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On Time-space Fractional Reaction-diffusion Equations with Nonlocal Initial Conditions*

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Abstract This paper investigates the nonlinear time-space fractional reaction-diffusion equations with nonlocal initial conditions. Based on the operator semigroup theory, we transform the time-space fractional reaction-diffusion equation into an abstract evolution equation. The existence and uniqueness of mild solution to the reaction-diffusion equation are obtained by solving the abstract evolution equation. Finally, we verify the Mittag-Leffler-Ulam stabilities of the nonlinear time-space fractional reaction-diffusion equations with nonlocal initial conditions. The results in this paper improve and extend some related conclusions to this topic.

Keywords Time-space fractional reaction-diffusion equation, Nonlocal initial condition, Mild solution, Existence and uniqueness, Mittag-Leffler-Ulam stability.

MSC(2010) 35R11, 47J35.

1. Introduction

In this paper, we study the nonlocal initial-boundary value problem for the time-space fractional reaction-diffusion equations (FRDE for short) with fractional Laplacian

$$\begin{cases} {}^C D_t^{\alpha} u(x,t) + (-\Delta)^{\beta} u(x,t) = f(x,t,u(x,t)), \qquad (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) + \sum\limits_{k=1}^{p} c_k u(x,t_k), \qquad (x,t_k) \in \Omega \times [0,T], \end{cases} \tag{1.1}$$

where ${}^C D_t^{\alpha}$ is the Caputo fractional derivative of the order $\alpha \in (0,1)$, $(-\Delta)^{\beta}$ is a fractional Laplacian with $\beta \in (0,1)$, Ω is an open bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$, $0 < t_1 < t_2 < \cdots < t_p < T$, $T > 0$ is a constant, $p \in \mathbb{N}$, $c_k \neq 0 (k = 1, 2, ..., p)$ are real numbers, $u_0 : \Omega \to \mathbb{R}$, and the nonlinear term $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is Carathéodory continuous.

FRDE is a subject of extensive research on fractional calculus, which has a wide range of applications in modeling such as mechanics of materials, fluid mechanics,

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signal processing and control as well as biology (see [4, 10, 15, 22, 28, 32–34] for details). In recent years, many scholars have been committed to the research of time-fractional or space-fractional partial differential equation (see [11–13, 38–40, 43]). On the other hand, there are numerous works that have been devoted to fractional diffusion equations. We only list a number of the numerous papers on the analysis for fractional diffusion equation. Eidelman and Kochubei [16] considered an evolution equation with the regularized fractional derivative of an order $\alpha \in (0,1)$. Jia and Li [19] obtained the Maximum principles of time-space fractional diffusion equation with Riemann-Liouville time-fractional derivative. Kemppainen, Siljander and Zacher [21] studied the Cauchy problem for a nonlocal heat equation, which is of fractional order both in space and time. In [23], Li, Liu and Wang established $L^r - L^q$ estimates and weighted estimates of the fundamental solutions, and obtained the existence and uniqueness of mild solutions to Keller-Segel type time-space fractional diffusion equation. The paper [24] constructed the iterative sequence of mild solution for time-space fractional diffusion equation with delay. A class of time fractional diffusion equation representation of solutions was derived by using Laplace transform in [35].

In 1990, Byszewski and Lakshmikantham [8] first investigated the existence of mild solution for nonlocal differential equations. Since the nonlocal initial conditions had been applied physics with better effect than the classical initial conditions, researchers began to study differential equations with nonlocal conditions and obtained some fundamental results (see [5, 6, 8, 14, 25, 41] for more comments and citations). As some results are in relation to the time-space FRDE (1.1), when $c_k = 0$, $k = 1, 2, ..., p$, one can see [24] and [30]. Whereas, there are few articles studying time-space FRDEs with nonlocal initial conditions. The aim of this paper is to extend the current results of the classical initial conditions considered into time-space fractional diffusion equations with nonlocal initial conditions. This extension is not just a mathematical problem, but is caused by numerous physical applications. As there are multitudinous works describing the significance of fractional and nonlocal models in the anomalous diffusion, one can refer to [30] and its references for details.

In the theory of functional equations, there are some special kinds of data dependence such as Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin and Aoki-Rassias [9, 18, 20]. Motivated by the results of [36], we further study the Mittag-leffler-Ulam stability of the time-space FRDE (1.1), and obtain some new and interesting stability results.

The main results with respect to the time-space FRDE (1.1) involving nonlocal initial conditions of this paper are as follows.

Theorem 1.1. *Assuming that the nonlinear function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory type condition and the following hypotheses,*
(H1) there exist a constant $q \in [0, \alpha)$ and a function $m(x,t)$ satisfying $\|m(t)\|_{\mathbb{H}^\beta(\Omega)} \in L^{1/q}([0,T], \mathbb{R}^+)$ with $M = \big(\int_0^T \|m(t)\|^{1/q}_{\mathbb{H}^\beta(\Omega)}\, dt\big)^q$ such that $|f(x,t,u)| \leqslant m(x,t)$ for all $x \in \Omega$, $t \in [0,T]$, $u \in \mathbb{R}$, and the norm $\|\cdot\|_{\mathbb{H}^\beta(\Omega)}$ of Sobolev space $\mathbb{H}^\beta(\Omega)$ is introduced in the following section;
(H2) $c_k > 0$, $k = 1, 2, ..., p$ and $\sum\limits_{k=1}^{p} c_k < 1$ are satisfied, then FRDE (1.1) has at least one mild solution $u \in C(\Omega \times [0,T], \mathbb{R})$.

