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Existence of Nonoscillatory Solutions for a Rational Difference Equation of Higher Order*

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Abstract In this paper, we investigate a rational difference equation of higher order for the existence of nonoscillatory solutions. To prove our main results, we use an inclusion theorem stated and proved in [4]. In this way, we give an answer of an open problem formulated in [3].

Keywords Rational difference equation, Nonoscillatory solution, Appropriate equation, Inclusion theorem.

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1 Introduction and preliminaries

In their paper, Amleh et al., [2] investigated the global stability, boundedness and periodicity of the positive solutions for the following rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, n = 0, 1, \cdots, \tag{1.1}$$

where α is a positive real constant, and the initial conditions x_{-1}, x_0 are positive real numbers. Under the condition that α is a negative real number and the initial conditions x_{-1}, x_0 are negative numbers, Hamza has recently studied the global stability, permanence and oscillation of equation (1.1) in [13]. However, none of the researchers above considered the existence of non-oscillatory solutions of equation (1.1).

In [11], DeVault, Kent and Kosmala considered the behavior of positive solutions to rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, n = 0, 1, \cdots, \tag{1.2}$$

where $\alpha > 0$, and $k \in N$ is a fixed positive integer. Among other things, they have proved that all nonoscillatory solutions of equation (1.2) converge to the positive equilibrium $\bar{x} = \alpha + 1$. However, it was not shown that such solutions do exist,

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which is an interesting problem. Therefore, they presented the open problem as follows.

Open problem 1. Do there exist nonoscillatory solutions of equation (1.2)?

To the best of our knowledge, there have been no results for the above open problem. Our main aim in this note is to solve the above problem in a more general framework. More generally and precisely speaking, we will investigate the existence of nonoscillatory solutions for the following higher-order rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{\sum_{i=0}^{k-1} b_i x_{n-i}}, n = 0, 1, \cdots, \tag{1.3}$$

where α is a positive real number, $k \in N$ is a fixed positive integer, $b_0 > 0, b_i \geq 0, i = 1, 2, \cdots k-1$, and the initial values $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are nonnegative real numbers such that $\sum_{i=0}^{k-1} b_i x_{-i} > 0$. Without loss of generality, we may assume $\sum_{i=0}^{k-1} b_i = 1$.

It is easy to see that equation (1.3) with the equilibrium point $\bar{x} = \alpha+1$ contains equation (1.1) and equation (1.2) as its special cases.

Rational difference equation is a kind of typical nonlinear difference equation, which has always been a hot spot among the subjects studied in recent years. It is more important for one to find some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one comes from the results for rational difference equations. For the systematical investigations of this aspect, the monographs [1, 17, 18], the papers [3, 6, 8, 12–16, 20–22, 24, 25, 30] and the references cited therein are hereby referred to.

Berg's inclusion theorem [4] is the main tool to prove our main results in this paper. For its proof, we refer the reader to [5].

The paper is organized as follows. In Section 2, we give some auxiliary results needed for the proof of our main results. In Section 3, we formulate and prove our main results. In Section 4, we give examples for our main result. In Section 5, concluding remarks are given.

2 Auxiliary results

Consider the following general real nonlinear difference equation with order $m \geq 1$,

$$F(x_n, x_{n+1}, \cdots, x_{n+m}) = 0, \tag{2.1}$$

where $F : R^{m+1} \mapsto R$, $n \in N_0$.

Suppose that φ_n and ψ_n are two consequences satisfying $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$. Then, (maybe under certain additional conditions), for any given $\epsilon > 0$, there are a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (2.1) and an $n_0(\epsilon) \in N$ such

$$\varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n, \quad n \geq n_0(\epsilon). \tag{2.2}$$

Define

$$X(\epsilon) = \{x_n : \varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n, \quad n \geq n_0(\epsilon)\},$$

which is called an asymptotic stripe. Therefore, if x_n belongs to $X(\epsilon)$, then there exists a real sequence C_n such that $x_n = \varphi_n + C_n\psi_n$ and $|C_n| \leq \epsilon$ for $n \geq n_0(\epsilon)$.

The main result in [4] is the following theorem, which is called inclusion theorem.

Theorem 2.1. Let $F(\omega_0, \omega_1, \cdots, \omega_m)$ be continuously differential, when $\omega_i = y_{n+i}$, for $i = 0, 1, \cdots, m$, and $y_n \in X(1)$. Let the partial derivatives of F satisfy

$$F_{\omega_i}(y_n, y_{n+1}, \cdots, y_{n+m}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+m})$$

as $n \to \infty$ uniformly in C_j for $|C_j| \leq 1$, $n \leq j \leq n+m$, as far as $F_{\omega_i} \not\equiv 0$. Assume that there exist a sequence $f_n > 0$ and constants $A_0, A_1, \cdots, A_m$ such that both

$$F(\varphi_n, \cdots, \varphi_{n+m}) = o(f_n)$$

and

$$\psi_{n+i} F_{w_i}(\varphi_n, \cdots, \varphi_{n+m}) \sim A_i f_n$$

for $i = 0, 1, \cdots, m$ as $n \to \infty$, and suppose that there exists an integer k with $0 \leq k \leq m$, such that

$$|A_0| + \cdots + |A_{k-1}| + |A_{k+1}| + \cdots + |A_m| < |A_k|.$$

Then, for sufficiently large n, there exists a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (2.1) satisfying (2.2).

