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Polynomial Homogeneous Maps and Their Periods*

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Abstract We study the set of periods of the homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ of degree $m > 1$. For these complex maps, we also describe the number of invariant straight lines through the origin by f^k for $k = 1, 2, \ldots$ and the dynamics of f^k over them.

Keywords Homogeneous polynomial map, Period, Set of period, Self-map of a sphere.

MSC(2010) 37F10, 37C25.

1. Introduction and statement of the main results

We consider discrete dynamical systems given by a real or complex homogeneous polynomial map defined in $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively. For discrete dynamical systems, the periodic orbits play an important role in understanding their dynamics. Perhaps, the best known example in this direction are the results contained in the paper entitled *"Period three implies chaos"* for continuous self-maps on the interval (see [21] or the book [2]).

The real homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ have been studied by many authors (see, for instance, the survey of Aliashvili [1] and the references quoted therein). However, not too much attention has been paid to the study of their periodic orbits with the exception of their fixed points (see, for instance, [8, 19, 27]).

Let $\mathbb{CP}^n$ be the complex projective space of dimension n. The complex homogeneous polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$ also has been considered in [1] and the references quoted there, but again not too much attention was paid to their periodic orbits. On the other hand, for the complex homogeneous polynomial maps $f : \mathbb{CP}^n \to \mathbb{CP}^n$, their set of periods have been studied (see Fornaes and Sibony [11], or [10]). On the other hand, these maps have also been studied from the point of view of their degrees (see [17]).

The study on the set of periods of the real homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ is our objective for such maps, while for complex homogeneous polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$, our main goal is to study their invariant straight lines through

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the origin of coordinates by f^k for $k = 1, 2, \ldots$ and the dynamics of the map f^k restricted to these straight lines.

Let $\mathbb{F}$ be the set of all real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, and let $\mathbb{F}[x_1, \ldots, x_n]$ be the ring of all polynomials in the n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{F}$. A polynomial $P(x_1, \ldots, x_n)$ of degree m is homogeneous, if

$$P(\lambda x_1, \ldots, \lambda x_n) = \lambda^m (x_1, \ldots, x_n) \text{ for all } \lambda \in \mathbb{F} \setminus \{0\}.$$

A homogeneous polynomial map $f : \mathbb{F}^n \to \mathbb{F}^n$ of degree m is a map $f = (f_1, \ldots, f_n)$, where $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ is a homogeneous polynomial of degree m for all $i = 1, \ldots, n$.

Here, f^k denotes the composition of the map f with itself k times. A point $\mathbf{x} \in \mathbb{F}^n$ is fixed by the map f, if $f(\mathbf{x}) = \mathbf{x}$. Let $k > 1$ be a positive integer. A point $\mathbf{x} \in \mathbb{F}^n$ is k–periodic or periodic point of period k by the map f, if $f^k(\mathbf{x}) = \mathbf{x}$ and $f^j(\mathbf{x}) \neq \mathbf{x}$ for $j = 1, \ldots, k-1$. The set

$$\{\mathbf{x}, f(\mathbf{x}), f^2(\mathbf{x}), \ldots, f^{k-1}(\mathbf{x})\}$$

is the *periodic orbit of* $\mathbf{x}$.

We say that the fixed points of f have period 1. We shall denote by $\mathrm{Per}(f)$, the set of periods of all periodic points of the map f. Clearly, $\mathrm{Per}(f)$ is a subset of the set $\mathbb{N}$ of all positive integers.

The sets $\mathrm{Per}(f)$ when f is a homogeneous polynomial map of degree m change completely, if the homogeneous polynomial map is real or complex. If the degree $m = 1$, then the map is linear, and its dynamics is easy to study. Here, we only consider homogeneous polynomial maps of degree $m > 1$.

