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Asymptotic Behavior of a Stochastic Predator-prey Model with Beddington-DeAngelis Functional Response and Lévy Jumps*

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Abstract A stochastic two-prey-one-predator model with Beddington-DeAngelis functional response and Lévy jumps is proposed and investigated in this paper. First of all, we prove the existence and uniqueness of the global positive solution, and stochastic ultimate boundedness of the solution. Next, under a simple assumption, by using Itô formula and other important inequalities, some sufficient conditions are established to ensure the extinction and persistence in the mean of the system. The results show that neither strong white noise nor Lévy noise is conducive to the persistence of the population. Finally, the theoretical results are verified by numerical simulations.

Keywords Stochastic predator-prey model, Beddington-DeAngelis functional response, Lévy jump, Extinction, Persistence.

MSC(2010) 34K50, 60H10, 93E03.

1. Introduction

In natural ecology, there are many complex relationships and interactions between organisms, which constitute a biological population system together. However, the interaction between predator-prey population is considered to be the most important one. Therefore, predator-prey model has become an important topic for many scholars. In addition, it is a common phenomenon in nature that predators often feed on some competing prey. So far, many scholars have carried out extensive research on deterministic two-prey-one-predator systems (see [1, 8, 9, 12, 14, 15] and other references).

Functional response has always been an important component of predator-prey dynamics. Functional responses are generally divided into two categories. One is prey-dependent functional responses, the most common of which are Holling-I, Holling-II and Holling-III, but these functional responses only consider the density of prey. The other is predator-dependent functional responses, which generally include Beddington-DeAngelis type [4, 7], Crowley-Martin type [5] and Hassell-Varley type [10]. They consider both prey density and predator density. In ecology, species

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not only compete, but also interfere with each other. Therefore, we consider using Beddington-DeAngelis functional response in this paper. In 1975, Beddington [4] and DeAngelis [7] first introduced a predator-prey model with Beddington-DeAngelis functional response

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = r_1 x_1 - \alpha_1 x_1^2 - \dfrac{c_1 x_1 x_2}{a_1 + a_2 x_1 + a_3 x_2}, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -r_2 x_2 - \alpha_2 x_2^2 + \dfrac{c_2 x_1 x_2}{a_1 + a_2 x_1 + a_3 x_2}, \end{cases} \tag{1.1}$$

where $\frac{c_1 x_1}{a_1+a_2x_1+a_3x_2}$ becomes the Beddington-DeAngelis functional response. $x_i(i = 1, 2)$ is the size of the ith population at time t, r_1 denotes the intrinsic growth rate of the prey, r_2 denotes death rate of the predator, c_1 represents the effect of capture rate, a_1 is a saturation constant, a_2 is the effect of handing time, and a_3 represents the magnitude of interference among predators.

On one hand, in real life, the population is inevitably affected by environmental noise. Therefore, it is very necessary to study the dynamic impact of white noise on population system. So far, a large number of scholars have proposed a variety of stochastic population models (see [6,13,16,18,19,22,26–30]). Liu et al., [22] derived some conditions for species to be stochastically permanent. They also showed that the species will become extinct with probability one, if the noise is sufficiently large. Das and Samanta [6] came to a conclusion that the environmental noise plays an important role in the extinction and persistence of prey and predator populations.

On the other hand, in ecology, sudden environmental disturbances, such as tsunamis, volcanic eruptions, avian influenza and infectious diseases, also have a very important impact on the population system. However, due to the abruptness and intensity of these events, the sampling path will be discontinuous. Therefore, these phenomena cannot be accurately described by Brownian motion. In this case, some scholars (see [2,24,25,31,32]) pointed out that Lévy jump can be introduced into the model for modeling. Zhu and Li [32] dealt with a predator-prey model of Beddington-DeAngelis type functional response with Lévy jumps. They proved that the variation of Lévy jumps can affect the asymptotic property of the system.

Inspired by the above discussion and references, we assume that the intrinsic growth rate of preys and the mortality of predator are affected by Lévy noise. That is,

$$r_1 \mathrm{d}t \to r_1 \mathrm{d}t + \sigma_1 \mathrm{d}B_1(t) + \int_{\mathbb{Y}} \gamma_1(u) \tilde{N}(\mathrm{d}t, \mathrm{d}u),$$

$$r_2 \mathrm{d}t \to r_2 \mathrm{d}t + \sigma_2 \mathrm{d}B_2(t) + \int_{\mathbb{Y}} \gamma_2(u) \tilde{N}(\mathrm{d}t, \mathrm{d}u),$$

$$-r_3 \mathrm{d}t \to -r_3 \mathrm{d}t + \sigma_3 \mathrm{d}B_3(t) + \int_{\mathbb{Y}} \gamma_3(u) \tilde{N}(\mathrm{d}t, \mathrm{d}u),$$

where σ_i represents the intensity of the white noise and $\sigma_i > 0$, and $B_i(t)$ $(i = 1, 2, 3)$ is independent standard Brownian motion.

According to (1.1), based on the fact that the most common system in the ecosystem is two-prey-one-predator system, we propose the following stochastic two-prey-one-predator system with the Beddington-DeAngelis functional response and

Lévy jumps

$$
\begin{cases}
\mathrm{d}x_1(t) = x_1(t)[r_1 - \alpha_1 x_1(t) - \dfrac{c_1 x_3(t)}{a_1 + a_2 x_1(t) + a_3 x_3(t)} - \beta_1 x_2(t)]\mathrm{d}t \\
\qquad + \sigma_1 x_1(t)\mathrm{d}B_1(t) + x_1(t^-)\displaystyle\int_{\mathbb{Y}} \gamma_1(u)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \\
\mathrm{d}x_2(t) = x_2(t)[r_2 - \alpha_2 x_2(t) - \dfrac{c_2 x_3(t)}{b_1 + b_2 x_2(t) + b_3 x_3(t)} - \beta_2 x_1(t)]\mathrm{d}t \\
\qquad + \sigma_2 x_2(t)\mathrm{d}B_2(t) + x_2(t^-)\displaystyle\int_{\mathbb{Y}} \gamma_2(u)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \\
\mathrm{d}x_3(t) = x_3(t)[-r_3 - \alpha_3 x_3(t) + \dfrac{e_1 x_1(t)}{a_1 + a_2 x_1(t) + a_3 x_3(t)} \\
\qquad + \dfrac{e_2 x_2(t)}{b_1 + b_2 x_2(t) + b_3 x_3(t)}]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}B_3(t) \\
\qquad + x_3(t^-)\displaystyle\int_{\mathbb{Y}} \gamma_3(u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),
\end{cases}
\tag{1.2}
$$

where $x_i(t)$ is the size of the ith population at time t, $x_i(t^-)$ is the left limit of $x_i(t)$, $r_i(i = 1, 2)$ represents the intrinsic growth rate of prey, r_3 denotes the mortality of predator, α_i is the intra-specific competition coefficient of the ith population, β_1 and β_2 are the competitive coefficients of x_1 and x_2 respectively, $\frac{e_j}{c_j}(j = 1, 2)$ represents conversion factor denoting the newly born predator for each captured prey, and all of the coefficients $r_i, \alpha_i, a_i, b_i, c_j, \beta_j$ and $e_j(i = 1, 2, 3, j = 1, 2)$ are positive constants. N is a Poisson counting measure with the characteristic measure λ on a measurable subset $\mathbb{Y}$ of $(0, +\infty)$ with $\lambda(\mathbb{Y}) < +\infty$, $\tilde{N}(\mathrm{d}t,\mathrm{d}u) = N(\mathrm{d}t,\mathrm{d}u) - \lambda(\mathrm{d}u)\mathrm{d}t$. Throughout this paper, we further assume that $B_i(t)$ and $\mathbb{N}$ are independent.

