dot v_j(s)\|ds$. Thus, according to the fundamental inequality and Cauchy inequality, we have

$$\begin{aligned}\|\dot v_i(t)\|^2 \le\ & 4\alpha^2 N_1\sum_{j\in\mathcal{N}_1}\|w_{ji}(t)\|^2+4\kappa^2 N_2\sum_{j\in\mathcal{N}_2}\|w_{ji}(t)\|^2\\ &+4(\alpha^2N_1+\kappa^2N_2)\tau P_\tau(t).\end{aligned}\tag{2.15}$$

An argument similar to the one used in the above shows that for $i\in\mathcal{N}_2$,

$$\begin{aligned}\|\dot v_i(t)\|^2 \le\ & 4\kappa^2 N_1\sum_{j\in\mathcal{N}_1}\|w_{ji}(t)\|^2+4\beta^2 N_2\sum_{j\in\mathcal{N}_2}\|w_{ji}(t)\|^2\\ &+4(\kappa^2N_1+\beta^2N_2)\tau P_\tau(t).\end{aligned}\tag{2.16}$$

Therefore, it can be directly obtained (2.13) by simple calculation based on (2.15), (2.16) and (2.2). The proof is completed. □

Next, we turn to prove Theorem 2.1.

Proof. The proof is divided into two steps: one is to prove the exponential decay of the velocity fluctuation based on the Lyapunov stability analysis, and the other is to estimate the position diameter of the particle swarm by contradiction.

Step 1. We prove that the velocity fluctuation decays exponentially.

Motivated by [19], consider the following candidate energy function

$$Q(t)=\frac{1}{2}\|w(t)\|^2+\theta\int_{t-\tau}^{t}e^{-(t-s)}\int_s^t\sum_{i=1}^N\|\dot v_i(\sigma)\|^2d\sigma ds,\ t\ge\tau,\tag{2.17}$$

where $\theta=\frac{C_3e^\tau}{1-4C_5\tau(e^\tau-1)}$. Noting that $\tau\le\min\{\ln M_0,(8C_5M_0)^{-1}\}$ with $M_0=1+\nu(16C_3C_4)^{-1}$, we have $e^\tau\le M_0$ and $\tau M_0\le(8C_5)^{-1}$, and consequently, $\tau e^\tau\le(8C_5)^{-1}$, which means $\theta>0$. Further, the derivative of $Q(t)$ along the solution trajectory of system (1.1) is

$$\begin{aligned}\frac{d}{dt}Q(t)\Big|_{(1.1)}&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2-\theta\int_{t-\tau}^{t}e^{-(t-s)}\int_s^t\sum_{i=1}^N\|\dot v_i(\sigma)\|^2\mathrm{d}\sigma\mathrm{d}s\\&\quad+\theta\int_{t-\tau}^{t}e^{-(t-s)}\sum_{i=1}^N\|\dot v_i(t)\|^2\mathrm{d}s-\theta e^{-\tau}\int_{t-\tau}^{t}\sum_{i=1}^N\|\dot v_i(\sigma)\|^2\mathrm{d}\sigma\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2-\theta\int_{t-\tau}^{t}e^{-(t-s)}\int_s^t\sum_{i=1}^N\|\dot v_i(\sigma)\|^2\mathrm{d}\sigma\mathrm{d}s\\&\quad+\theta(1-e^{-\tau})\sum_{i=1}^N\|\dot v_i(t)\|^2-\theta e^{-\tau}\int_{t-\tau}^{t}\sum_{i=1}^N\|\dot v_i(\sigma)\|^2\mathrm{d}\sigma.\end{aligned}\tag{2.18}$$

Denote $M_1:=\theta\int_{t-\tau}^{t}e^{-(t-s)}\int_s^t\sum_{i=1}^N\|\dot v_i(\sigma)\|^2\mathrm{d}\sigma\mathrm{d}s$. Thus, (2.18) is simplified as

$$\frac{d}{dt}Q(t)\Big|_{(1.1)}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2-M_1+\theta(1-e^{-\tau})\sum_{i=1}^N\|\dot v_i(t)\|^2-\theta e^{-\tau}P_\tau(t),\tag{2.19}$$