In order to state our result for the complex homogeneous polynomial maps of degree $m > 1$, we need some preliminary notions.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree m. For each $\mathbf{x} \in \mathbb{C}^n \setminus \{0\}$, we define the straight line $\mathcal{L}_{\mathbf{x}}$ through the origin of $\mathbb{C}^n$ as

$$\mathcal{L}_{\mathbf{x}} = \{\mu \mathbf{x} : \text{ for all } \mu \in \mathbb{C}\},$$

and we say that $\mathbf{x}$ is a director vector of $\mathcal{L}_{\mathbf{x}}$. A straight line through the origin of $\mathbb{C}^n$ with the director vector $\mathbf{x}$ is invariant by f, if $f(\mathbf{x}) = \lambda \mathbf{x}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then, $f(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$.

It is clear that every k–periodic point $\mathbf{x}$ of f is on the invariant straight line $\mathcal{L}_{\mathbf{x}}$ of f^k, i.e., $f^k(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$. Moreover, the set of straight lines

$$\{\mathcal{L}_{\mathbf{x}}, \mathcal{L}_{f(\mathbf{x})}, \mathcal{L}_{f^2(\mathbf{x})}, \ldots, \mathcal{L}_{f^{k-1}(\mathbf{x})}\}$$

is a k–periodic orbit of f in the set of all invariant straight lines through the origin, i.e., $f^k(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$ and $f^\ell(\mathcal{L}_{\mathbf{x}}) \neq \mathcal{L}_{\mathbf{x}}$ for $\ell = 1, \ldots, k-1$.

The Möbius function $\mu : \mathbb{N} \to \mathbb{N}$ is defined as

$$\mu(r) = \begin{cases} 1, & \text{if } r = 1, \\ 0, & \text{if some } k^2 | r \text{ for some } k \in \mathbb{N}, \\ (-1)^s, & \text{if } r = p_1 \cdots p_s \text{ (distinct primes)}. \end{cases}$$

Theorem 1.1. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree $m > 1$ such that $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$. Let k be a positive integer.*

(a) The number of invariant straight lines through the origin for the map f^k is $(m^{kn}-1)/(m^k-1)$ taking into account their multiplicities.

(b) The number of ℓ–periodic invariant straight lines through the origin for the map f^k is

$$\sum_{r|\ell} \mu(r) \frac{m^{k\ell n/r}-1}{m^{k\ell/r}-1},$$

where $\mu : \mathbb{N} \to \mathbb{N}$ is the Möbius function, and if all the multiplicities of the invariant straight lines of statement (a) are simple.

(c) If $\mathcal{L}_{\mathbf{x}}$ is an invariant straight line of f^k, then $\mathrm{Per}(f^k|_{\mathcal{L}_{\mathbf{x}}}) = \mathbb{N}$.

(d) All the periodic points of the map $f^k|_{\mathcal{L}_{\mathbf{x}}}$ with the exception of the origin are on a circle centered at the origin of coordinates, and these periodic points are repelling. In fact, this circle is the Julia set of the map $f^k|_{\mathcal{L}_{\mathbf{x}}} : \mathbb{C} \to \mathbb{C}$, where we have identified $\mathcal{L}_{\mathbf{x}}$ with $\mathbb{C}$.

(e) The number of ℓ–periodic points of the map $f^k|_{\mathcal{L}_{\mathbf{x}}}$ is

$$\sum_{r|\ell} \mu(r) m^{k\ell/r},$$

where $\mu : \mathbb{N} \to \mathbb{N}$ is the Möbius function.

The multiplicities mentioned in Theorem 1.1 are in the sense of the multiplicities of the Bezout Theorem (see [7,12,26] for more details). For a definition of the Julia set, see [9] for instance.

Theorem 1.1 is proved in Section 2.

By Theorem 1.1, every homogeneous polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ of degree $m > 1$ has k–periodic points for all positive integer k. Hence, from the point of view of the periodicity, these maps are not very interesting. The situation is completely different from homogeneous polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$.

The homogeneous polynomial maps $f : \mathbb{CP}^n \to \mathbb{CP}^n$ of degree d have been studied by Fornaes and Sibony [11]. Their results are related to the ones presented in statement (a) of Theorem 1.1 for the homogeneous polynomial maps $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of degree m, because the periodic points of the homogeneous polynomial maps of $\mathbb{CP}^n$ are the periodic straight lines through the origin of $\mathbb{C}^{n+1}$ (see Corollary 3.2 of [11] for more details, and also see [23] and [25]).