From a biological point of view, we suppose $1 + \gamma_i(u) > 0$, where $\gamma_i(u) > 0$ means the increase of the species (e.g., planting), and $-1 < \gamma_i(u) < 0$ means the decrease of the species (e.g., harvesting and epidemics), $u \in \mathbb{Y}, i = 1, 2, 3$.

The rest parts of the paper are organized as follows. Some preliminaries are given in Section 2. In Section 3, we prove that there is a unique global positive solution for system (1.2) with the initial value $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}}$, and the solution is stochastically ultimately bounded. In Section 4, we present sufficient conditions for the extinction and persistence in the mean of the system, which is the most important part of this paper. In Section 5, we verify our theoretical results by numerical simulation. Finally, the conclusions are given in Section 6.

2. Preliminaries

In this section, we will provide some information which helps establish our main results. For the sake of convenience and simplicity in the following discussion, we always use the notations

$$\mu_i = \frac{\sigma_i^2}{2} - \int_{\mathbb{Y}} [\ln(1 + \gamma_i(u)) - \gamma_i(u)]\lambda(\mathrm{d}u), \quad i = 1, 2, 3;$$

$$N_i(t) = \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(u))\tilde{N}(\mathrm{d}s,\mathrm{d}u), \quad i = 1, 2, 3;$$

$$g_* = \liminf_{t\to+\infty} g(t), \qquad g^* = \limsup_{t\to+\infty} g(t), \qquad \langle g(t)\rangle = \frac{1}{t}\int_0^t g(s)\mathrm{d}s,$$

where $g(t)$ is a continuous bounded function.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), B_2(t), B_3(t))^{\mathrm{T}}$ be 3-dimensional independent standard Brownian motions defined on this probability space. Let $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_i \geq 0, 1 \leq i \leq 3\}$. We define the norm as $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Assumption 2.1. Assume that there exists a positive constant c such that

$$\int_{\mathbb{Y}} [\ln(1+\gamma_i(u)) \vee (\ln(1+\gamma_i(u)))^2]\lambda(\mathrm{d}u) < c, i = 1, 2, 3.$$

This assumption means that the intensities of Lévy jumps will not be too large.

Definition 2.1. ([20]) For any given initial value $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} \in \mathbb{R}^3_+$, the solution $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ of system (1.2) is
(i) extinctive, if $\lim_{t\to+\infty} x(t) = 0$ a.s;
(ii) persistent in the mean, if $\liminf_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\mathrm{d}s > 0$ a.s.

Definition 2.2. ([17]) For any given initial value $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} \in \mathbb{R}^3_+$, the solution $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ of system (1.2) is said to be stochastically and ultimately bounded for any $\varepsilon \in (0, 1)$, if there is a positive constant $\eta = \eta(\varepsilon)$ such that the solution $x(t)$ of system (1.2) has the property that

$$\limsup_{t\to+\infty} \mathbb{P}\{|x(t)| > \eta\} < \varepsilon.$$

Lemma 2.1 ([3])**.** *Letting Assumption 2.1 hold, for any initial value $x(0) \in \mathbb{R}^3_+$, the solution $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ of system* (1.2) *has the property that*

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{t} \leq 0 \quad a.s., i = 1, 2, 3.$$

Lemma 2.2 ([21])**.** *Suppose $Z(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$ and let Assumption 2.1 hold.*

(1) If there exist two positive constants T and δ_0 such that

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + \sigma B(t) + \sum_{i=1}^3 \delta_i \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\tilde{N}(\mathrm{d}s, \mathrm{d}u) \quad a.s.,$$

for all $t \geq T$, where σ and δ_i are constants. Then,

$$\begin{cases} \langle Z\rangle^* \leq \frac{\delta}{\delta_0} \quad a.s., if \delta \geq 0; \\ \lim_{t\to+\infty} Z(t) = 0 \quad a.s., if \delta < 0. \end{cases}$$

(2) If there exist three positive constants T, δ and δ_0 such that

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + \sigma B(t) + \sum_{i=1}^3 \delta_i \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\tilde{N}(\mathrm{d}s, \mathrm{d}u) \quad a.s.,$$

for all $t \geq T$. Then, $\langle Z\rangle_* \geq \frac{\delta}{\delta_0}$ a.s.

Lemma 2.3 ([23]). *(Chebyshev's inequality) For $p \in (0, +\infty)$, let $L^p = L^p(\Omega; \mathbb{R}^d)$ be the family of $\mathbb{R}^d$-valued random variables X with $\mathbb{E}|X|^p < +\infty$. If $c > 0, p > 0, X \in L^p$. Then, $\mathbb{P}\{\omega : |X(\omega)| \geq c\} \leq c^{-p}\mathbb{E}|X|^p$.*

3. Existence and uniqueness of the global positive solution

In this section, we will prove the existence and uniqueness of the global positive solution of system (1.2), and the theorem is stated as follows.

Theorem 3.1. *For any given initial value $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} \in \mathbb{R}^3_+$, system (1.2) has a unique solution $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ a.s.*

Proof. Since the coefficients of system (1.2) are locally Lipschitz continuous, for any given initial value $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} \in \mathbb{R}^3_+$, there is a unique local solution $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in \mathbb{R}^3_+$, for $t \in [0, \tau_{\mathrm{e}})$, where τ_{e} is the explosion time. To show this solution is global, we need to show that $\tau_{\mathrm{e}} = +\infty$ a.s. Let $m_0 > 0$ be sufficiently large such that $(x_1(0), x_2(0), x_3(0))^{\mathrm{T}}$ lie within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\{t \in [0, \tau_{\mathrm{e}}) : x_1(t) \notin (\frac{1}{m}, m) \quad \text{or} \quad x_2(t) \notin (\frac{1}{m}, m) \quad \text{or} \quad x_3(t)) \notin (\frac{1}{m}, m)\}.$$

Throughout this paper, we set $\inf \emptyset = +\infty$ (as usual $\emptyset$ is the empty set). Obviously, τ_m is increasing as $m \to +\infty$. Let $\tau_\infty = \lim_{m\to+\infty} \tau_m$. Thus, $\tau_\infty \leq \tau_{\mathrm{e}}$ a.s. If we can show $\tau_\infty = +\infty$, then $\tau_{\mathrm{e}} = +\infty$ a.s. If this assertion is false, there exist a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon.$$

Hence, there is an integer $m_1 \geq m_0$, such that $\mathbb{P}\{\tau_m \leq T\} \geq \varepsilon$, for all $m \geq m_1$.

We write $x_1(t) = x_1$, $x_2(t) = x_2$, $x_3(t) = x_3$ and define a function V:$\mathbb{R}^3_+ \to \mathbb{R}$ by

$$V(x_1, x_2, x_3) = x_1 - 1 - \ln x_1 + x_2 - 1 - \ln x_2 + x_3 - 1 - \ln x_3,$$

for all $t \in [0, \tau_m]$, from $u - 1 - \ln u \geq 0$, for any $u > 0$. Obviously, $V(x_1, x_2, x_3)$ is nonnegative. Applying Itô formula to V, we have