3 Main results

Our main results in this paper are as follows.

Theorem 3.1. Let $\bar{x}$ be the positive equilibrium point of equation (1.3). Assume that $\alpha > 0$. Then, equation (1.3) possesses nonoscillatory solutions.

As an application, the following corollary is derived.

Corollary 3.2. Both equation (1.1) and equation (1.2) possess nonoscillatory solutions.

Hence, our result presents positive confirmation to open problem 1, and also gives the existence result for nonoscillatory solutions of equation (1.1), which has not been studied yet.

Proof of Theorem 3.1 Put $y_n = x_n - \bar{x}$. Then, equation (1.3) is transformed into

$$\begin{aligned} &y_{n+k+1} \textstyle\sum_{i=0}^{k-1} b_i y_{n+k-i} + \bar{x} y_{n+k+1} + \sum_{i=0}^{k-1} b_i y_{n+k-i} - y_n = 0, \\ &n = -k, -k+1, -k+2, \cdots . \end{aligned} \tag{3.1}$$

Equation (3.1) can be rewritten in the following form

$$\bar{x} y_{n+k+1} + \sum_{i=0}^{k-1} b_i y_{n+k-i} - y_n = 0, n = -k, -k+1, -k+2, \cdots \tag{3.2}$$

provided that $y_n \to 0$ as $n \to \infty$. The general solution of equation (3.2) is

$$y_n = \sum_{i=1}^{k+1} c_i t_i^n,$$

where $c_i \in C, i = 1, 2, \cdots, k+1$ and $t_i \in C, i = 1, 2, \cdots, k+1$ are the $k+1$ roots of the polynomial equation

$$P(t) = \bar{x}t^{k+1} + \sum_{i=0}^{k-1} b_i t^{k-i} - 1 = 0.$$

The known assumption $\alpha > 0$ implies $P(0)P(1) < 0$. Therefore, $P(t) = 0$ has a positive root t lying in the interval $(0, 1)$. Now, choose the solution $z_n = t^n$ for this $t \in (0, 1)$. For some $\lambda \in (1, 2)$, define the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ as follows

$$\varphi_n = t^n \quad \text{and} \quad \psi_n = t^{\lambda n}. \tag{3.3}$$

Obviously, $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$.

Now, define the function again

$$F(\omega_0, \omega_1, \omega_2, \cdots, \omega_{k+1}) = \omega_{k+1}(\sum_{i=0}^{k-1} b_i \omega_{k-i} + \bar{x}) + \sum_{i=0}^{k-1} b_i \omega_{k-i} - \omega_0.$$

Then, the partial derivatives of F with respect to $\omega_0, \omega_1, \cdots, \omega_{k+1}$ are respectively

$$\begin{aligned} &F_{\omega_0} = -1, \\ &F_{\omega_i} = (\omega_{k+1} + 1)b_{k-i}, i = 1, \cdots, k, \\ &F_{\omega_{k+1}} = \textstyle\sum_{i=0}^{k-1} b_i \omega_{k-i} + \bar{x}. \end{aligned} \tag{3.4}$$

When $y_n \in X(1)$, we have $y_n \sim \varphi_n$. Hence, one has

$$F_{\omega_i}(y_n, y_{n+1}, \cdots, y_{n+k+1}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}), i = 0, 1, \cdots, k+1,$$

as $n \to \infty$ uniformly in C_j for $|C_j| \leq 1$, $n \leq j \leq n+k+1$.

Moreover, according to the definitions of the function F, (3.3) and (3.4), it follows

$$\begin{aligned} &F(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}) = t^{2n+k+1} \textstyle\sum_{i=0}^{k-1} b_i t^{k-i}, \\ &\psi_n F_{\omega_0}(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}) = -t^{\lambda n}, \\ &\psi_{n+i} F_{\omega_i}(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}) = b_{k-i} t^{\lambda(n+i)}(t^{n+k+1} + 1), i = 1, \cdots, k, \\ &\psi_{n+k+1} F_{\omega_{k+1}}(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}) = t^{\lambda(n+k+1)}(\textstyle\sum_{i=0}^{k-1} b_i t^{n+k-i} + \bar{x}). \end{aligned}$$

Let us choose $f_n = t^{\lambda n}$. Then, it is not difficult to derive both

$$F(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}) = o(f_n)$$

and

$$\psi_{n+i} F_{\omega_i}(\varphi_n, \varphi_{n+1}, \cdots, \varphi_{n+k+1}) \sim A_i f_n, i = 0, 1, \cdots, k+1,$$

where $A_0 = -1, A_i = b_{k-i} t^{\lambda i}$, $i = 1, \cdots, k$, $A_{k+1} = t^{\lambda(k+1)} \bar{x}$.