We denote the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ by $||\mathbf{x}||$. Let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$, i.e., $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous polynomial map of degree m such that $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$. Then, the map

$$F = F_f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \text{ is defined by } F(\mathbf{x}) = \frac{f(\mathbf{x})}{||f(\mathbf{x})||}.$$

Let $\mathbb{Z}$ be the set of all integer numbers. The number of elements of the set

$$\left\{ (x_1, \ldots, x_n) \in \mathbb{Z}^n : 0 \le x_i < m,\ \sum_{i=1}^{n} x_i = \frac{1}{2}(m-1)n \right\}$$

is denoted by $\pi_n(m)$. This number was introduced by Arnold [3] (also see Khovanskii [18]), and for low values of n are

$$\pi_2(m) = m,$$

$$\pi_3(m) = \begin{cases} (1+3m^2)/4, & \text{if } m \text{ is odd},\\ 0, & \text{if } m \text{ is even}, \end{cases}$$

$$\pi_4(m) = (m+2m^3)/3,$$

$$\pi_5(m) = \begin{cases} (27+50m^2+115m^4)/192 & \text{if } m \text{ is odd},\\ 0, & \text{if } m \text{ is even}, \end{cases}$$

$$\pi_6(m) = (4m+5m^3+11m^5)/20,$$

$$\pi_7(m) = \begin{cases} (1125+1813m^2+2695m^4+5887m^6)/11520, & \text{if } m \text{ is odd},\\ 0, & \text{if } m \text{ is even}, \end{cases}$$

$$\pi_8(m) = (45m+49m^3+70m^5+151m^7)/315.$$

For more details about the number $\pi_n(m)$, see the appendix of the paper of Cima, Gasull and Torregrosa [6].

For the real homogeneous polynomial maps of degree $m > 1$, we have the following results.

Theorem 1.2. *The following statements hold.*

(a) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a homogeneous polynomial map of degree* m *such that* $f(\mathbf{x}) \neq 0$*, if* $\mathbf{x} \neq 0$*.*

  (a.1) Then, $\mathrm{Per}(f) = \mathrm{Per}(F) \cup \{1\}$*, and*

  (a.2) the topological degree of F *is an integer* d *congruent with* m *module* 2 *satisfying* $|d| \leq \pi_n(m)$*.*

(b) Let d *be an integer congruent with* m *module* 2 *satisfying* $|d| \leq \pi_n(m)$*. Then, there exists a homogeneous polynomial map of degree* m *such that* $f(\mathbf{x}) \neq 0$*, if* $\mathbf{x} \neq 0$*, for which the topological degree of* F_f *is* d*.*

In fact, statements (a.2) and (b) of Theorem 1.2 follow directly from [3] and [18] respectively (for more details, see Corollary 1 of [18] and its proof). Here, we will only prove statement (a) of Theorem 1.2 in Section 3.

The set of periods of the homogeneous polynomial maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ is characterized in the next theorem, but we need some definitions first.

We recall the Sharkovskii ordering on the set $\mathbb{N}$, and this total ordering from the largest one to the smallest one is

$$3, 5, 7, \ldots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \ldots, 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, \ldots, 2^\ell\cdot 3, 2^\ell\cdot 5, 2^\ell\cdot 7, \ldots, 2^3, 2^2, 2, 1.$$

For $k \in \mathbb{N}$, the set $S(k)$ is formed by k and all the elements of $\mathbb{N}$, which follows k in the Sharkovskii ordering. Thus, for instance, $S(3) = \mathbb{N}$. We also define the set $S(2^\infty) = \{1, 2, 4, 8, \ldots\}$.

Let $a, b \in \mathbb{R}$. For $a \leq b$, we define the set

$$M(a,b) = \{k \in \mathbb{N} : a < \ell/k < b\},$$

i.e., $M(a,b)$ is formed by the positive denominators of all rational numbers, which are in the interior of the interval $[a,b]$. We note that we do not assume that ℓ and k are coprime. Of course, $M(a,b) = \emptyset$, if $a = b$.