where $P_\tau(t)=\sum_{j=1}^N\int_{t-\tau}^t\|\dot v(s)\|^2ds$. Reviewing Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}\frac{d}{dt}Q(t)\bigg|_{(1.1)} \le\ & -\frac{\nu}{2}\|w(t)\|^2+C_3\tau P_\tau(t)-M_1-\theta e^{-\tau}P_\tau(t)\\ &+\theta(1-e^{-\tau})\left(4C_4\|w(t)\|^2+4C_5\tau P_\tau(t)\right)\\ =\ & -\left(\frac{\nu}{2}-4\theta C_4(1-e^{-\tau})\right)\|w(t)\|^2-M_1\\ &+\left(C_3-\theta e^{-\tau}+4C_5\theta(1-e^{-\tau})\tau\right)P_\tau(t).\end{aligned}\tag{2.20}$$

Due to the definition of δ in (2.17), we have $C_3-\theta e^{-\tau}+4C_5\theta(1-e^{-\tau})\tau=0$. Thus, (2.20) is rewritten as

$$\frac{d}{dt}Q(t)\bigg|_{(1.1)}\le -C_*\|w(t)\|^2-\theta\int_{t-\tau}^t e^{-(t-s)}\int_s^t\sum_{i=1}^N\|\dot v_i(\sigma)\|^2\mathrm{d}\sigma\mathrm{d}s,\tag{2.21}$$

where $C_*=\frac{\nu}{2}-4\theta C_4(1-e^{-\tau})$. Noting that $\tau\le\min\{\ln M_0,(8C_5M_0)^{-1}\}\le\ln M_0$, we have $e^\tau\le M_0=1+\nu(16C_3C_4)^{-1}$. Thus, we get $C_*>0$, according to the following derivations

$$4\theta C_4(1-e^{-\tau})=\frac{4C_3C_4(e^\tau-1)}{1-4C_5\tau(e^\tau-1)}<\frac{4C_3C_4(e^\tau-1)}{1-4C_5\tau e^\tau}<8C_3C_4(e^\tau-1)\le\frac{\nu}{2}.$$

Therefore, there exists a positive constant $r=\min\{C_*,1\}>0$ such that

$$\frac{d}{dt}Q(t)\bigg|_{(1.1)}\le -rQ(t).\tag{2.22}$$

To conclude, it suffices to write that

$$\|w(t)\|^2\le 2Q(t)\le c_0e^{-rt},\ t\ge\tau.\tag{2.23}$$

where $c_0=2Q(\tau)e^{r\tau}$, and further (2.23) shows that $\|v_{ij}(t)\|^2=\|w_{ij}(t)\|^2\le 2N\|w(\tau)\|^2\le 2Nc_0e^{-rt}$ for $i,\ j\in\mathcal{N}$ and $t\ge\tau$. Hence, $D_v(t)\le\sqrt{2Nc_0}e^{-\frac{rt}{2}}$ can be obtained, which means that the velocity fluctuation decays exponentially.

Step 2. We prove the boundedness of the position diameter $D_x(t)$.

First, we define a set

$$\mathcal{T}:=\left\{t>\tau\ \middle|\ D_x(s)<D_x(\tau)+2\sqrt{2Nc_0}r^{-1}e^{\frac{r}{2}\tau},\ s\in[\tau,t]\right\},$$

where $C_*>0$ is shown in Step 1. Let $D_*=D_x(\tau)+2\sqrt{2Nc_0}r^{-1}e^{\frac{r}{2}\tau}$. It follows from $D_x(0)<D_*$ that $\mathcal{T}\ne\emptyset$. Thus, we claim $\sup\mathcal{T}=+\infty$. If not, then we have $T_1=\sup\mathcal{T}<+\infty$ and $\lim\limits_{t\to T_1-}D_x(t)=D_*$, which means that there exist $p,\ q\in\mathcal{N}$ such that

$$\lim_{t\to T_1-}\|x_p(t)-x_q(t)\|=D_*.\tag{2.24}$$

From the definitions of $D_v,\ D_x$ and the Cauchy inequality, it can be obtained that

$$\begin{aligned}\|x_p(t)-x_q(t)\|&\le\|x_p(\tau)-x_q(\tau)\|+\int_\tau^t\|v_p(s)-v_q(s)\|ds\\ &\le D_x(\tau)+\int_\tau^t\sqrt{2Nc_0}e^{-\frac{rt}{2}}ds\\ &=D_x(\tau)+\frac{2\sqrt{2Nc_0}}{r}\left(e^{\frac{r}{2}\tau}-e^{-\frac{r}{2}t}\right),\ t\in[\tau,T_1).\end{aligned}\tag{2.25}$$