Noticing $\lambda \in (1,2)$ and $t \in (0,1)$, one has

$$\sum_{i=1}^{k+1} |A_i| = \sum_{i=1}^{k} b_{k-i} t^{\lambda i} + t^{\lambda(k+1)} \bar{x} < \sum_{i=1}^{k} b_{k-i} t^{i} + t^{k+1} \bar{x} = 1 = |A_0|.$$

This implies that all conditions of Theorem 2.1 are satisfied. Accordingly, we see that, for arbitrary $\epsilon \in (0,1)$ and for sufficiently large n, saying $n \geq N_0 \in N$, equation (3.1) has a solution $\{y_n\}_{n=-1}^{\infty}$ in the stripe

$$\varphi_n - \epsilon\psi_n \leq y_n \leq \varphi_n + \epsilon\psi_n, \quad n \geq N_0,$$

where φ_n and ψ_n are defined in (3.3). It is easy to see $y_n > 0$ for $n \geq N_0$, because $\varphi_n - \epsilon\psi_n > \varphi_n - \psi_n = t^n - t^{\lambda n} > 0$. Thus, equation (1.3) has a solution $\{x_n\}_{n=-1}^{\infty}$ satisfying $x_n = y_n + \bar{x} > \bar{x}$ for $n \geq N_0$. Since equation (1.3) is an autonomous equation, and $\{x_{n+N_0+1}\}_{n=-1}^{\infty}$ is still its solution, evidently satisfying $x_{n+N_0+1} > \bar{x}$ for $n \geq -1$. Therefore, the proof is complete.

Remark 3.3. Equation (1.3) has also solutions with a single negative semicyle. In fact, φ_n is taken as $-t^n$ in (3.3), then $\varphi_n + \epsilon\psi_n < -t^n + t^{\lambda n} < 0$, which indicates that equation (1.3) possesses the solution $\{x_n\}_{n=-1}^{\infty}$, remaining below its equilibrium for all $n \geq -1$.

Remark 3.4. Consider the following difference equation

$$x_{n+1} = \frac{a + \sum_{i=0}^{k} a_i x_{n-i}}{b + \sum_{i=0}^{k} b_i x_{n-i}}, n = 0, 1, \cdots, \tag{3.5}$$

where the initial conditions $x_{-k}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers, k is a positive integer number, and the parameters a, b, a_i, b_i for $i = 0, 1, \cdots, k$ are nonnegative with $a + \sum_{i=0}^{k} a_i > 0$ and $b + \sum_{i=0}^{k} b_i > 0$ such that the denominator is always positive.

The authors in [33] obtained the existence of nonoscillatory solutions of equation (3.5) by using the inclusion theorem, too. Nevertheless, one can see that equation (1.3) and equation (3.5) are two different difference equations.

4 Examples

Example 4.1. Consider the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, n = 0, 1, \cdots,$$

where α is a positive real constant, and the initial conditions x_{-1}, x_0 are positive real numbers.

Obviously, this difference equation, i.e., equation (1.1) is a special case of equation (1.3) under the conditions $k = 1$ and $b_0 = 1$. According to Theorem 2.1, this difference equation has nonoscillatory solutions.

Example 4.2. In this note, we investigate the existence of nonoscillatory solutions for a rational difference equation of higher order. As a corollary, our result solves an open problem presented in [11]. The higher-order rational difference equation we study in this paper is different from the one considered in [33].

The investigations on rational difference equations are still interesting for many readers. Some problems for rational difference equations, such as dichotomy [9,10], trichotomy [23], bifurcation [26,28,31,32] and chaos [31,32], are worthy of further consideration under the difference equation as follows

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, n = 0, 1, \cdots,$$

where $\alpha > 0$ and $k \in N$ is a fixed positive integer.

Equation (1.3) with $b_0 = 1, b_1 = b_2 = \cdots = b_{k-1} = 0$ is reduced to the above difference equation, namely, equation (1.2). Therefore, in view of Theorem 2.1, this difference equation possesses nonoscillatory solutions.

Combining the results in the above two examples, one can see the correctness of Corollary 3.2.

5 Conclusions and discussions

In this note, we investigate the existence of nonoscillatory solutions for a rational difference equation of higher order. As a corollary, our result solves an open problem presented in [11]. The higher-order rational difference equation we study in this paper is different with the one considered in [33].

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