Every continuous map $F : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1 has associated with a closed rotation interval $[a,b]$, and eventually a and b can be equal (see [2] for a precise definition of the rotation interval).

If $a = \ell/k$ is rational with ℓ and k coprime, then we define the set $S(a,r) = kS(r)$ for some $r \in \mathbb{N} \cup \{2^\infty\}$. Here, $kS(r)$ is the set formed by all the elements of the set $S(r)$ multiplied by k. If a is irrational, then we define the set $S(a,r) = \emptyset$.

Theorem 1.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homogeneous polynomial map of degree $m > 1$ such that $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$, and let $d \in [-m, m]$ be the topological degree of $F = F_f$. If $d = 1$, we denote by $[a,b]$ as the rotation interval of f. Then,* $\mathrm{Per}(f) = \mathrm{Per}(F) \cup \{1\}$ *and* $\mathrm{Per}(F)$ *are equal to*

$\mathbb{N}$, *if* $d \neq -2, -1, 0, 1$;

$\mathbb{N} \setminus \{2\}$ or $\mathbb{N}$, *if* $d = -2$;

$S(k)$ *for some* $k \in \mathbb{N} \cup \{2^\infty\}$, *if* $d = -1, 0$;

$S(a,r) \cup M(a,b) \cup S(b,s)$ *for some* $r, s \in \mathbb{N} \cup \{2^\infty\}$, *if* $d = 1$.

Theorem 1.3 is proved in Section 4.

From Theorem 1.3, it follows that $\mathrm{Per}(F) \neq \emptyset$, except if $d = 1$ and $a = b \notin \mathbb{Q}$.

In general, we can say almost nothing about the set of periods of the homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$, when $n > 2$, except if these maps are transversal. Then, we can say many things.

Clearly, the map $F = F_f$, associated to a homogeneous polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$, is of class $\mathcal{C}^1$.

A $\mathcal{C}^1$ map $F : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is called transversal, if for all $k \in \mathbb{N}$, the graph of F^k intersects transversally, and the diagonal of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ at each point (x,x) such that x is a fixed point of f^k. In other words, the Jacobian matrix $DF^k(x)$ does not have the eigenvalue 1 for all fixed point x of f^k.

Theorem 1.4. *For $n > 2$, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous polynomial map of degree $m > 1$ such that $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$, and let $d \in [-\pi_n(m), \pi_n(m)]$ be the topological degree of $F = F_f$. Assume that F is transversal. Then, we have* $\mathrm{Per}(f) = \mathrm{Per}(F) \cup \{1\}$ *and*

$1 \in \mathrm{Per}(F)$, *if* $d = 0$;

$\mathrm{Per}(F)$ *can be empty, if n is odd and* $d = 1$;

$1 \in \mathrm{Per}(F)$, *if n is even and* $d = 1$;

$1 \in \mathrm{Per}(F)$, *if n is odd and* $d = -1$;

$\{1,2\} \cap \mathrm{Per}(f) \neq \emptyset$, *if n is even and* $d = -1$;

$\mathrm{Per}(F) \supset \{1, 3, 5, 7, \ldots\}$, *and for every k, even if* $k \notin \mathrm{Per}(F)$, *then*

$\mathrm{Per}(F) \supset \{k/2, 2k\}$, *if* $d \notin \{-1, 0, 1\}$.

Theorem 1.4 is proved in Section 5.

Here, we have considered homogeneous polynomial maps. However, if we consider polynomial maps, many other questions on these maps can be studied (see, for instance, [15, 16, 20, 28]).

2. Proof of Theorem 1.1

The set of invariant straight lines through the origin of a homogeneous polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ of degree m has been studied by the authors of [23] and by Rong [25] (also see [11]). The following results were proved by the authors under generic assumptions (see Theorem 1 of [23]), and in general in Theorem 1 of [25]. It also follows from Corollary 3.2 of [11].