Let $t \to T_1-$ be in (2.25), and we have

$$\lim_{t\to T_1-} \|x_p(t) - x_q(t)\| \le D_x(\tau) + \frac{2\sqrt{2Nc_0}}{r}\left(e^{\frac{r}{2}\tau} - e^{-\frac{r}{2}T_1}\right) < D_x(\tau) + \frac{2\sqrt{2Nc_0}}{r} e^{\frac{r}{2}\tau},$$

which is contrary to (2.24). Hence, $\sup_{t>\tau} D_x(t) < D_*$. The proof is completed. □

3. Flocking conditions in scenarios without delay

In this section, we will analyze the flocking conditions of system (1.1) with $\tau = 0$, and the model considered is as follows

$$\begin{cases} \dot{x}_i(t) = v_i(t),\ i \in \mathcal{N}, \\ \dot{v}_i(t) = \alpha \sum_{j\in\mathcal{N}_1} \psi(r_{ji}) v_{ji}(t) + \kappa \sum_{j\in\mathcal{N}_2} a_{ij} v_{ji}(t),\ i \in \mathcal{N}_1, \\ \dot{v}_i(t) = \kappa \sum_{j\in\mathcal{N}_1} a_{ij} v_{ji}(t) + \beta \sum_{j\in\mathcal{N}_2} \psi(r_{ji}) v_{ji}(t),\ i \in \mathcal{N}_2, \end{cases} \tag{3.1}$$

where $x_i,\ v_i \in \mathbb{R}^d (i \in \mathcal{N})$, $r_{ji} = \|x_j - x_i\|$, $v_{ji} = v_j - v_i$. The other parameters in (3.1) have the same meaning as (1.1), and the initial data $(x_i(0), v_i(0))$ are denoted by (x_{i0}, v_{i0}). Denote $v_c = \frac{1}{N}\sum_{i=1}^N v_i$ and let $\hat{v} = (\hat{v}_1, \cdots, \hat{v}_N)$ defined by $\hat{v}_i = v_i - v_c$, for $i \in \mathcal{N}$. Thus, we have $\sum_{i=1}^N \hat{v}_i = 0$ and

$$\|\hat{v}(t)\|^2 = \sum_{i=1}^N \|\hat{v}_i(t)\|^2 = \frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N \|\hat{v}_i(t) - \hat{v}_j(t)\|^2, \tag{3.2}$$

and further, $\|\hat{v}_i(t) - \hat{v}_j(t)\| \le \sqrt{2N}\|\hat{v}(t)\|$ for $i, j \in \mathcal{N}$. For simplicity, we introduce the following quantities

$$\begin{aligned} d_X(t) &= \max_{1\le i,j\le N}\{\|x_i(t) - x_j(t)\|\},\ t \ge 0; \\ d_V(t) &= \max_{1\le i,j\le N}\{\|v_i(t) - v_j(t)\|\},\ t \ge 0; \\ c_1 &= (\alpha + \beta)N;\ c_2 = 2\kappa\sqrt{2NN_1N_2}. \end{aligned} \tag{3.3}$$

Based on the above preliminaries, we know that there exist some appropriate coupling strengths satisfying $c_1 > c_2$ to ensure the enforceability of following discussions. Under this premise, the conditions that allow the flocking behavior of system (3.1) to occur can be described as follows.

Theorem 3.1. *Let $\{x_i(t), v_i(t)\} (i \in \mathcal{N})$ be a solution to system (3.1) for a given initial data. If there exists $d_0 > d_X(0)$ such that $c_2 \le c_1\psi(d_0)$ and*

$$\sqrt{2N}\|\hat{v}(0)\| < \int_{d_X(0)}^{d_0} \rho(r)dr, \tag{3.4}$$

where $\rho(r) = c_1\psi(r) - c_2$. Then, system (3.1) converges to a flock. In addition, there exists $d^* < d_0$ such that $\sup_{t\ge 0} d_X(t) < d^*$ and $d_V(t) \le \sqrt{2N}\|\hat{v}(0)\|e^{-\rho(d^*)t}$, where $\|\hat{v}\|$ is indicated in (3.2); $c_i (i = 1, 2)$, d_X and d_V are shown in (3.3).