$$\begin{aligned}
LV(x_1, x_2, x_3) = {} & (x_1 - 1)[r_1 - \alpha_1 x_1 - \tfrac{c_1 x_3}{a_1 + a_2 x_1 + a_3 x_3} - \beta_1 x_2] + \tfrac{\sigma_1^2}{2} + \tfrac{\sigma_2^2}{2} + \tfrac{\sigma_3^2}{2} \\
& + \textstyle\int_{\mathbb{Y}} [\gamma_1(u) - \ln(1 + \gamma_1(u))]\lambda(\mathrm{d}u) + (x_2 - 1)[r_2 - \alpha_2 x_2 \\
& - \tfrac{c_2 x_3}{b_1 + b_2 x_2 + b_3 x_3} - \beta_2 x_1] + \textstyle\int_{\mathbb{Y}} [\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(\mathrm{d}u) \\
& + (x_3 - 1)[-r_3 - \alpha_3 x_3 + \tfrac{e_1 x_1}{a_1 + a_2 x_1 + a_3 x_3} + \tfrac{e_2 x_2}{b_1 + b_2 x_2 + b_3 x_3}] \\
& + \textstyle\int_{\mathbb{Y}} [\gamma_3(u) - \ln(1 + \gamma_3(u))]\lambda(\mathrm{d}u) \\
= {} & r_1 x_1 - \alpha_1 x_1^2 - \tfrac{c_1 x_1 x_3}{a_1 + a_2 x_1 + a_3 x_3} - \beta_1 x_1 x_2 - r_1 + \alpha_1 x_1 \\
& + \tfrac{c_1 x_3}{a_1 + a_2 x_1 + a_3 x_3} + \beta_1 x_2 + \tfrac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} + r_2 x_2 - \alpha_2 x_2^2 \\
& - \tfrac{c_2 x_2 x_3}{b_1 + b_2 x_2 + b_3 x_3} - \beta_2 x_1 x_2 - r_2 + \alpha_2 x_2 + \tfrac{c_2 x_3}{b_1 + b_2 x_2 + b_3 x_3} \\
& + \beta_2 x_1 - r_3 x_3 - \alpha_3 x_3^2 + \tfrac{e_1 x_1 x_3}{a_1 + a_2 x_1 + a_3 x_3} + \tfrac{e_2 x_2 x_3}{b_1 + b_2 x_2 + b_3 x_3} \\
& + r_3 + \alpha_3 x_3 - \tfrac{e_1 x_1}{a_1 + a_2 x_1 + a_3 x_3} - \tfrac{e_2 x_2}{b_1 + b_2 x_2 + b_3 x_3}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{Y}}[\gamma_1(u)-\ln(1+\gamma_1(u))]\lambda(\mathrm{d}u)\\
&+\int_{\mathbb{Y}}[\gamma_2(u)-\ln(1+\gamma_2(u))]\lambda(\mathrm{d}u)\\
&+\int_{\mathbb{Y}}[\gamma_3(u)-\ln(1+\gamma_3(u))]\lambda(\mathrm{d}u)\\
\le\ & r_1x_1-\alpha_1x_1^2+\alpha_1x_1+\tfrac{c_1}{a_3}+\beta_1x_2+r_2x_2-\alpha_2x_2^2+\alpha_2x_2\\
&+\tfrac{c_2}{b_3}+\beta_2x_1-\alpha_3x_3^2+r_3+\alpha_3x_3+\tfrac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{2}\\
&+\int_{\mathbb{Y}}[\gamma_1(u)-\ln(1+\gamma_1(u))]\lambda(\mathrm{d}u)\\
&+\int_{\mathbb{Y}}[\gamma_2(u)-\ln(1+\gamma_2(u))]\lambda(\mathrm{d}u)\\
&+\int_{\mathbb{Y}}[\gamma_3(u)-\ln(1+\gamma_3(u))]\lambda(\mathrm{d}u).
\end{aligned}
$$

Let

$$
\begin{aligned}
n=&\int_{\mathbb{Y}}[\gamma_1(u)-\ln(1+\gamma_1(u))]\lambda(\mathrm{d}u)\vee\int_{\mathbb{Y}}[\gamma_2(u)-\ln(1+\gamma_2(u))]\lambda(\mathrm{d}u)\\
&\vee\int_{\mathbb{Y}}[\gamma_3(u)-\ln(1+\gamma_3(u))]\lambda(\mathrm{d}u).
\end{aligned}
$$

It is a fact that there is a positive constant K such that

$$
\begin{aligned}
LV(x_1,x_2,x_3)\le\ & r_1x_1-\alpha_1x_1^2+\alpha_1x_1+\tfrac{c_1}{a_3}+\beta_1x_2+r_2x_2-\alpha_2x_2^2+\alpha_2x_2+\tfrac{c_2}{b_3}+\beta_2x_1\\
&-\alpha_3x_3^2+r_3+\alpha_3x_3+\tfrac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{2}+3n\\
\le\ & K.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{d}V(x_1,x_2,x_3)\le& K\mathrm{d}t+(x_1-1)\sigma_1\mathrm{d}B_1(t)+\int_{\mathbb{Y}}[\gamma_1(u)x_1-\ln(1+\gamma_1(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u)\\
&+(x_2-1)\sigma_2\mathrm{d}B_2(t)+\int_{\mathbb{Y}}[\gamma_2(u)x_2-\ln(1+\gamma_2(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u)\\
&+(x_3-1)\sigma_3\mathrm{d}B_3(t)+\int_{\mathbb{Y}}[\gamma_3(u)x_3-\ln(1+\gamma_3(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}
\tag{3.1}
$$

Integrating both sides of inequality (3.1) from 0 to $\tau_m\wedge T$, where $\tau_m\wedge T=\min\{\tau_m,T\}$, we have

$$
\begin{aligned}
\int_0^{\tau_m\wedge T}\mathrm{d}V(x_1,x_2,x_3)\le&\int_0^{\tau_m\wedge T}K\mathrm{d}t+\int_0^{\tau_m\wedge T}(x_1-1)\sigma_1\mathrm{d}B_1(t)\\
&+\int_0^{\tau_m\wedge T}(x_2-1)\sigma_2\mathrm{d}B_2(t)+\int_0^{\tau_m\wedge T}(x_3-1)\sigma_3\mathrm{d}B_3(t)\\
&+\int_0^{\tau_m\wedge T}\int_{\mathbb{Y}}[\gamma_1(u)x_1-\ln(1+\gamma_1(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u)\\
&+\int_0^{\tau_m\wedge T}\int_{\mathbb{Y}}[\gamma_2(u)x_2-\ln(1+\gamma_2(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u)\\
&+\int_0^{\tau_m\wedge T}\int_{\mathbb{Y}}[\gamma_3(u)x_3-\ln(1+\gamma_3(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}
$$

Taking expectation, we can obtain

$$
\mathbb{E}V\left(x_1(\tau_m\wedge T),x_2(\tau_m\wedge T),x_3(\tau_m\wedge T)\right)\le V\left(x_1(0),x_2(0),x_3(0)\right)+K\mathbb{E}(\tau_m\wedge T).
\tag{3.2}
$$

Set $\Omega_m = \{\tau_m \leq T\}$. From $\mathbb{P}(\tau_m \leq T) \geq \varepsilon$, then we have $\mathbb{P}(\Omega_m) \geq \varepsilon$. For each $\omega \in \Omega_m$, $x_1(\tau_m, \omega), x_2(\tau_m, \omega)$ or $x_3(\tau_m, \omega)$ equals either m or $\frac{1}{m}$, and

$$V\left(x_1(\tau_m, \omega), x_2(\tau_m, \omega), x_3(\tau_m, \omega)\right) \geq \min\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\}.$$

Therefore, from (3.2), it is not difficult to see

$$\begin{aligned} V(x_1(0), x_2(0), x_3(0)) + K\mathbb{E}(\tau_m \wedge T) =& V(x_1(0), x_2(0), x_3(0)) + KT \\ \geq& \mathbb{E}[I_{\Omega_m}(\omega) V\left(x_1(\tau_m, \omega), x_2(\tau_m, \omega), x_3(\tau_m, \omega)\right)] \\ \geq& \mathbb{P}\left(\Omega_m(\omega)\right) \min\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\} \\ \geq& \varepsilon \min\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\}, \end{aligned}$$

where I_{Ω_m} is the indicator function of Ω_m. Letting $m \to +\infty$ leads to the contradiction

$$+\infty > V\left(x_1(0), x_2(0), x_3(0)\right) + KT \geq +\infty.$$

Therefore, we show $\tau_\infty = +\infty$ a.s. Then, $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the unique global positive solution of system (1.2). Here, the proof of this theorem is completed. □