Theorem 2.1. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree m. Assume that f has finitely many straight lines $\mathcal{L}_{\mathbf{x}}$ such that $f(\mathcal{L}_{\mathbf{x}}) \subset \mathcal{L}_{\mathbf{x}}$. Then, f has $1+m+m^2+\cdots+m^{n-1} = (m^n-1)/(m-1)$ of these straight lines through the origin taking into account their multiplicities.*

From Theorem 2.1, it follows immediately the next result.

Corollary 2.1. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree m such that $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$. Assume that f has finitely many invariant straight lines $\mathcal{L}_{\mathbf{x}}$ such that $f(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$. Then, f has $(m^n-1)/(m-1)$ invariant straight lines through the origin taking into account their multiplicities.*

In 1964, Baker [4] completely described the periodic set of any polynomial map $P : \mathbb{C} \to \mathbb{C}$. More precisely he proved as follows.

Theorem 2.2. *Let P be a polynomial of degree at least two and suppose that $P : \mathbb{C} \to \mathbb{C}$ has no periodic points of period k. Then, $k = 2$, and P is conjugate to the polynomial $z^2 - z$.*

The proof of this theorem can also be found in [5] and [10]. In this last paper, Theorem 2.2 has been proved by using Lefschetz numbers.

Proof. [Proof of Theorem 1.1] Statement (a) follows immediately from Corollary 2.1.

For a map h having finitely many periodic points of every period, we define $\mathcal{F}(k)$ as the number of fixed points of h^ℓ and $\mathcal{P}(\ell)$ as the number of periodic points of period ℓ of h. Then, it is known that

$$\mathcal{P}(\ell) = \sum_{r|\ell} \mu(r)\mathcal{F}(\ell/r). \tag{2.1}$$

For more details, see [24]. Then, for the map f acting on the straight lines through the origin, we have $\mathcal{F}(\ell) = (m^{\ell n}-1)/(m^\ell-1)$. In short, from (2.1), it immediately follows statement (b) of Theorem 1.1.

We shall prove the rest of the statements of Theorem 1.1 for $k = 1$, the proof for $k > 1$ is completely similar recalling that the degree of f^k for $k > 1$ is m^k.

If f has infinitely many invariant straight lines, let $\mathcal{L}_{\mathbf{x}}$ be one of them. Otherwise, since $f(\mathbf{x}) \neq 0$, if $\mathbf{x} \neq 0$, by Corollary 2.1, f has invariant straight lines. Let $\mathcal{L}_{\mathbf{x}}$ be one invariant straight line of f.

Changing the coordinates from $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ to $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{C}^n$ and doing a rotation through the origin of coordinates which passes the invariant straight line $\mathcal{L}_{\mathbf{x}}$ to the y_1–axis of $\mathbb{C}^n$, the homogeneous polynomial map f of degree m becomes a homogeneous polynomial map g of degree m having the y_1–axis invariant. Hence, we have $g(y_1, 0, \dots, 0) = (ay_1^m, 0, \dots, 0)$ with $a \in \mathbb{C} \setminus \{0\}$. In other words, the map $f|_{\mathcal{L}_{\mathbf{x}}}$ can be identified with the polynomial map $P : \mathbb{C} \to \mathbb{C}$ defined by $P(z) = az^m$. Therefore, all fixed points of P are the origin and all roots of the equation $z^{m-1} = (1/a)^{1/(m-1)}$.

By Theorem 2.2, and since $m > 1$ and the polynomial map $P(z) = az^m$ is not conjugated to the map $z \mapsto z^2 - z$, it follows that $\mathrm{Per}(P) = \mathbb{N}$. Consequently, $\mathrm{Per}(f|_{\mathcal{L}_{\mathbf{x}}}) = \mathbb{N}$, and statement (c) of Theorem 1.1 is proved.