Remark 3.1. If the interaction weight function in (3.1) is defined as $\psi(r) = (1 + r^2)^{-s}$ proposed in [9,10], then a key feature is the role that s plays in $\psi(r)$. When $s \leq \frac{1}{2}$, the population converges to flocking over time, it is regardless of its initial states. More interestingly, it also showed that $s > \frac{1}{2}$ convergence to a flock occurs, supposed that the initial state satisfies some certain conditions, and it may fail to occur otherwise. Roughly speaking, the particle swarm cannot be too unaligned and too dispersed initially. Therefore, the case $s > \frac{1}{2}$ (i.e., weak interactions) is closer to biological populations, and not all the wild geese will ultimately align or end up in a line.

Before proving Theorem 3.1, inspired by [19], we first construct the following dissipation inequality.

Lemma 3.1. *Let $\{x_i(t), v_i(t)\}(i \in \mathcal{N})$ be a solution to system (3.1) for a given initial data. Then, we have*

$$\frac{d}{dt}\|\hat{v}(t)\| \leq -\rho(d_X(t))\|\hat{v}(t)\|, \ a.e. \ t > 0, \tag{3.5}$$

where $\hat{v} = (\hat{v}_1, \cdots, \hat{v}_N)$; $\hat{v}_i = v_i - v_c$ for $i \in \mathcal{N}$; $v_c = \frac{1}{N}\sum_{i=1}^N v_i$; $\rho(r) = c_1\psi(r) - c_2$; c_1, c_2 and d_X are shown in (3.3).

Proof. Although the proof of the lemma is trivial, it is of vital significance to the subsequent flocking conditions. We omit here, and refer to Lemma 2.1 in [19] for details. □

Now, we are in a position to prove Theorem 3.1.

Proof. Consider the following candidate Lyapunov function

$$E(t) = \|\hat{v}(t)\| + \frac{1}{\sqrt{2N}}\int_{d_X(0)}^{d_X(t)} \rho(r)dr, \ t > 0,$$

where $\rho(r) = c_1\psi(r) - c_2$; c_1, c_2 and d_X are shown in (2.3). Based on the above preliminaries and the conditions in Theorem 3.1, we first have $\rho(r) > 0$, and then the derivative of $E(t)$ along the trajectory of system (3.1) is as follows

$$\frac{d}{dt}E(t)\bigg|_{(3.1)} = \frac{d}{dt}\|\hat{v}(t)\| + \frac{\rho(d_X)}{\sqrt{2N}}\frac{d}{dt}d_X(t). \tag{3.6}$$

According to the definition of d_V in (3.3), we take $d_V = \|v_p(t) - v_q(t)\| = \|\hat{v}_p(t) - \hat{v}_q(t)\|$, and then we get $d_V^2(t) = \|\hat{v}_p(t) - \hat{v}_q(t)\|^2 \leq \sum_{i=1}^N\sum_{j=1}^N \|\hat{v}_i(t) - \hat{v}_j(t)\|^2 = 2N\|\hat{v}(t)\|^2$, which means that $d_V(t) \leq \sqrt{2N}\|\hat{v}(t)\|$. It further follows from Lemma 3.1 and (3.6) that $\frac{d}{dt}E\big|_{(3.1)} \leq 0$, which implies that $E(t)$ does not increase with respect to t (i.e., $E(t) \leq E(0)$ for $t > 0$), and further we have

$$\int_{d_X(0)}^{d_X(t)} \rho(r)dr \leq \sqrt{2N}\|\hat{v}(0)\|. \tag{3.7}$$

Moreover, reviewing (3.4) produces

$$\int_{d_X(0)}^{d_X(t)} \rho(r)dr < \int_{d_X(0)}^{d_0} \rho(r)dr, \ t > 0. \tag{3.8}$$