Lemma 3.1. *For any given initial value $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} \in \mathbb{R}^3_+$ and $p > 0$, there is a constant Q such that the solution $x(t)$ of system (1.2) satisfies $\limsup_{t\to+\infty} \mathbb{E}|x(t)|^p \leq Q$. That is, the solution is stochastically and ultimately bounded.*

Proof. Define a Lyapunov function $V(x) = x_1^p + x_2^p + x_3^p, p > 0$. Making use of the Itô's formula to $e^t V(x)$, we obtain

$$\begin{aligned} \mathrm{d}(e^t V(x)) =& e^t V(x) + e^t L(V(x)) + pe^t[\sigma_1 x_1^p \mathrm{d}B_1(t) + \sigma_2 x_2^p \mathrm{d}B_2(t) + \sigma_3 x_3^p \mathrm{d}B_3(t)] \\ &+ e^t \int_{\mathbb{Y}} x_1^p[(1 + \gamma_1(u)^p) - 1]\tilde{N}(\mathrm{d}t, \mathrm{d}u) \\ &+ e^t \int_{\mathbb{Y}} x_2^p[(1 + \gamma_2(u)^p) - 1]\tilde{N}(\mathrm{d}t, \mathrm{d}u) \\ &+ e^t \int_{\mathbb{Y}} x_3^p[(1 + \gamma_3(u)^p) - 1]\tilde{N}(\mathrm{d}t, \mathrm{d}u). \end{aligned} \tag{3.3}$$

Integrating from 0 to t on both sides of (3.3) and taking the expectation, then

$$\mathbb{E}(e^t V(x(t))) = V(x(0)) + \mathbb{E}\int_0^t e^s[V(x(s)) + LV(x(s))]\mathrm{d}s,$$

where

$$\begin{aligned} LV(x) =& px_1^p[r_1 - \alpha_1 x_1 - \frac{c_1 x_3}{a_1 + a_2 x_1 + a_3 x_3} - \beta_1 x_2] + px_2^p[r_2 - \alpha_2 x_2 \\ &- \frac{c_2 x_3}{b_1 + b_2 x_2 + b_3 x_3} - \beta_2 x_1] + px_3^p[-r_3 - \alpha_3 x_3 + \frac{e_1 x_1}{a_1 + a_2 x_1 + a_3 x_3} \\ &+ \frac{e_2 x_2}{b_1 + b_2 x_2 + b_3 x_3}] + \frac{p(p-1)\sigma_1^2}{2} x_1^p + \frac{p(p-1)\sigma_2^2}{2} x_2^p + \frac{p(p-1)\sigma_3^2}{2} x_3^p \\ &+ \int_{\mathbb{Y}} [(x_1 + \gamma_1(u) x_1)^p - x_1^p - p\gamma_1(u) x_1^p]\lambda(\mathrm{d}u) \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{Y}}[(x_2+\gamma_2(u)x_2)^p-x_2^p-p\gamma_2(u)x_2^p]\lambda(\mathrm{d}u)\\
&+\int_{\mathbb{Y}}[(x_3+\gamma_3(u)x_3)^p-x_3^p-p\gamma_3(u)x_3^p]\lambda(\mathrm{d}u)\\
=&x_1^p\left[pr_1+\frac{p(p-1)\sigma_1^2}{2}+\int_{\mathbb{Y}}[(1+\gamma_1(u))^p-1-p\gamma_1(u)]\lambda(\mathrm{d}u)\right]\\
&+x_2^p\left[pr_2+\frac{p(p-1)\sigma_2^2}{2}+\int_{\mathbb{Y}}[(1+\gamma_2(u))^p-1-p\gamma_2(u)]\lambda(\mathrm{d}u)\right]\\
&+x_3^p\left[-pr_3+\frac{p(p-1)\sigma_3^2}{2}+\int_{\mathbb{Y}}[(1+\gamma_3(u))^p-1-p\gamma_3(u)]\lambda(\mathrm{d}u)\right]\\
&-p\alpha_1x_1^{p+1}-p\alpha_2x_2^{p+1}-p\alpha_3x_3^{p+1}-p\beta_1x_1^px_2-p\beta_2x_1x_2^p\\
&-\frac{pc_1x_1^px_3}{a_1+a_2x_1+a_3x_3}-\frac{pc_2x_2^px_3}{b_1+b_2x_2+b_3x_3}\\
&+\frac{pe_1x_1x_3^p}{a_1+a_2x_1+a_3x_3}+\frac{pe_2x_2x_3^p}{b_1+b_2x_2+b_3x_3}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
V(x)+LV(x)\le&-p\alpha_1x_1^{p+1}+x_1^p\{1+pr_1+\frac{p(p-1)\sigma_1^2}{2}\}-p\alpha_2x_2^{p+1}\\
&+x_2^p\{1+pr_2+\frac{p(p-1)\sigma_2^2}{2}\}-p\alpha_3x_3^{p+1}\\
&+x_3^p\{1-pr_3+\frac{p(p-1)\sigma_3^2}{2}+\frac{pe_1}{a_2}+\frac{pe_2}{b_2}\}\\
&+x_1^p\int_{\mathbb{Y}}[(1+\gamma_1(u))^p-1-p\gamma_1(u)]\lambda(\mathrm{d}u)\\
&+x_2^p\int_{\mathbb{Y}}[(1+\gamma_2(u))^p-1-p\gamma_2(u)]\lambda(\mathrm{d}u)\\
&+x_3^p\int_{\mathbb{Y}}[(1+\gamma_3(u))^p-1-p\gamma_3(u)]\lambda(\mathrm{d}u).
\end{aligned}
$$

For any $p\in[0,1]$, according to the inequality $x^r\le 1+r(x-1), x\ge 0, 0\le r\le 1$,

$$\int_{\mathbb{Y}}[(1+\gamma_i(u))^p-1-p\gamma_i(u)]\lambda(\mathrm{d}u)\le 0, i=1,2,3.$$

For $\alpha_i>0$, we can deduce that there exists a constant $Q(p)>0$ such that

$$V(x)+LV(x)\le Q(p).$$

Therefore,

$$\mathbb{E}(e^tV(x_1(t),x_2(t),x_3(t)))\le V(x_1(0),x_2(0),x_3(0))+Q(p)(e^t-1),$$

which implies

$$\limsup_{t\to+\infty}\mathbb{E}(x_1^p(t)+x_2^p(t)+x_3^p(t))\le Q(p).$$

By fundamental inequality,

$$3^{(1-\frac{p}{2})\wedge 0}|x|^p\le\sum_{i=1}^3 x_i^p\le 3^{(1-\frac{p}{2})\vee 0}|x|^p,$$

for all $p > 0, x \in \mathbb{R}^3_+$, we can find a constant $Q = \frac{Q(p)}{3^{(1-\frac{p}{2})\wedge 0}}$, and this yields $\limsup_{t\to+\infty} \mathbb{E}|x(t)|^p \leq Q$. For any $\varepsilon > 0$, let $\eta(\varepsilon) = (\frac{Q}{\varepsilon})^{\frac{1}{p}}$, and by the Chebyshev inequality,

$$\limsup_{t\to+\infty} \mathbb{P}(|x(t)| > \eta) \leq \eta^{-p}\mathbb{E}|x(t)|^p \leq \frac{Q}{\eta^p} < \varepsilon.$$

□

4. Extinction and persistence

In this section, we will further derive sufficient conditions for the extinction and persistence in the mean of system (1.2).