Since $f^\ell|_{\mathcal{L}_{\mathbf{x}}}$ can be identified with the polynomial map $P^\ell : \mathbb{C} \to \mathbb{C}$ of the form $P^\ell(z) = a^{(m^\ell-1)/(m-1)} z^{m^\ell}$, all periodic points of period ℓ or of period a divisor of ℓ of P are solutions of the equation $a^{(m^\ell-1)/(m-1)} z^{m^\ell} = z$. Then, for the map P, the origin $z = 0$ is a fixed point, and the other periodic points with period a divisor of ℓ are the roots of the equation

$$z^{m^\ell-1} = \frac{1}{a^{\frac{m^\ell-1}{m-1}}}.$$

Therefore, all periodic points of $f^\ell|_{\mathcal{L}_{\mathbf{x}}}$ are on the circle centered at the origin of radius $(1/a)^{1/(m-1)}$ contained in $\mathcal{L}_{\mathbf{x}}$, and the unique exception is the fixed point localized at the origin.

Assume $f(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$. Therefore, if $\mathbf{y} \in \mathcal{L}_{\mathbf{x}} \setminus \{0\}$, we have $f(\mathbf{y}) = \lambda(\mathbf{y})\mathbf{y}$ with $\lambda(\mathbf{y}) \neq 0$, and consequently,

$$f^k(\mathbf{y}) = \lambda(\mathbf{y})^{1+m+m^2+\cdots+m^{k-1}}\mathbf{y}.$$

Therefore, the fixed point 0 at the origin of the invariant straight line $\mathcal{L}_{\mathbf{x}} = \mathbb{C}$ is stable for the map $f|_{\mathcal{L}_{\mathbf{x}}}$, and its bassin of attraction is the set of points $\mathbf{y} \in \mathcal{L}_{\mathbf{x}}$ such that $|\lambda(\mathbf{y})| < 1$. Noting that for the points $\mathbf{y}$ in the circle of radius $(1/a)^{1/(m-1)}$ centered at the origin, we have their $|\lambda(\mathbf{y})| = 1$, because the set of all periodic points of $f|_{\mathcal{L}_{\mathbf{x}}}$ is dense in this circle. Hence, all the points of $\mathcal{L}_{\mathbf{x}}$ in the interior of the region limited by the circle containing the periodic points of $f|_{\mathcal{L}_{\mathbf{x}}}$ tend to the origin under the iteration by $f|_{\mathcal{L}_{\mathbf{x}}}$, and the points outside the region limited by this circle tend to infinity for their $|\lambda(\mathbf{y})| > 1$. Therefore, all periodic points of the circle are repelling, and since they are dense in the circle, its closure (the Julia set) is the circle. In short, statement (d) is proved.

Applying formula (2.1) to the map $f|_{\mathcal{L}_{\mathbf{x}}}$, we have $\mathcal{F}(\ell) = m^\ell$. Therefore, it immediately follows statement (e) of Theorem 1.1. □

Remark 2.1. If $f(\mathbf{x}) = \lambda(\mathbf{x})\mathbf{x}$ for some $\lambda(\mathbf{x}) \in \mathbb{C} \setminus \{0\}$, then for every $\mathbf{y} \in \mathbb{C}$, we have $f(\mathbf{y}) = \lambda(\mathbf{y})\mathbf{y}$, where $\lambda(\mathbf{y}) = c^{m-1}\lambda(\mathbf{x})$ with $c = \mathbf{y}/\mathbf{x}$.

3. Proof of Theorem 1.2

We denote by $\mathbb{R}^+$, the set of all positive real numbers. For each $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, we define the *ray* $L_{\mathbf{x}}$ as

$$L_{\mathbf{x}} = \{\lambda\mathbf{x} : \text{ for all } \lambda \in \mathbb{R}^+\}.$$

Therefore, it is clear that $f(L_{\mathbf{x}}) = L_{f(\mathbf{x})}$.

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous polynomial map of degree $m > 1$. If $L_{\mathbf{x}} = L_{f(\mathbf{x})}$, then there is a unique fixed point in the ray $L_{\mathbf{x}}$.*

Proof. Since the ray $L_{\mathbf{x}}$ is invariant by the map f, working as in the proof of Theorem 1.1, we can identify the map $f|_{L_{\mathbf{x}}}$ to the polynomial map $P : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $P(z) = az^m$, but now with $a > 0$.