Theorem 1.2. *Assuming that the nonlinear function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies*

condition (H2) and the following condition (H3),
(H3) there exists a positive constant $L < \frac{(1-\sum_{k=1}^{p} c_k)\Gamma(\alpha+1)}{T^\alpha}$, *where* $\Gamma(\cdot)$ *is Gamma function, such that* $|f(x,t,u) - f(x,t,v)| \leqslant L|u-v|$ *for all* $x \in \Omega$, $t \in [0,T]$, $u, v \in \mathbb{R}$, *then FRDE* (1.1) *has a unique mild solution* $u \in C(\Omega \times [0,T], \mathbb{R})$.

Theorem 1.3. *Assuming that all assumptions of Theorem 1.2 are satisfied, then FRDE* (1.1) *is Mittag-Leffler-Ulam-Hyers stable.*

Theorem 1.4. *Suppose that the function* $\varphi \in C([0,T], \mathbb{R}^+)$ *is increasing, and there exists* $\eta > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds \leqslant \eta\varphi(t), \quad t \in [0,T],$$

where $\Gamma(\cdot)$ *is Gamma function. If the conditions of Theorem 1.2 are satisfied, then FRDE* (1.1) *is generalized Mittag-Leffler-Ulam-Hyers-Rassias stablity with respect to* ηE_α, *where* E_α *is Mittag-Leffler function.*

The paper is organized as follows. In Section 2, we first introduce some notations and preliminaries which are used to prove our main results. In Section 3, we prove the existence and uniqueness of the mild solutions of FRDE (1.1), namely, Theorem 1.1 and Theorem 1.2. In Section 4, we verify the Mittag-Leffler-Ulam-Hyers stability of FRDE (1.1), namely, Theorem 1.3 and Theorem 1.4.

2. Preliminaries

In this section, we briefly give some notations and definitions for the sake of readers' convenience.

We adopt the $\beta \in (0,1)$ order Dirichlet Laplacian $(-\Delta)^\beta$ by Balakrishnan formula [3]. Let us begin by reviewing the classical spectral Laplacian $-\Delta$ with zero Dirichlet boundary condition. As is known to all, $-\Delta : D(-\Delta) \subset L^2(\Omega) \to L^2(\Omega)$ is unbounded, positive and closed with dense domain $D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, from [2], one knows that $-\Delta$ has a compact resolvent in $L^2(\Omega)$, and there exists a non-decreasing sequence $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$ called eigenvalues for $-\Delta$, $\lambda_j \to \infty$ as $j \to \infty$. Denoting the orthonormal eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ associated with $\{\lambda_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ as well as an orthogonal basis of $H_0^1(\Omega)$. That is,

$$\begin{cases} -\Delta\phi_j = \lambda_j\phi_j, \quad x \in \Omega, \\ \phi_j = 0, \quad x \in \partial\Omega. \end{cases}$$

Thus, we introduce the fractional power Dirichlet Laplacian.

Definition 2.1. The spectral fractional Laplacian is defined on the space $C_0^\infty(\Omega)$ as

$$(-\Delta)^\beta u := \sum_{j=1}^{\infty} \lambda_j^\beta u_j \phi_j, \quad u_j := \int_\Omega u\phi_j\, dx, \quad j \in \mathbb{N}.$$

The fractional power Laplacian from spectral theory and the semigroups theory is

$$(-\Delta)^\beta u = \frac{\sin \beta\pi}{\pi} \int_0^\infty \lambda^{\beta-1} (\lambda I - \Delta)^{-1} (-\Delta) u\, d\lambda$$

by Balakrishnan formula in [3]. It can be seen from [7] that the spectral definition is equivalent to the Balakrishnan definition in $L^2(\Omega)$.

We introduce the following definitions related to fractional-order Sobolev space on $\Omega \subset \mathbb{R}^n$, which can be found in [2,27,29]. We denote the norm of $L^2(\Omega)$ by $\|\cdot\|_2$ and the $\beta \in (0,1)$ order Sobolev space

$$H^\beta(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\beta}}\, dx\, dy < \infty \right\}$$

endowed with the natural norm

$$\|u\|_{H^\beta(\Omega)} = \left(\|u\|_2^2 + |u|^2_{H^\beta(\Omega)} \right)^{\frac{1}{2}}, \quad u \in H^\beta(\Omega),$$

where the seminorm $|u|^2_{H^\beta(\Omega)}$ is defined as follows

$$|u|^2_{H^\beta(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\beta}}\, dx\, dy, \quad u \in H^\beta(\Omega).$$