Theorem 4.1. *Let Assumption 2.1 hold, we have the following discussions about system* (1.2)*.*

(i) If $r_i - \mu_i < 0$, $i = 1, 2$, and $-r_3 - \mu_3 < 0$, then all populations become extinct;

(ii) If $r_1 - \mu_1 < 0$, then the population $x_1(t)$ becomes extinct. Furthermore, if $r_2 - \mu_2 > \max\{0, \frac{c_2}{b_3}\}$ and $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} > 0$, then the populations $x_2(t), x_3(t)$ are persistent in the mean. That is,

$$\frac{r_2 - \mu_2 - \frac{c_2}{b_3}}{\alpha_2} \leq \langle x_2(t)\rangle_* \leq \langle x_2(t)\rangle^* \leq \frac{r_2 - \mu_2}{\alpha_2} \quad a.s.$$

$$\frac{-r_3 - \mu_3}{\alpha_3} \leq \langle x_3(t)\rangle_* \leq \langle x_3(t)\rangle^* \leq \frac{-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2}}{\alpha_3} \quad a.s.;$$

(iii) If $r_2 - \mu_2 < 0$, then the population $x_2(t)$ becomes extinct. In addition, if $r_1 - \mu_1 > \max\{0, \frac{c_1}{a_3}\}$ and $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} > 0$, then the populations $x_1(t), x_3(t)$ are persistent in the mean, namely,

$$\frac{r_1 - \mu_1 - \frac{c_1}{a_3}}{\alpha_1} \leq \langle x_1(t)\rangle_* \leq \langle x_1(t)\rangle^* \leq \frac{r_1 - \mu_1}{\alpha_1} \quad a.s.$$

$$\frac{-r_3 - \mu_3}{\alpha_3} \leq \langle x_3(t)\rangle_* \leq \langle x_3(t)\rangle^* \leq \frac{-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2}}{\alpha_3} \quad a.s.;$$

(iv) If $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} < 0$, then the population $x_3(t)$ becomes extinct. Moreover, if $r_1 - \mu_1 > \max\{0, \frac{c_1}{a_3} + \beta_1\frac{r_2-\mu_2}{\alpha_2}\}$ and $r_2 - \mu_2 > \max\{0, \frac{c_2}{b_3} + \beta_2\frac{r_1-\mu_1}{\alpha_1}\}$, then the populations $x_1(t), x_2(t)$ are persistent in the mean. That is,

$$\frac{r_1 - \mu_1 - \frac{c_1}{a_3} - \beta_1\frac{r_2-\mu_2}{\alpha_2}}{\alpha_1} \leq \langle x_1(t)\rangle_* \leq \langle x_1(t)\rangle^* \leq \frac{r_1 - \mu_1}{\alpha_1} \quad a.s.$$

$$\frac{r_2 - \mu_2 - \frac{c_2}{b_3} - \beta_2\frac{r_1-\mu_1}{\alpha_1}}{\alpha_2} \leq \langle x_2(t)\rangle_* \leq \langle x_2(t)\rangle^* \leq \frac{r_2 - \mu_2}{\alpha_2} \quad a.s.;$$

(v) If $-r_3-\mu_3+\frac{e_1}{a_2}+\frac{e_2}{b_2}>0$, $r_1-\mu_1>\max\{0,\frac{c_1}{a_3}+\beta_1\frac{r_2-\mu_2}{\alpha_2}\}$, $r_2-\mu_2>\max\{0,\frac{c_2}{b_3}+\beta_2\frac{r_1-\mu_1}{\alpha_1}\}$, *then all populations are persistent in the mean. That is,*

$$\frac{r_1-\mu_1-\frac{c_1}{a_3}-\beta_1\frac{r_2-\mu_2}{\alpha_2}}{\alpha_1}\le\langle x_1(t)\rangle_*\le\langle x_1(t)\rangle^*\le\frac{r_1-\mu_1}{\alpha_1}\quad a.s.$$

$$\frac{r_2-\mu_2-\frac{c_2}{b_3}-\beta_2\frac{r_1-\mu_1}{\alpha_1}}{\alpha_2}\le\langle x_2(t)\rangle_*\le\langle x_2(t)\rangle^*\le\frac{r_2-\mu_2}{\alpha_2}\quad a.s.$$

$$\frac{-r_3-\mu_3}{\alpha_3}\le\langle x_3(t)\rangle_*\le\langle x_3(t)\rangle^*\le\frac{-r_3-\mu_3+\frac{e_1}{a_2}+\frac{e_2}{b_2}}{\alpha_3}\quad a.s.$$

Proof. By the Itô formula, we derive from (1.2) that

$$\begin{aligned}\mathrm{d}\ln x_1(t)=&[r_1-\alpha_1x_1(t)-\frac{c_1x_3(t)}{a_1+a_2x_1(t)+a_3x_3(t)}-\beta_1x_2(t)-\frac{\sigma_1^2}{2}\\&+\int_{\mathbb{Y}}[\ln(1+\gamma_1(u))-\gamma_1(u)]\lambda(\mathrm{d}u)]\mathrm{d}t+\sigma_1\mathrm{d}B_1(t)\\&+\int_{\mathbb{Y}}[\ln(1+\gamma_1(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u),\end{aligned}\tag{4.1}$$

$$\begin{aligned}\mathrm{d}\ln x_2(t)=&[r_2-\alpha_2x_2(t)-\frac{c_2x_3(t)}{b_1+b_2x_2(t)+b_3x_3(t)}-\beta_2x_1(t)-\frac{\sigma_2^2}{2}\\&+\int_{\mathbb{Y}}[\ln(1+\gamma_2(u))-\gamma_2(u)]\lambda(\mathrm{d}u)]\mathrm{d}t+\sigma_2\mathrm{d}B_2(t)\\&+\int_{\mathbb{Y}}[\ln(1+\gamma_2(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u),\end{aligned}\tag{4.2}$$

$$\begin{aligned}\mathrm{d}\ln x_3(t)=&[-r_3-\alpha_3x_3(t)+\frac{e_1x_1(t)}{a_1+a_2x_1(t)+a_3x_3(t)}+\frac{e_2x_2(t)}{b_1+b_2x_2(t)+b_3x_3(t)}\\&-\frac{\sigma_3^2}{2}+\int_{\mathbb{Y}}[\ln(1+\gamma_3(u))-\gamma_3(u)]\lambda(\mathrm{d}u)]\mathrm{d}t+\sigma_3\mathrm{d}B_3(t)\\&+\int_{\mathbb{Y}}[\ln(1+\gamma_3(u))]\tilde{N}(\mathrm{d}t,\mathrm{d}u).\end{aligned}\tag{4.3}$$

Integrating both sides of inequalities (4.1), (4.2) and (4.3) on the interval $[0,t]$ and dividing them by t, we can obtain

$$\begin{aligned}\frac{\ln x_1(t)-\ln x_1(0)}{t}=&r_1-\mu_1-\alpha_1\langle x_1(t)\rangle-\langle\frac{c_1x_3(t)}{a_1+a_2x_1(t)+a_3x_3(t)}\rangle-\beta_1\langle x_2(t)\rangle\\&+\frac{\sigma_1B_1(t)}{t}+\frac{N_1(t)}{t},\end{aligned}\tag{4.4}$$

$$\begin{aligned}\frac{\ln x_2(t)-\ln x_2(0)}{t}=&r_2-\mu_2-\alpha_2\langle x_2(t)\rangle-\langle\frac{c_2x_3(t)}{b_1+b_2x_2(t)+b_3x_3(t)}\rangle-\beta_2\langle x_1(t)\rangle\\&+\frac{\sigma_2B_2(t)}{t}+\frac{N_2(t)}{t},\end{aligned}\tag{4.5}$$

$$\frac{\ln x_3(t) - \ln x_3(0)}{t} = - r_3 - \mu_3 - \alpha_3\langle x_3(t)\rangle + \langle \frac{e_1 x_1(t)}{a_1 + a_2 x_1(t) + a_3 x_3(t)}\rangle + \langle \frac{e_2 x_2(t)}{b_1 + b_2 x_2(t) + b_3 x_3(t)}\rangle + \frac{\sigma_3 B_3(t)}{t} + \frac{N_3(t)}{t}. \tag{4.6}$$