Since the equation $az^m = z$ has a unique real solution, namely, $z = (1/a)^{1/(m-1)}$, it follows that the map $f|_{L_{\mathbf{x}}}$ has a unique fixed point. Hence, the lemma is proved. □

Under the assumptions of Lemma 3.1, it is easy to check that the fixed point y on the ray $L_{\mathbf{x}} = \mathbb{R}^+ = (0, +\infty)$ is unstable. More precisely, the points smaller than y tends to the origin, and the points larger than y tends to $+\infty$.

Proof. (Proof of statement (a) of Theorem 1.2) We claim that if $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ is a k–periodic point of f, then $\mathbf{x}/||\mathbf{x}|| \in \mathbb{S}^{n-1}$ is a k–periodic point of F. Now, we shall prove the claim.

First, by induction, we shall see

$$F^j\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right) = \frac{f^j(\mathbf{x})}{||f^j(\mathbf{x})||} \text{ for all } j = 1, 2, \ldots \text{ and } \mathbf{x} \neq 0. \tag{3.1}$$

For $j = 1$, equality (3.1) follows from

$$F\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right) = \frac{f\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)}{||f\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||} = \frac{\frac{1}{||\mathbf{x}||^m} f(\mathbf{x})}{||\frac{1}{||\mathbf{x}||^m} f(\mathbf{x})||} = \frac{f(\mathbf{x})}{||f(\mathbf{x})||}.$$

Assume that (3.1) holds for $1, 2, \ldots, j$, and we shall prove it for $j + 1$. Indeed, we have

$$\begin{aligned} F^{j+1}\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right) &= F\left(\frac{f^j(\mathbf{x})}{||f^j(\mathbf{x})||}\right) = \frac{f\left(\frac{f^j(\mathbf{x})}{||f^j(\mathbf{x})||}\right)}{||f\left(\frac{f^j(\mathbf{x})}{||f^j(\mathbf{x})||}\right)||} \\ &= \frac{\frac{1}{||f^j(\mathbf{x})||^m} f^{j+1}(\mathbf{x})}{\frac{1}{||f^j(\mathbf{x})||^m} ||f^{j+1}(\mathbf{x})||} = \frac{f^{j+1}(\mathbf{x})}{||f^{j+1}(\mathbf{x})||}. \end{aligned}$$

Hence, (3.1) is proved.

Now, we prove the claim. Let $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ be a k–periodic point of f. First, from (3.1), we have

$$F^k\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right) = \frac{f^k(\mathbf{x})}{||f^k(\mathbf{x})||} = \frac{\mathbf{x}}{||\mathbf{x}||}.$$

Therefore, $\mathbf{x}/||\mathbf{x}||$ is fixed by F^k. It remains to show

$$F^j\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right) \neq \frac{\mathbf{x}}{||\mathbf{x}||} \text{ for all } j = 1, \ldots, k-1.$$

Assume

$$F^j\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right) = \frac{\mathbf{x}}{||\mathbf{x}||} \text{ for some } j \in \{1, \ldots, k-1\}.$$

Then, $f^j(\mathbf{x})$ and $\mathbf{x}$ are in the ray $L_{\mathbf{x}}$, and this ray is invariant by f^k, i.e., $L_{\mathbf{x}} = L_{f^k(\mathbf{x})}$. Therefore, f^k has two fixed points in the ray $L_{\mathbf{x}}$, namely, $f^j(\mathbf{x})$ and $\mathbf{x}$, which are in contradiction with Lemma 3.1.