Next, we prove $\rho(d_X(t)) = c_1\psi(d_X(t)) - c_2 > 0$ for all $t > 0$. If not, then there exists $0 < t_1 < +\infty$ such that $\rho(d_X(t_1)) \leq 0$. It follows from $\rho(d_0) > 0$ and $d_X(0) < d_0$ that $\rho(d_X(0)) \geq \rho(d_0) > 0$ according to the monotonous of $\rho(r)$. Thus, there exists $t_2 \in (0, t_1)$ such that $\rho(d_X(t_2)) = 0$ and $\rho(d_X(t)) > 0$, for $t \in (0, t_2)$. Furthermore, we get $\rho(d_X(t_1)) \leq \rho(d_X(t_2)) < \rho(d_0) \leq \rho(d_X(0))$, which implies $d_X(0) \leq d_0 \leq d_X(t_2) \leq d_X(t_1)$ and

$$\int_{d_X(0)}^{d_X(t_2)} \rho(r)dr = \int_{d_X(0)}^{d_0} \rho(r)dr + \int_{d_0}^{d_X(t_2)} \rho(r)dr \geq \int_{d_X(0)}^{d_0} \rho(r)dr,$$

yielding a contradiction with (3.8). Thus, $\rho(d_X(t)) > 0$ for $t > 0$, and then according to (3.8), there exists $d^* < d_0$ such that $d_X(t) \leq d^* < d_0$ for all $t > 0$, which implies that the position diameter is bounded. Given this, we can further get $0 < \rho(d^*) \leq \rho(d_X(t))$ for $t > 0$, and then we apply (3.5) in Lemma 3.1 to get

$$\frac{d}{dt}\|\hat{v}(t)\| \leq -\rho(d^*)\|\hat{v}(t)\|. \tag{3.9}$$

Thus, we get $\|\hat{v}(t)\| \leq \|\hat{v}(0)\|e^{-\rho(d^*)t}$ with $\rho(d^*) > 0$ according to Gronwall inequality. Since $2N\|\hat{v}(t)\|^2 = \sum_{i=1}^N \sum_{j=1}^N \|v_i(t) - v_j(t)\|^2$, we obtain $\|v_i(t) - v_j(t)\| \leq \sqrt{2N}\|\hat{v}(0)\|e^{-\rho(d^*)t}$ for $i, j \in \mathcal{N}$, which implies $\lim_{t\to+\infty} \|v_i(t) - v_j(t)\| = 0$ for $i, j \in \mathcal{N}$. The proof is completed. □

Remark 3.2. As a direct application of Theorem 3.1, we get that the upper bound on the position diameter d_f satisfy $d_X(0) < d_f$ and $\sqrt{2N}\|\hat{v}(0)\| = \int_{d_X(0)}^{d_f} \rho(r)dr$.

4. Specific cases

In this section, we visualize the effects of inter-group interaction and time delay on the flocking behavior of system (1.1).

Consider a system coupled by two groups with $N_1 = 3$ and $N_2 = 5$, as shown in Figure 1. For simplicity, the intra-group interaction and inter-group interaction in system (1.1) are set as $\psi(r) = (1+r^2)^{-0.6}$ and $a_{ij} = a_{ji} = 1$ for $i = 1, 2, 3,\ j = 4, ...8$. The initial conditions are $(x_i(t), v_i(t)) = (t\sin(it), 1 + \sin(it)), for\ i = 1, 2, 3,\ t \in [-\tau, 0]$ and $(x_i(t), v_i(t)) = (t\sin(it), 1 - \cos(it)), for\ i = 4, ...8,\ t \in [-\tau, 0]$. Other parameters are given in specific examples.

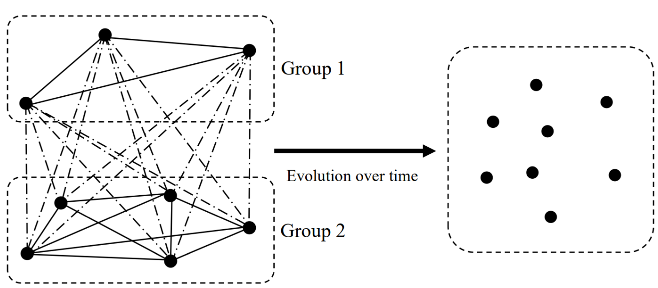

Figure 1. A flock schematic of an eight-particle system coupled by two groups

• **Case 1:** $\alpha = \beta = 0.03$, $\kappa = 0$, $\tau = 0$ (see Figure 2). It can be intuitively seen from Figure 2 that system (1.1) with the above parameters cannot achieve flocking behavior, because the velocity fluctuation converges to a positive constant, and the position diameter diverges.