Let

$$H_0^\beta(\Omega) := \{ u \in H^\beta(\Omega) : \ \Delta u(x)|_{\partial\Omega} = 0 \}$$

and

$$H_{00}^{\frac{1}{2}}(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) : \ \int_\Omega \frac{u^2(x)}{\operatorname{dist}(x,\partial\Omega)}\, dx < \infty \right\}$$

with the norm

$$\|u\|_{H_{00}^{\frac{1}{2}}(\Omega)} = \left(\|u\|^2_{H^{\frac{1}{2}}(\Omega)} + \int_\Omega \frac{u^2(x)}{\operatorname{dist}(x,\partial\Omega)}\, dx \right)^{\frac{1}{2}}.$$

From [2], we also define the fractional Sobolev space

$$\mathbb{H}^\beta(\Omega) := \left\{ u = \sum_{j=1}^\infty u_j \phi_j \in L^2(\Omega) : \ \|u\|^2_{\mathbb{H}^\beta(\Omega)} := \sum_{j=1}^\infty \lambda_j^\beta u_j^2 < \infty \right\},$$

where λ_j are the eigenvalues of $-\Delta$ with zero Dirichlet boundary conditions on Ω, and ϕ_j are eigenfunctions with respect to λ_j and

$$u_j = \int_\Omega u \phi_j \, dx.$$

The characterization of the Sobolev space $\mathbb{H}^\beta(\Omega)$ is as follows

$$\mathbb{H}^\beta(\Omega) = \begin{cases} H^\beta(\Omega) = H_0^\beta(\Omega), & \beta \in (0, \frac{1}{2}), \\ H_{00}^{\frac{1}{2}}(\Omega), & \beta = \frac{1}{2}, \\ H_0^\beta(\Omega), & \beta \in (\frac{1}{2}, 1). \end{cases}$$

Let $C([0,T], \mathbb{H}^\beta(\Omega))$ be a Banach space of all $\mathbb{H}^\beta(\Omega)$-value continuous functions on $[0,T]$ with the norms $\|u\|_C := \sup_{t\in[0,T]} \|u(t)\|_{\mathbb{H}^\beta(\Omega)}$ and $Au = -\Delta u$, $u \in \mathbb{H}^\beta(\Omega)$. It is well-known that a positive, uniformly bounded analytic semigroup $T(t)(t \geqslant 0)$ has been generated by $-A$ from [1] and [31], since $-A$ has compact resolvent in

$L^2(\Omega)$, and then $T(t)(t \geqslant 0)$ is also a compact semigroup. From [42], based on the equivalence of spectral definition and Balakrishnan definition, we know that $-A^\beta$ generates a positive, compact and uniformly bounded analytic semigroup $T_\beta(t)(t \geqslant 0)$. Moreover, $T_\beta(t)(t \geqslant 0)$ is a contractive semigroup.

For $u \in \mathbb{H}^\beta(\Omega)$, define the two operators $\mathscr{T}_{\alpha,\beta}(t)(t \geqslant 0)$ and $\mathscr{S}_{\alpha,\beta}(t)(t \geqslant 0)$ by

$$\mathscr{T}_{\alpha,\beta}(t)u = \int_0^\infty h_\alpha(s) T_\beta(t^\alpha s)u\, ds, \qquad \mathscr{S}_{\alpha,\beta}(t)u = \alpha \int_0^\infty s h_\alpha(s) T_\beta(t^\alpha s)u\, ds,$$

where

$$h_\alpha(s) = \frac{1}{\pi\alpha} \sum_{n=1}^\infty (-s)^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \qquad s \in (0,\infty)$$

is a function of Wright type defined on $(0,\infty)$ satisfying

$$h_\alpha(s) \geqslant 0,\ s \in (0,\infty), \qquad \int_0^\infty h_\alpha(s) ds = 1,$$

and

$$\int_0^\infty s^\gamma h_\alpha(s) ds = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)},\ \gamma \in [0,1].$$

The following lemma can be found in [17, 37].

Lemma 2.1. *The operators $\mathscr{T}_{\alpha,\beta}(t)(t \geqslant 0)$ and $\mathscr{S}_{\alpha,\beta}(t)(t \geqslant 0)$ have the following properties.*

- *(i) The operators $\mathscr{T}_{\alpha,\beta}(t)(t \geqslant 0)$ and $\mathscr{S}_{\alpha,\beta}(t)(t \geqslant 0)$ are strongly continuous, which means that for any $u \in \mathbb{H}^\beta(\Omega)$ and $0 \leqslant t_1 \leqslant t_2 \leqslant T$, as $t_2 - t_1 \to 0$,*

$$\|\mathscr{T}_{\alpha,\beta}(t_2)u - \mathscr{T}_{\alpha,\beta}(t_1)u\|_{\mathbb{H}^\beta(\Omega)} \to 0, \quad \|\mathscr{S}_{\alpha,\beta}(t_2)u - \mathscr{S}_{\alpha,\beta}(t_1)u\|_{\mathbb{H}^\beta(\Omega)} \to 0.$$