Case 1. We will prove the conclusion of Case (i). By (4.4),

$$\frac{\ln x_1(t) - \ln x_1(0)}{t} \leq r_1 - \mu_1 - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}.$$

From the condition $r_1 - \mu_1 < 0$, and by Case (1) in Lemma 2.2, we obtain $\lim_{t\to+\infty} x_1(t) = 0$. For (4.5), we know

$$\frac{\ln x_2(t) - \ln x_2(0)}{t} \leq r_2 - \mu_2 - \alpha_2\langle x_2(t)\rangle + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}.$$

Noticing that $r_2 - \mu_2 < 0$, and by Case (1) in Lemma 2.2, we get $\lim_{t\to+\infty} x_2(t) = 0$. Thus, we have

$$\langle|\frac{x_1(t)}{a_1 + a_2 x_1(t) + a_3 x_3(t)}|\rangle \leq \langle|x_1(t)|\rangle < \varepsilon_1,$$
$$\langle|\frac{x_2(t)}{b_1 + b_2 x_2(t) + b_3 x_3(t)}|\rangle \leq \langle|x_2(t)|\rangle < \varepsilon_2.$$

Let us take $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, for sufficiently large t, where $0 < \varepsilon < \frac{r_3+\mu_3}{e_1+e_2}$. Then, for (4.6), we have

$$\frac{\ln x_3(t) - \ln x_3(0)}{t} \leq -r_3 - \mu_3 - \alpha_3\langle x_3(t)\rangle + (e_1 + e_2)\varepsilon + \frac{\sigma_3 B_3(t)}{t} + \frac{N_3(t)}{t}.$$

We note that $-r_3 - \mu_3 < 0$ and $0 < \varepsilon < \frac{r_3+\mu_3}{e_1+e_2}$. Hence, by Case (1) in Lemma 2.2, we have $\lim_{t\to+\infty} x_3(t) = 0$.

Case 2. We will prove the conclusion in Case (ii). From (4.4),

$$\frac{\ln x_1(t) - \ln x_1(0)}{t} \leq r_1 - \mu_1 - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}.$$

According to $r_1 - \mu_1 < 0$ and Case (1) of Lemma 2.2, we get

$$\lim_{t\to+\infty} x_1(t) = 0.$$

It follows from (4.5) that

$$\frac{\ln x_2(t) - \ln x_2(0)}{t} \leq r_2 - \mu_2 - \alpha_2\langle x_2(t)\rangle + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}.$$

By virtue of $r_2 - \mu_2 > 0$ and Case (1) of Lemma 2.2, we deduce

$$\langle x_2(t)\rangle^* \leq \frac{r_2 - \mu_2}{\alpha_2}.$$

From $\lim_{t\to+\infty} x_1(t) = 0$, we have $|\langle x_1(t)\rangle| < \varepsilon$, for sufficiently large t, where $0 < \varepsilon < \frac{r_2-\mu_2-\frac{c_2}{b_3}}{\beta_2}$. Then, for (4.5), we have

$$\frac{\ln x_2(t) - \ln x_2(0)}{t} \geq r_2 - \mu_2 - \frac{c_2}{b_3} - \alpha_2\langle x_2(t)\rangle - \beta_2\varepsilon + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}.$$

By Case (2) in Lemma 2.2, we obtain

$$\langle x_2(t)\rangle_* \geq \frac{r_2 - \mu_2 - \frac{c_2}{b_3} - \beta_2\varepsilon}{\alpha_2}.$$

According to the arbitrariness of ε, we have

$$\frac{r_2 - \mu_2 - \frac{c_2}{b_3}}{\alpha_2} \leq \langle x_2(t)\rangle_* \leq \langle x_2(t)\rangle^* \leq \frac{r_2 - \mu_2}{\alpha_2}.$$

Next, for (4.6), we obtain

$$\frac{\ln x_3(t) - \ln x_3(0)}{t} \leq -r_3 - \mu_3 - \alpha_3\langle x_3(t)\rangle + \frac{e_1}{a_2} + \frac{e_2}{b_2} + \frac{\sigma_3 B_3(t)}{t} + \frac{N_3(t)}{t}.$$

Through $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} > 0$, using Case (1) in Lemma 2.2, we have

$$\langle x_3(t)\rangle^* \leq \frac{-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2}}{\alpha_3}.$$

Again, by (4.6),

$$\frac{\ln x_3(t) - \ln x_3(0)}{t} \geq -r_3 - \mu_3 - \alpha_3\langle x_3(t)\rangle + \frac{\sigma_3 B_3(t)}{t} + \frac{N_3(t)}{t}.$$

Using Case (2) in Lemma 2.2, we have

$$\langle x_3(t)\rangle_* \geq \frac{-r_3 - \mu_3}{\alpha_3}.$$

To sum up, we can get

$$\frac{-r_3 - \mu_3}{\alpha_3} \leq \langle x_3(t)\rangle_* \leq \langle x_3(t)\rangle^* \leq \frac{-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2}}{\alpha_3}.$$

Case 3. We will give the proof of Case (iii). By (4.5),

$$\frac{\ln x_2(t) - \ln x_2(0)}{t} \leq r_2 - \mu_2 - \alpha_2\langle x_2(t)\rangle + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}.$$

We note that $r_2 - \mu_2 < 0$. Hence, by Case (1) in Lemma 2.2,

$$\lim_{t\to+\infty} x_2(t) = 0.$$

From (4.4),

$$\frac{\ln x_1(t) - \ln x_1(0)}{t} \leq r_1 - \mu_1 - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}.$$

Through $r_1 - \mu_1 > 0$ and Case (1) in Lemma 2.2, we have

$$\langle x_1(t)\rangle^* \leq \frac{r_1 - \mu_1}{\alpha_1}.$$

From $\lim_{t\to+\infty} x_2(t) = 0$, we obtain $\langle |x_2(t)| \rangle < \varepsilon$, for sufficiently large t, where $0 < \varepsilon < \frac{r_1-\mu_1-\frac{c_1}{a_3}}{\beta_1}$. Then, for (4.4),

$$\frac{\ln x_1(t) - \ln x_1(0)}{t} \geq r_1 - \mu_1 - \frac{c_1}{a_3} - \beta_1\varepsilon - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}.$$

By Case (2) in Lemma 2.2, we deduce

$$\langle x_1(t)\rangle_* \geq \frac{r_1 - \mu_1 - \frac{c_1}{a_3} - \beta_1\varepsilon}{\alpha_1}.$$

Therefore, in view of the arbitrariness of ε, we can get

$$\frac{r_1 - \mu_1 - \frac{c_1}{a_3}}{\alpha_1} \leq \langle x_1(t)\rangle_* \leq \langle x_1(t)\rangle^* \leq \frac{r_1-\mu_1}{\alpha_1}.$$

The proof of persistent of $x_3(t)$ is the same as (ii). Therefore, we ignore it.