In short, we have proved that if $x \in \mathrm{Per}(f)$, then $x \in \mathrm{Per}(F)$. On the other hand, letting O be the origin of $\mathbb{R}^n$, since $f(O) = O$, it follows that $1 \in \mathrm{Per}(f)$. Therefore, we prove that

$$\mathrm{Per}(f) \subset \mathrm{Per}(F) \cup \{1\}.$$

Now, we claim that if $\mathbf{x} \in \mathbb{S}^{n-1}$ is a k–periodic point of F, then there exists $\mathbf{y} \in L_{\mathbf{x}}$ such that $\mathbf{y}$ is a k–periodic point of f. We prove the claim. From (3.1), we have

$$\mathbf{x} = F^k(\mathbf{x}) = \frac{f^k(\mathbf{x})}{||f^k(\mathbf{x})||}.$$

Hence, $f^k(\mathbf{x}) = ||f^k(\mathbf{x})||\mathbf{x}$. Then, $L_{\mathbf{x}} = L_{f^k(\mathbf{x})}$. Therefore, by Lemma 3.1, there exists $\mathbf{y} \in L_{\mathbf{x}}$ such that $f^k(\mathbf{y}) = \mathbf{y}$. From (3.1), if follows that $L_{f^j(\mathbf{x})} = L_{F^j(\mathbf{x})}$ for $j = 1, 2, \ldots$, then since $L_{\mathbf{x}} = L_{\mathbf{y}}$, the rays $L_{f^j(\mathbf{y})} = L_{f^j(\mathbf{x})} = L_{F^j(\mathbf{x})}$ for $j = 1, 2, \ldots$ Since the rays $L_{F^j(\mathbf{x})}$ are different for $j = 0, 1, \ldots, k-1$, it follows that $\mathbf{y}$ is a k–periodic point of f. Hence, the claim is proved. Consequently,

$$\mathrm{Per}(F) \subset \mathrm{Per}(f).$$

This completes the proof. □

4. Continuous maps of the circle $\mathbb{S}^1$

The following result is proved in [2].

Theorem 4.1. *Let $F : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $d = 1$, we denote by $[a, b]$, the rotation interval of F. Then,* $\mathrm{Per}(F)$ *is equal to*

$\mathbb{N}$, *if* $d \neq -2, -1, 0, 1$;

$\mathbb{N} \setminus \{2\}$ or $\mathbb{N}$, *if* $d = -2$;

$S(k)$ *for some* $k \in \mathbb{N} \cup \{2^\infty\}$, *if* $d = 0, 1$;

$S(a, r) \cup M(a, b) \cup S(b, s)$ *for some* $r, s \in \mathbb{N} \cup \{2^\infty\}$, *if* $d = 1$.

Theorem 1.3 immediately follows from Theorem 4.1.

5. Transversal maps on the sphere $\mathbb{S}^{n-1}$

For a $\mathcal{C}^1$ transversal map $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, we define its minimal set of periods in the class of $\mathcal{C}^1$ transversal self–maps of $\mathbb{S}^{n-1}$ as the set

$$\mathrm{MPer}(f) := \bigcap_g \mathrm{Per}(g),$$

where g runs over all $\mathcal{C}^1$ transversal self–maps of $\mathbb{S}^{n-1}$ of the same degree than f.

For $\mathcal{C}^1$ transversal self–maps of $\mathbb{S}^{n-1}$, we have the following result (see Theorem 3 of [22]).

Theorem 5.1. *Let $F : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a transversal map of degree d.*

(a) If $d = 0$*, then* $\mathrm{MPer}(F) = \{1\}$.

(b) If $n - 1$ *is odd and* $d = 1$*, then* $\mathrm{MPer}(F) = \emptyset$.

(c) If $n - 1$ *is even and* $d = 1$*, then* $\mathrm{MPer}(F) = \{1\}$.

(d) If $n - 1$ *is odd and* $d = -1$*, then* $\mathrm{MPer}(F) = \{1\}$.

(e) If $n - 1$ *is even and* $d = -1$*, then* $\{1, 2\} \cap \mathrm{MPer}(F) \neq \emptyset$.

(f) If $d \notin \{-1, 0, 1\}$*, then* $\mathrm{MPer}(F) \supset \{1, 3, 5, 7, \ldots\}$*, and for* k *even if* $k \notin \mathrm{MPer}(F)$*, then* $\mathrm{MPer}(F) \supset \{k/2, 2k\}$.

Now, Theorem 1.4 is an easy corollary of Theorem 5.1.

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