- *(ii) If C_0-semigroup $T_\beta(t)(t \geqslant 0)$ is uniformly bounded and contractive, then for any fixed $t \geqslant 0$, $\mathscr{T}_{\alpha,\beta}(t)$ and $\mathscr{S}_{\alpha,\beta}(t)$ are linear and bounded operators, which means that for any $u \in \mathbb{H}^\beta(\Omega)$,*

$$\|\mathscr{T}_{\alpha,\beta}(t)u\|_{\mathbb{H}^\beta(\Omega)} \leqslant \|u\|_{\mathbb{H}^\beta(\Omega)}, \qquad \|\mathscr{S}_{\alpha,\beta}(t)u\|_{\mathbb{H}^\beta(\Omega)} \leqslant \frac{1}{\Gamma(\alpha)} \|u\|_{\mathbb{H}^\beta(\Omega)}.$$

- *(iii) If C_0-semigroup $T_\beta(t)(t \geqslant 0)$ is compact, then $\mathscr{T}_{\alpha,\beta}(t)$ and $\mathscr{S}_{\alpha,\beta}(t)$ are compact operators for every $t > 0$.*
- *(iv) If C_0-semigroup $T_\beta(t)(t \geqslant 0)$ is continuous by operator norm for every $t > 0$, then $\mathscr{T}_{\alpha,\beta}(t)$ and $\mathscr{S}_{\alpha,\beta}(t)$ are uniformly continuous for every $t > 0$.*

By the contraction of semigroup $T_\beta(t)(t \geqslant 0)$ and condition (H2), we have

$$\Big\| \sum_{k=1}^p c_k \mathscr{T}_{\alpha,\beta}(t_k) \Big\|_{\mathbb{H}^\beta(\Omega)} \leqslant \sum_{k=1}^p c_k < 1.$$

According to the operator spectral theorem, we know that the operator

$$\mathscr{B} := \Big(I - \sum_{k=1}^p c_k \mathscr{T}_{\alpha,\beta}(t_k) \Big)^{-1}$$

exists and is bounded. Moreover, by Neumann expression, $\mathscr{B}$ can be expressed by

$$\mathscr{B} = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t_k) \Big)^n. \tag{2.1}$$

Therefore,

$$\begin{aligned} \|\mathscr{B}\|_{\mathbb{H}^\beta(\Omega)} &\leqslant \sum_{n=0}^{\infty} \Big\| \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t_k) \Big\|^n_{\mathbb{H}^\beta(\Omega)} = \frac{1}{1 - \| \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t_k) \|_{\mathbb{H}^\beta(\Omega)}} \\ &\leqslant \frac{1}{1 - \sum_{k=1}^{p} c_k}. \end{aligned} \tag{2.2}$$

Lemma 2.2. *For $\sigma \in (0,1]$ and $0 < a \leqslant b$, we have*

$$|a^\sigma - b^\sigma| \leqslant (b-a)^\sigma.$$

Theorem 2.1 (Theorem 1.4, [26])**.** *For any $t \in [0,T)$, if*

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\alpha_i - 1} u(s)\, ds,$$

where not all the functions are negative continuous. The constants $\alpha_i > 0$, $b_i(i = 1, 2, ..., n)$ are the bounded and monotonic increasing functions on $[0,T)$, then

$$u(t) \leqslant \sum_{k=1}^{\infty} \left(\sum_{1',2',\cdots,k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t)\Gamma(\alpha_{i'})]}{\Gamma(\sum_{i=1}^{k} \alpha_{i'})} \int_0^t (t-s)^{\sum_{i=1}^{k} \alpha_{i'} - 1} a(s)\, ds \right).$$

Remark 2.1. For $n = 2$, if the constants $b_1, b_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$, $a(t)$ is nonnegative and locally intergrable on $0 \leqslant t < T$, and $u(t)$ is nonnegative and locally integrable on $0 \leqslant t < T$ with

$$u(t) \leqslant a(t) + b_1 \int_0^t (t-s)^{\alpha_1 - 1} u(s)\, ds + b_2 \int_0^t (t-s)^{\alpha_2 - 1} u(s)\, ds.$$

Then,

$$u(t) \leqslant a(t) + \sum_{k=1}^{\infty} \left[\frac{(b_1\Gamma(\alpha_1))^k}{\Gamma(k\alpha_1)} \int_0^t (t-s)^{k\alpha_1 - 1} a(s)\, ds + \frac{(b_2\Gamma(\alpha_2))^k}{\Gamma(k\alpha_2)} \int_0^t (t-s)^{k\alpha_2 - 1} a(s)\, ds \right].$$

Remark 2.2. Under the hypothesis of Remark 2.1, let $a(t)$ be a nondecreasing function on $0 \leqslant t < T$. Then, we have

$$u(t) \leqslant a(t)(E_{\alpha_1}[b_1\Gamma(\alpha_1)t^{\alpha_1}] + E_{\alpha_2}[b_2\Gamma(\alpha_2)t^{\alpha_2}]),$$

where $E_\alpha[\cdot]$ is the Mittag-Leffler function defined by $E_\alpha[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z \in \mathbb{C}$.