Case 4. We will proof Case (iv). From (4.6),

$$\frac{\ln x_3(t) - \ln x_3(0)}{t} \leq -r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} - \alpha_3\langle x_3(t)\rangle + \frac{\sigma_3 B_3(t)}{t} + \frac{N_3(t)}{t}.$$

Using Case (1) in Lemma 2.2 and $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} < 0$, we obtain

$$\lim_{t\to+\infty} x_3(t) = 0.$$

By (4.5),

$$\frac{\ln x_2(t) - \ln x_2(0)}{t} \leq r_2 - \mu_2 - \alpha_2\langle x_2(t)\rangle + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}.$$

By virtue of $r_2 - \mu_2 \geq 0$ and Case (1) of Lemma 2.2, we deduce

$$\langle x_2(t)\rangle^* \leq \frac{r_2-\mu_2}{\alpha_2}. \tag{4.7}$$

From (4.4),

$$\frac{\ln x_1(t) - \ln x_1(0)}{t} \leq r_1 - \mu_1 - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}.$$

By $r_1 - \mu_1 > 0$, it follows from Case (1) in Lemma 2.2, we have

$$\langle x_1(t)\rangle^* \leq \frac{r_1-\mu_1}{\alpha_1}. \tag{4.8}$$

Combining inequality (4.7) and Lemma 2.1, by (4.4), we can obtain

$$\begin{aligned}\alpha_1\langle x_1(t)\rangle_* \geq& \liminf_{t\to+\infty}\{r_1 - \mu_1 - \frac{\ln x_1(t) - \ln x_1(0)}{t} - \frac{c_1}{a_3}\\ &- \beta_1\langle x_2(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}\}\\ \geq& r_1 - \mu_1 - \frac{c_1}{a_3} - \limsup_{t\to+\infty}\frac{\ln x_1(t)}{t} - \beta_1\langle x_2(t)\rangle^*\\ \geq& r_1 - \mu_1 - \frac{c_1}{a_3} - \beta_1\frac{r_2-\mu_2}{\alpha_2}.\end{aligned} \tag{4.9}$$

To sum up, we can get

$$\frac{r_1-\mu_1-\frac{c_1}{a_3}-\beta_1\frac{r_2-\mu_2}{\alpha_2}}{\alpha_1}\leq\langle x_1(t)\rangle_*\leq\langle x_1(t)\rangle^*\leq\frac{r_1-\mu_1}{\alpha_1}\quad a.s.$$

Combining inequality (4.8) and Lemma 2.1, by (4.5), we can get

$$\begin{aligned}\alpha_2\langle x_2(t)\rangle_*\geq&\liminf_{t\to+\infty}\{r_2-\mu_2-\frac{\ln x_2(t)-\ln x_2(0)}{t}-\frac{c_2}{b_3}\\&-\beta_2\langle x_1(t)\rangle+\frac{\sigma_2B_2(t)}{t}+\frac{N_2(t)}{t}\}\\\geq&r_2-\mu_2-\frac{c_2}{b_3}-\limsup_{t\to+\infty}\frac{\ln x_2(t)}{t}-\beta_2\langle x_1(t)\rangle^*\\\geq&r_2-\mu_2-\frac{c_2}{b_3}-\beta_2\frac{r_1-\mu_1}{\alpha_1}.\end{aligned}\tag{4.10}$$

Thus, $\langle x_2(t)\rangle_*\geq\frac{r_2-\mu_2-\frac{c_2}{b_3}-\beta_2\frac{r_1-\mu_1}{\alpha_1}}{\alpha_2}$. To sum up, we can get

$$\frac{r_2-\mu_2-\frac{c_2}{b_3}-\beta_2\frac{r_1-\mu_1}{\alpha_1}}{\alpha_2}\leq\langle x_2(t)\rangle_*\leq\langle x_2(t)\rangle^*\leq\frac{r_2-\mu_2}{\alpha_2}\quad a.s.$$

Case 5. We shall prove Case (v). Through (4.6), we obtain

$$\frac{\ln x_3(t)-\ln x_3(0)}{t}\leq-r_3-\mu_3-\alpha_3\langle x_3(t)\rangle+\frac{e_1}{a_2}+\frac{e_2}{b_2}+\frac{\sigma_3B_3(t)}{t}+\frac{N_3(t)}{t}.$$

Through $-r_3-\mu_3+\frac{e_1}{a_2}+\frac{e_2}{b_2}>0$ and from Case (1) in Lemma 2.2, we have

$$\langle x_3(t)\rangle^*\leq\frac{-r_3-\mu_3+\frac{e_1}{a_2}+\frac{e_2}{b_2}}{\alpha_3}.$$

Again, for (4.6),

$$\frac{\ln x_3(t)-\ln x_3(0)}{t}\geq-r_3-\mu_3-\alpha_3\langle x_3(t)\rangle+\frac{\sigma_3B_3(t)}{t}+\frac{N_3(t)}{t}.$$

From Case (2) in Lemma 2.2, we have

$$\langle x_3(t)\rangle_*\geq\frac{-r_3-\mu_3}{\alpha_3}.$$

Therefore, we have

$$\frac{-r_3-\mu_3}{\alpha_3}\leq\langle x_3(t)\rangle_*\leq\langle x_3(t)\rangle^*\leq\frac{-r_3-\mu_3+\frac{e_1}{a_2}+\frac{e_2}{b_2}}{\alpha_3}.$$

For $x_1(t)$ and $x_2(t)$, the estimates of the ultimate infimum and ultimate supremum are the same as those in Case (iv). Therefore, they are omitted. This proof is completed. □

5. Numerical simulations

In this section, we will demonstrate our theoretical results with the help of computer simulations through the Milstein method [11]. We choose the same initial value $(x_1(0), x_2(0), x_3(0))^{\mathrm{T}} = (0.84, 0.68, 0.15)^{\mathrm{T}}$. Other parameters always choose $r_1 = 0.86$, $r_2 = 0.78$, $r_3 = 0.02$, $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.2$, $c_1 = 0.32$, $c_2 = 0.3$, $a_1 = 0.8$, $a_2 = 1.12$, $a_3 = 0.86$, $b_1 = 0.74$, $b_2 = 0.76$, $b_3 = 0.84$, $\beta_1 = 0.14$, $\beta_2 = 0.1$, $e_1 = 0.24$, $e_2 = 0.25$, $\mathbb{Y} = (0, +\infty)$, $\lambda(\mathbb{Y}) = 1$.

Next, we will reveal the effects of white noise and Lévy noise to system (1.2) by considering the following five examples.

Example 5.1. Let $\gamma_1 = 0.8$, $\gamma_2 = 0.6$, $\gamma_3 = 0.5$, $\sigma_1 = 1.30$, $\sigma_2 = 1.24$, $\sigma_3 = 1.12$. After simple calculation, $r_1 - \mu_1 = -0.1972 < 0$, $r_2 - \mu_2 = -0.1188$, $-r_3 - \mu_3 = -0.7414 < 0$, then the condition of (i) in Theorem 4.1 is satisfied. Hence, all species go to extinction (see Figure 1).

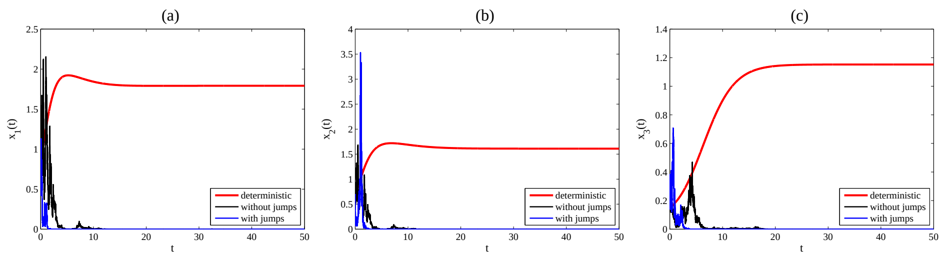

Figure 1. (a), (b) and (c) respectively represent the sample paths of $x_1(t)$, $x_2(t)$ and $x_3(t)$ in the stochastic model (1.2). The red lines represent the solution of deterministic model, the blue lines represent the solution of model (1.2), and the black lines represent the solution of the corresponding system without Lévy noise.

Example 5.2. Let $\gamma_1 = 0.8$, $\gamma_2 = 0.3$, $\gamma_3 = 0.1$, $\sigma_1 = 1.2$, $\sigma_2 = 0.24$, $\sigma_3 = 0.08$. After calculation, $r_1 - \mu_1 = -0.0722 < 0$, $r_2 - \mu_2 = 0.7136 > \max\{0, 0.3571\}$, $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} = 0.5153 > 0$, then the condition of (ii) in Theorem 4.1 is satisfied. Therefore, the prey population $x_1(t)$ becomes extinct, and the predator populations $x_2(t), x_3(t)$ are persistent in the mean in Figure 2.