- **Case 2:** $\alpha = \beta = 0.03$, $\kappa = 0.02$, $\tau = 0$ (see Figure 3). The purpose is to explore the effect of inter-group interaction on flocking dynamics by comparing with Case 1. Under the above parameters, the system exhibits flocking behavior, as shown in Figure 3. Compared with Case 1, it can be concluded that under certain conditions, the exchange of information between groups has a positive effect on the emergence of flocking behavior, which is consistent with practical experience.

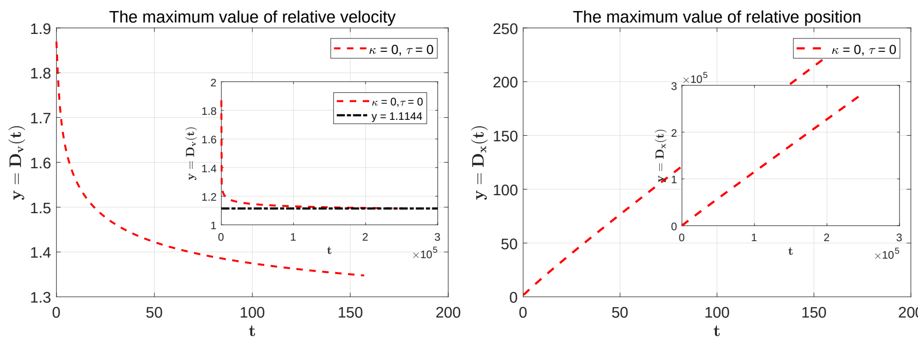

Figure 2. Schematic diagram of velocity fluctuation and position diameter evolution of system (1.1) without inter-group interactions and time delays

- **Case 3:** $\alpha = \beta = 0.03$, $\kappa = 0.02$, $\tau = 0.01\pi$ (see Figure 3). The purpose is to investigate the effect of time delay on the two-group coupling system under the premise of information interaction between groups and to demonstrate the validity of the theoretical results in Theorem 2.1.

Theoretically, according to (2.3), the required parameters can be calculated as $C_1 = 0.19$, $C_2 = 0.21$, $C_3 = 1.9052$, $C_4 = 0.0104$, $C_5 = 0.0426$ and $\nu = 0.0631$. Consequently, we get

$$\frac{1}{8C_5} - \tau e^{\tau} = 2.9019 > 0, \ \ln\left(\frac{\nu}{16C_3C_4} + 1\right) - \tau = 0.1502 > 0,$$

which means that $\tau \leq \tau_c$ in Theorem 2.1 is satisfied. Therefore, Theorem 2.1 guarantees that flocking behavior can occur in system (1.1), which is also confirmed in Figure 3. In addition, we can intuitively observe from Figure 3 that the velocity fluctuation and the position diameter for a given time are larger in Case 3 than in Case 2. This result is consistent with practical experience, because the time delay will increase the convergence time to a certain extent.

- **Case 4:** $\alpha = \beta = 0.03$, $\kappa = 0.06$, $\tau = 0.01\pi$ (see Figure 3). Using the same argumentation and analysis as in Case 3, the theoretical analysis shows that the parameters in this case can ensure the emergence of flocking behavior in system (1.1), since the conditions in Theorem 2.1 are satisfied. The required parameters involved are calculated as follows: $C_1 = 0.39$, $C_2 = 0.33$, $C_3 = 2.1035$, $C_4 = 0.036$, $C_5 = 0.1386$, $\nu = 0.1861$ and

$$\frac{1}{8C_5} - \tau e^{\tau} = 0.8695 > 0, \ \ln\left(\frac{\nu}{16C_3C_4} + 1\right) - \tau = 0.1115 > 0.$$

The simulation results in Figure 3 also verify the above theoretical analysis.

In addition, comparing Case 3 and Case 4, it can be seen from Figure 3 that the more timely the information interaction between individuals or the stronger the coupling between groups is, the more likely the flocking behavior is to occur in the system under certain conditions, which is in line with practical experience.