3. Existence and uniqueness of mild solutions

In this section, we will give a proof of the existence and uniqueness of mild solutions to the time-space FRDE (1.1). That is to say, we will prove Theorem 1.1 and Theorem 1.2. Let $u(t) = u(\cdot,t)$, $f(t,u(t)) = f(\cdot,t,u(\cdot,t))$, $u_0 + \sum_{k=1}^{p} c_k u(t_k) = u_0(\cdot) + \sum_{k=1}^{p} c_k u(\cdot,t_k)$. Then, FRDE (1.1) can be rewritten as an abstract form in $C([0,T],\mathbb{H}^\beta(\Omega))$ of the fractional evolution equation with nonlocal conditions

$$\begin{cases} {}^C D_t^\alpha u(t) + A^\beta u(t) = f(t,u(t)), \quad t \in [0,T], \\ u(0) = u_0 + \sum\limits_{k=1}^{p} c_k u(t_k). \end{cases} \tag{3.1}$$

Definition 3.1. A function $u \in C([0,T],\mathbb{H}^\beta(\Omega))$ is a mild solution to (3.1), if it satisfies

$$\begin{aligned} u(t) = \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}u_0 + \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds \\ + \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds. \end{aligned} \tag{3.2}$$

Proof of Theorem 1.1. From the above, we know that FRDE (1.1) can be transformed into the abstract nonlocal evolution (3.1). In what follows, we prove the existence of the mild solution to the nonlocal problem (3.1) by applying fixed point theorem. We set

$$\mathcal{B}_{R,T} = \big\{u \in C([0,T],\mathbb{H}^\beta(\Omega)) \mid \|u\|_C \leqslant R, R > 0\big\}$$

as a nonempty, closed and convex subset in $C([0,T],\mathbb{H}^\beta(\Omega))$. Then, for any $u \in \mathcal{B}_{R,T}$, we consider the operator P on $C([0,T],\mathbb{H}^\beta(\Omega))$ defined by

$$\begin{aligned} (Pu)(t) =& \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}u_0 + \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds \\ &+ \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds. \end{aligned} \tag{3.3}$$

From Definition 3.1, the mild solution of the nonlocal problem (3.1) on $[0,T]$ is equivalent to the fixed point of the operator P defined by (3.3). Next, we prove that the operator P has a fixed point via Schauder fixed point theorem.

First, we demonstrate that P is continuous on $\mathcal{B}_{R,T}$. Let $\{u_n\}_{n=1}^\infty \subset C([0,T],\mathbb{H}^\beta(\Omega))$ be a sequence and $\lim_{n\to\infty} u_n = u$ in $C([0,T],\mathbb{H}^\beta(\Omega))$. Owing to the continuity of f, we know $\lim_{n\to\infty} f(s,u_n(s)) = f(s,u(s))$ for all $s \in [0,T]$. Therefore,

$$\sup_{s\in[0,T]} \|f(s,u_n(s)) - f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Then, for $t \in [0,T]$, we have

$$\begin{aligned} &\|(Pu_n)(t) - (Pu)(t)\|_{\mathbb{H}^\beta(\Omega)} \\ \leqslant& \frac{\sum_{k=1}^{p} c_k}{1-\sum_{k=1}^{p} c_k}\frac{1}{\Gamma(\alpha)}\int_0^{t_k}(t_k-s)^{\alpha-1}\|f(s,u_n(s)) - f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,u_n(s))-f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\,ds\\
\leqslant&\frac{\sum_{k=1}^p c_k T^\alpha}{(1-\sum_{k=1}^p c_k)\Gamma(1+\alpha)}\sup_{s\in[0,T]}\|f(s,u_n(s))-f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\\
&+\frac{T^\alpha}{\Gamma(1+\alpha)}\sup_{s\in[0,T]}\|f(s,u_n(s))-f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\\
=&\frac{T^\alpha}{(1-\sum_{k=1}^p c_k)\Gamma(1+\alpha)}\sup_{s\in[0,T]}\|f(s,u_n(s))-f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)},
\end{aligned}
$$

which means

$$
\|(Pu_n)-(Pu)\|_C\leqslant\frac{T^\alpha}{(1-\sum_{k=1}^p c_k)\Gamma(1+\alpha)}\sup_{s\in[0,T]}\|f(s,u_n(s))-f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}.
$$

By (3.4), we conclude

$$
\|(Pu_n)-(Pu)\|_C\to 0\quad\text{as}\quad n\to\infty.
$$

Namely, P is continuous on $\mathcal{B}_{R,T}$.