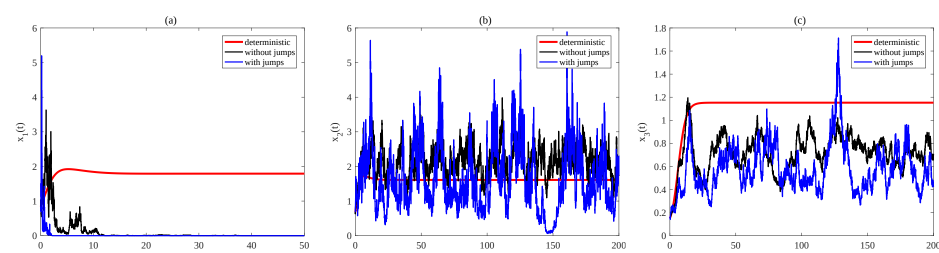

Figure 2. (a), (b) and (c) respectively represent the sample paths of $x_1(t)$, $x_2(t)$ and $x_3(t)$ in the stochastic model (1.2). The red lines represent the solution of deterministic model, the blue lines represent the solution of model (1.2), and the black lines represent the solution of the corresponding system without Lévy noise.

Example 5.3. Let $\gamma_1 = 0.3$, $\gamma_2 = 0.74$, $\gamma_3 = 0.1$, $\sigma_1 = 0.32$, $\sigma_2 = 1.24$, $\sigma_3 = 0.08$. After calculation, $r_1 - \mu_1 = 0.7712 > \max\{0, 0.3721\}$, $r_2 - \mu_2 = -0.1749 < 0$, $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} = 0.5153 > 0$, From the condition of (iii) in Theorem 4.1, the population $x_2(t)$ becomes extinct, and the populations $x_1(t), x_3(t)$ are persistent in the mean (see Figure 3).

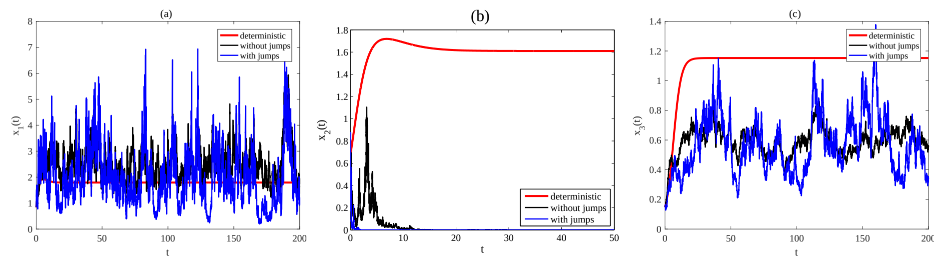

Figure 3. (a), (b) and (c) respectively represent the sample paths of $x_1(t)$, $x_2(t)$ and $x_3(t)$ in the stochastic model (1.2). The red lines represent the solution of deterministic model, the blue lines represent the solution of model (1.2), and the black lines represent the solution of the corresponding system without Lévy noise.

Example 5.4. Let $\gamma_1 = 0.3$, $\gamma_2 = 0.3$, $\gamma_3 = 0.68$, $\sigma_1 = 0.32$, $\sigma_2 = 0.24$, $\sigma_3 = 1.12$. By simple calculation, $r_1 - \mu_1 = 0.7712 > \max\{0, 0.7051\}$, $r_2 - \mu_2 = 0.7136 > \max\{0, 0.6142\}$, $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} = -0.2652 < 0$. Then, the condition of (iv) in Theorem 4.1 tells us that the population $x_3(t)$ becomes extinct, and the populations $x_1(t), x_2(t)$ are persistent in the mean (see Figure 4).

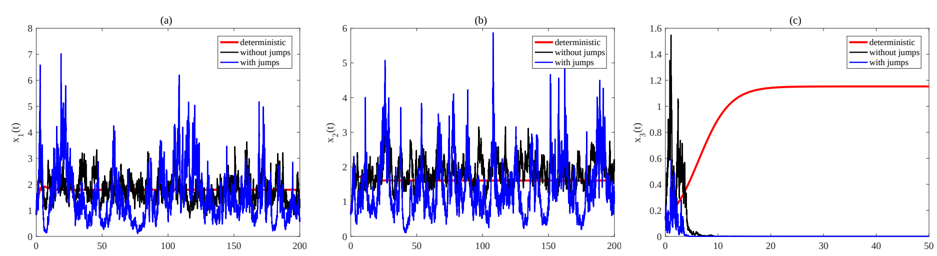

Figure 4. (a), (b) and (c) respectively represent the sample paths of $x_1(t)$, $x_2(t)$ and $x_3(t)$ in the stochastic model (1.2). The red lines represent the solution of deterministic model, the blue lines represent the solution of model (1.2) and the black lines represent the solution of the corresponding system without Lévy noise.

Example 5.5. Let $\gamma_1 = 0.3$, $\gamma_2 = 0.3$, $\gamma_3 = 0.1$, $\sigma_1 = 0.32$, $\sigma_2 = 0.24$, $\sigma_3 = 0.08$. By calculation, $r_1 - \mu_1 = 0.7712 > \max\{0, 0.7051\}$, $r_2 - \mu_2 = 0.7136 > \max\{0, 0.6142\}$, $-r_3 - \mu_3 + \frac{e_1}{a_2} + \frac{e_2}{b_2} = 0.5153 > 0$. Then, the condition of (v) in Theorem 4.1 is satisfied. Hence, we can know that all populations of system (1.2) are persistent in the mean (see Figure 5).

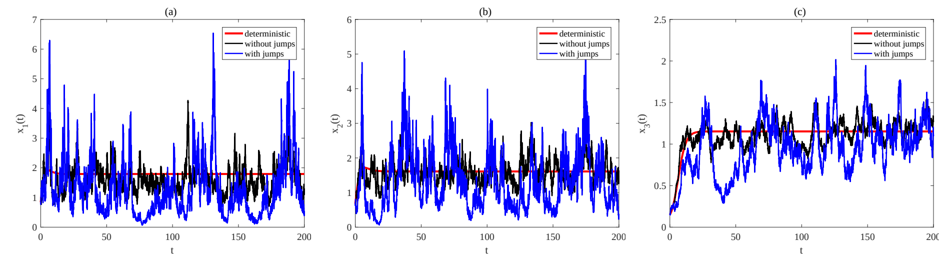

Figure 5. (a), (b) and (c) respectively represent the sample paths of $x_1(t)$, $x_2(t)$ and $x_3(t)$ in the stochastic model (1.2). The red lines represent the solution of deterministic model, the blue lines represent the solution of model (1.2) and the black lines represent the solution of the corresponding system without Lévy noise.

6. Conclusion

This paper is related to a stochastic two-prey one-predator model with Beddington-DeAngeli functional response and Lévy jumps. Under some sufficient conditions, we study the dynamical properties of system (1.2) such as extinction and persistence in the mean. In each case, we have proved that each species is either persistent or extinct (see Figures 1-5). Finally, the theoretical results are verified by numerical simulation. We conclude that both white noise and Lévy noise have a great impact on population dynamics. By Theorem 4.1, under Lévy jumps interference, if the species are extinct in the deterministic system, it will become persistent in the stochastic system (1.2), and we only need to ensure that $A = \min\{r_1 - \mu_1, r_2 - \mu_2, -r_3 - \mu_3\} > 0$. Under the interference of white noise, the persistent population in the deterministic system will become extinct, but the extinct population cannot become persistent, because the white noise σ_i can only be positive constant. In a word, strong white noise and Lévy noise will lead to the extinction of the population. However, relatively small white noise and Lévy jump can ensure the survival of species.

For the stochastic population model with Lévy jumps, we can also consider introducing impulses and time delay into the model, which is worthy of in-depth study in our future work.

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