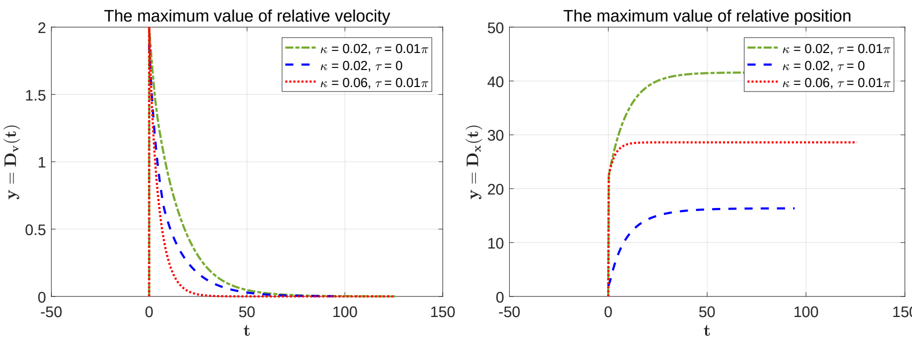

Figure 3. Schematic diagram of evolution of velocity fluctuation and position diameter evolution in Case 2, Case 3 and Case 4

5. Conclusions

From the perspective of mathematical mechanism, we have investigated the flocking dynamic evolution mechanism of a two-group coupling system with measurement delay and explored the flocking conditions related to the delay. We have established a sufficient framework that depends on the size of the lag (see Theorem 2.1) to ensure that flocking behavior occurs when the delay is less than a certain critical value related to the system parameters, which means that we remove the assumption that the time delay is sufficiently small in [5,35].

Appendix A: flocking conditions for (1.1) with $\kappa = 0$, $\tau = 0$

System (1.1) with $\kappa = 0$ and $\tau = 0$ can be considered as two independent Cucker-Smale systems, and the model degenerates into

$$\dot{v}_i(t) = \alpha \sum_{j\in\mathcal{N}_1} \psi\left(r_{ji}\right)\left(v_j(t) - v_i(t)\right), i \in \mathcal{N}_1 \tag{5.1}$$

and

$$\dot{v}_i(t) = \beta \sum_{j\in\mathcal{N}_2} \psi\left(r_{ji}\right)\left(v_j(t) - v_i(t)\right), i \in \mathcal{N}_2, \tag{5.2}$$

where $\alpha > 0, \beta > 0$; $\psi(r) = \left(1 + r^2\right)^{-s}$ with $s \geq 0$; $r_{ji} = \|x_j - x_i\|$. It can be intuitively seen that (5.1) and (5.2) are essentially the Cucker-Smale model proposed in [9,10]. We further denote $x_c^k(t) := \frac{1}{N_k}\sum_{j\in\mathcal{N}_k} x_j(t)$ and $v_c^k(t) := \frac{1}{N_k}\sum_{j\in\mathcal{N}_k} v_j(t)$, $k = 1, 2$. For brevity, we adopt $\hat{x}_i, \hat{v}_i (i \in \mathcal{N})$ to represent these two fluctuation variables. Based on the classical flocking results in [19], some sharp conditions for flocking behavior can be obtained directly, as is shown below.

Conclusion. Let $\{x_i(t), v_i(t)\}$ $(i \in \mathcal{N})$ be a solution to (5.1)–(5.2) for a given initial condition. If

$$\|\hat{v}_0\| < \xi \int_{\|\hat{x}_0\|}^{\infty} \psi(s)ds, \tag{5.3}$$

where $\xi = \min\{\alpha N_1^{-1}, \beta N_2^{-1}\} > 0$. Then, there exists $x_M \geq 0$ such that $\|\hat{v}_0\| = \xi \int_{\|\hat{x}_0\|}^{x_M} \psi(s)ds$, $\|\hat{x}(t)\| \leq \frac{x_M}{2}$ and $\|\hat{v}(t)\| \leq \|\hat{v}_0\| e^{-\xi\psi(x_M)t}$. In addition, the main results on flocking phenomena can be summarized as follows:

(i) if $v_c^1(0) = v_c^2(0)$ and $0 \leq s \leq \frac{1}{2}$, then unconditional flocking will achieve;

(ii) if $v_c^1(0) = v_c^2(0), s > \frac{1}{2}$, then systems (5.1) and (5.2) can converge to a flock with $v_\infty = v_c^1(0) = v_c^2(0)$;

(iii) if $v_c^1(0) \neq v_c^2(0)$, then the system will form two-cluster flocking asymptotically, and in addition, $v_\infty^1 = v_c^1(0), v_\infty^2 = v_c^2(0)$.

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