Next, we show that there exists a constant $R>0$ being determined latter such that $P(\mathcal{B}_{R,T})\subset\mathcal{B}_{R,T}$. For any $u\in\mathcal{B}_{R,T}$ and $t\in[0,T]$, following condition (H1), we have

$$
\begin{aligned}
&\|(Pu)(t)\|_{\mathbb{H}^\beta(\Omega)}\\
\leqslant&\|\mathscr{B}\|_{\mathbb{H}^\beta(\Omega)}\cdot\|u_0\|_{\mathbb{H}^\beta(\Omega)}+\sum_{k=1}^p c_k\cdot\|\mathscr{B}\|_{\mathbb{H}^\beta(\Omega)}\frac{1}{\Gamma(\alpha)}\int_0^{t_k}(t_k-s)^{\alpha-1}\|f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\,ds\\
\leqslant&\frac{\|u_0\|_{\mathbb{H}^\beta(\Omega)}}{1-\sum_{k=1}^p c_k}+\frac{\sum_{k=1}^p c_k}{1-\sum_{k=1}^p c_k}\frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_k}(t_k-s)^{\frac{\alpha-1}{1-q}}\,ds\Big)^{1-q}\Big(\int_0^{t_k}\|m(s)\|_{\mathbb{H}^\beta(\Omega)}^{1/q}\,ds\Big)^q\\
&+\frac{1}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}\,ds\Big)^{1-q}\Big(\int_0^t\|m(s)\|_{\mathbb{H}^\beta(\Omega)}^{1/q}\,ds\Big)^q\\
\leqslant&\frac{1}{1-\sum_{k=1}^p c_k}\Big(\|u_0\|_{\mathbb{H}^\beta(\Omega)}+\frac{MT^{\alpha-q}}{\Gamma(\alpha)}\big(\frac{1-q}{\alpha-q}\big)^{1-q}\Big):=R.
\end{aligned}\tag{3.5}
$$

Hence, $P:\mathcal{B}_{R,T}\to\mathcal{B}_{R,T}$ is bounded.

In what follows, we prove that the $P(\mathcal{B}_{R,T})$ is equicontinuous. For any $u\in\mathcal{B}_{R,T}$ and $0\leqslant t_1<t_2\leqslant T$, we have

$$
\begin{aligned}
(Pu)(t_2)-(Pu)(t_1)&\\
=&\mathscr{T}_{\alpha,\beta}(t_2)\mathscr{B}u_0-\mathscr{T}_{\alpha,\beta}(t_1)\mathscr{B}u_0\\
&+\sum_{k=1}^p c_k(\mathscr{T}_{\alpha,\beta}(t_2)-\mathscr{T}_{\alpha,\beta}(t_1))\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_2-s)f(s,u(s))\,ds\\
&+\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\mathscr{S}_{\alpha,\beta}(t_2-s)f(s,u(s))\,ds
\end{aligned}
$$

$$+\int_0^{t_1}((t_1-s)^{\alpha-1}(\mathscr{S}_{\alpha,\beta}(t_2-s)-\mathscr{S}_{\alpha,\beta}(t_1-s))f(s,u(s))\,ds$$

$$:=J_1+J_2+J_3+J_4+J_5.$$

Owing to

$$\|(Pu)(t_2)-(Pu)(t_1)\|_{\mathbb{H}^\beta(\Omega)}\leqslant\sum_{i=1}^{5}\|J_i\|_{\mathbb{H}^\beta(\Omega)},$$

we just need to examine that $\|J_i\|_{\mathbb{H}^\beta(\Omega)}\to 0(i=1,2,...,5)$ is independent of $u\in\mathcal{B}_{R,T}$, as $t_2-t_1\to 0$.
According to (i) of Lemma 2.1, $\|J_1\|_{\mathbb{H}^\beta(\Omega)}\leqslant\|\mathscr{T}_{\alpha,\beta}(t_2)-\mathscr{T}_{\alpha,\beta}(t_1)\|_{\mathbb{H}^\beta(\Omega)}\mathscr{B}u_0\to 0$,

$$\begin{aligned}\|J_2\|_{\mathbb{H}^\beta(\Omega)}\leqslant&\|\mathscr{T}_{\alpha,\beta}(t_2)-\mathscr{T}_{\alpha,\beta}(t_1)\|_{\mathbb{H}^\beta(\Omega)}\\&\times\sum_{k=1}^{p}c_k\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds\to 0,\end{aligned}$$

and $\|J_3\|_{\mathbb{H}^\beta(\Omega)}\to 0$ are available as $t_2\to t_1$.
For J_4, using the Lemma 2.1, Lemma 2.2 and the Hölder inequality,

$$\begin{aligned}\|J_4\|_{\mathbb{H}^\beta(\Omega)}\leqslant&\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big)\|m(s)\|_{\mathbb{H}^\beta(\Omega)}\,ds\\\leqslant&\frac{1}{\Gamma(\alpha)}\bigg(\int_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)^{\frac{1}{1-q}}\,ds\bigg)^{1-q}\\&\times\bigg(\int_0^{t_1}\|m(s)\|_{\mathbb{H}^\beta(\Omega)}^{1/q}\,ds\bigg)^{q}\\\leqslant&\frac{M}{\Gamma(\alpha)}\bigg(\int_0^{t_1}\big((t_1-s)^{\frac{\alpha-1}{1-q}}-(t_2-s)^{\frac{\alpha-1}{1-q}}\big)\,ds\bigg)^{1-q}\\\leqslant&\frac{M}{\Gamma(\alpha)}\Big(\frac{1-q}{\alpha-q}\Big)^{1-q}\big(t_1^{\frac{\alpha-q}{1-q}}-t_2^{\frac{\alpha-q}{1-q}}+(t_2-t_1)^{\frac{\alpha-q}{1-q}}\big)^{1-q}\\\leqslant&\frac{M}{\Gamma(\alpha)}\Big(\frac{1-q}{\alpha-q}\Big)^{1-q}\big(2(t_2-t_1)\big)^{\alpha-q}\to 0\quad\text{as}\quad t_2\to t_1.\end{aligned}$$

It is conspicuous that $\|J_5\|_{\mathbb{H}^\beta(\Omega)}=0$, when $t_1=0$, $0<t_2\leqslant T$. Now, we consider $t_1>0$ and $0<\varepsilon<t_1$, and based on the equicontinuity of the semigroup $T_\beta(t)(t\geqslant 0)$, we have

$$\begin{aligned}&\|J_5\|_{\mathbb{H}^\beta(\Omega)}\\\leqslant&\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\|\mathscr{S}_{\alpha,\beta}(t_2-s)-\mathscr{S}_{\alpha,\beta}(t_1-s)\|_{\mathbb{H}^\beta(\Omega)}\cdot\|f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\,ds\\&+\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\|\mathscr{S}_{\alpha,\beta}(t_2-s)-\mathscr{S}_{\alpha,\beta}(t_1-s)\|_{\mathbb{H}^\beta(\Omega)}\cdot\|f(s,u(s))\|_{\mathbb{H}^\beta(\Omega)}\,ds\\\leqslant&\sup_{s\in[0,t_1-\varepsilon]}\|\mathscr{S}_{\alpha,\beta}(t_2-s)-\mathscr{S}_{\alpha,\beta}(t_1-s)\|_{\mathbb{H}^\beta(\Omega)}\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\|m(s)\|_{\mathbb{H}^\beta(\Omega)}\,ds\\&+\frac{2}{\Gamma(\alpha)}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\|m(s)\|_{\mathbb{H}^\beta(\Omega)}\,ds\end{aligned}$$

$$\leqslant \sup_{s\in[0,t_1-\varepsilon]} \|\mathscr{S}_{\alpha,\beta}(t_2-s)-\mathscr{S}_{\alpha,\beta}(t_1-s)\|_{\mathbb{H}^\beta(\Omega)}\cdot M\Big(\frac{1-q}{\alpha-q}\Big)^{1-q}\Big(t_1^{\frac{\alpha-q}{1-q}}-\varepsilon^{\frac{\alpha-q}{1-q}}\Big)^{1-q}$$
$$+\frac{2M}{\Gamma(\alpha)}\Big(\frac{1-q}{\alpha-q}\Big)^{1-q}\varepsilon^{\alpha-q}\to 0 \quad \text{as} \quad t_2\to t_1, \varepsilon\to 0.$$

Therefore, $\|(Pu)(t_2)-(Pu)(t_1)\|_{\mathbb{H}^\beta(\Omega)}\to 0$ as $t_2\to t_1$, and it is not dependent on $u\in\mathcal{B}_{R,T}$, which means that $P:\mathcal{B}_{R,T}\to\mathcal{B}_{R,T}$ is equicontinuous.

Finally, we prove that the set $(P\mathcal{B}_{R,T})(t)$ is relatively compact in $\mathcal{B}_{R,T}$. For $\epsilon>0$, we define a set

$$(P_\epsilon\mathcal{B}_{R,T})(t)=\{(P_\epsilon u)(t)\mid u\in\mathcal{B}_{R,T}, t\in[0,T]\},$$

where $P_\epsilon u$ is defined as follows

$$\begin{aligned}(P_\epsilon u)(t)=&\mathscr{T}_{\alpha,\beta}(t+\epsilon)\mathscr{B}u_0+\sum_{k=1}^{p}c_k\mathscr{T}_{\alpha,\beta}(t+\epsilon)\mathscr{B}\\ &\times\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k+\epsilon-s)f(s,u(s))\,ds\\ &+\int_0^{t}(t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t+\epsilon-s)f(s,u(s))\,ds\\ =&\mathscr{T}_{\alpha,\beta}(\epsilon)\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}u_0+\mathscr{T}_{\alpha,\beta}(\epsilon)\mathscr{S}_{\alpha,\beta}(\epsilon)\sum_{k=1}^{p}c_k\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\\ &\times\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds\\ &+\mathscr{S}_{\alpha,\beta}(\epsilon)\int_0^{t}(t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds,\end{aligned}$$

for every $t\in[0,T]$. Since the operators $\mathscr{T}_{\alpha,\beta}(\epsilon)$ and $\mathscr{S}_{\alpha,\beta}(\epsilon)$ are compact in $\mathbb{H}^\beta(\Omega)$, then the set $(P_\epsilon\mathcal{B}_{R,T})(t)$ is relatively compact in $\mathbb{H}^\beta(\Omega)$. Since

$$\begin{aligned}&\|(Pu)(t)-(P_\epsilon u)(t)\|_{\mathbb{H}^\beta(\Omega)}\\ \leqslant&\|\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}u_0-\mathscr{T}_{\alpha,\beta}(\epsilon)\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}u_0\|_{\mathbb{H}^\beta(\Omega)}\\ &+\Big\|\sum_{k=1}^{p}c_k\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds\\ &-\mathscr{T}_{\alpha,\beta}(\epsilon)\mathscr{S}_{\alpha,\beta}(\epsilon)\sum_{k=1}^{p}c_k\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}\\ &+\Big\|\int_0^{t}(t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds\\ &-\mathscr{S}_{\alpha,\beta}(\epsilon)\int_0^{t}(t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}\\ \to&0, \quad \text{as} \quad \epsilon\to 0,\end{aligned}$$

which yields that $(P\mathcal{B}_{R,T})(t)$ is relatively compact. By the Arzela-Ascoli theorem, P is a compact continuous operator on $\mathcal{B}_{R,T}$.

Therefore, $P : \mathcal{B}_{R,T} \to \mathcal{B}_{R,T}$ has at least one fixed point via the Schauder fixed point theorem, which means the nonlocal problem (3.1) has mild solution $u \in C([0,T], \mathbb{H}^\beta(\Omega))$. Following the time-space FRDE (1.1), it can be transformed into the abstract evolution equation (3.1). Thus, FRDE (1.1) has at least one mild solution $u(x,t) \in C(\Omega \times [0,T], \mathbb{R})$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. From the proof of Theorem 1.1, we know that FRDE (1.1) can be transformed into the abstract nonlocal evolution equation (3.1) in Sobolev space $\mathbb{H}^\beta(\Omega)$, and the mild solutions of the abstract nonlocal evolution equation (3.1) are equivalent to the fixed point of operator $P : C([0,T], \mathbb{H}^\beta(\Omega)) \to C([0,T], \mathbb{H}^\beta(\Omega))$ defined by (3.3). By condition (H3), we get

$$\begin{aligned}\|f(t,u(t)) - f(t,v(t))\|_{\mathbb{H}^\beta(\Omega)} &\leqslant \Big(\sum_{j=1}^\infty \lambda_j^\beta \Big(\int_\Omega |f(t,u(t)) - f(t,v(t))| \phi_j \, dx \Big)^2 \Big)^{\frac{1}{2}} \\ &\leqslant \Big(\sum_{j=1}^\infty \lambda_j^\beta \Big(\int_\Omega L |u(t) - v(t)| \phi_j \, dx \Big)^2 \Big)^{\frac{1}{2}} \\ &= L \|u(t) - v(t)\|_{\mathbb{H}^\beta(\Omega)}.\end{aligned}$$

Then, for every $t \in [0,T]$ and any $u, v \in C([0,T], \mathbb{H}^\beta(\Omega))$,

$$\begin{aligned}&\|(Pu)(t) - (Pv)(t)\|_{\mathbb{H}^\beta(\Omega)} \\ \leqslant& \frac{\sum_{k=1}^p c_k}{1 - \sum_{k=1}^p c_k} \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} \|f(s,u(s)) - f(s,v(s))\|_{\mathbb{H}^\beta(\Omega)} \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u(s)) - f(s,v(s))\|_{\mathbb{H}^\beta(\Omega)} \, ds \\ \leqslant& \frac{\sum_{k=1}^p c_k T^\alpha}{(1 - \sum_{k=1}^p c_k)\Gamma(\alpha+1)} L \|u - v\|_C + \frac{T^\alpha}{\Gamma(\alpha+1)} L \|u - v\|_C \\ \leqslant& \frac{L}{(1 - \sum_{k=1}^p c_k)\Gamma(\alpha+1)} T^\alpha \|u - v\|_C < \|u - v\|_C,\end{aligned}$$

which means

$$\|Pu - Pv\|_C = \sup_{t \in [0,T]} \|(Pu)(t) - (Pv)(t)\|_{\mathbb{H}^\beta(\Omega)} < \|u - v\|_C.$$

Hence, $P : C([0,T], \mathbb{H}^\beta(\Omega)) \to C([0,T], \mathbb{H}^\beta(\Omega))$ is a contraction operator, and we know that P has a unique fixed point $u \in C([0,T], \mathbb{H}^\beta(\Omega))$. That is, the nonlocal problem (3.1) has a unique mild solution on $[0,T]$, then FRDE (1.1) has a unique mild solution $u(x,t) \in C(\Omega \times [0,T], \mathbb{R})$. This completes the proof of Theorem 1.2.□

4. Mittag-Leffler-Ulam stabilities

In this section, we consider the Mittag-leffler-Ulam stability of FRDE (1.1). From the discussion of the previous parts, we only need to study the stability of the nonlocal problem (3.1). Letting $\epsilon > 0$ and $\varphi \in C([0,T], \mathbb{R}^+)$, we consider the following inequalities

$$\|{}^C D_t^\alpha v(t) + A^\beta v(t) - f(t,v(t))\|_{\mathbb{H}^\beta(\Omega)} \leqslant \epsilon, \quad t \in [0,T]; \tag{4.1}$$

$$\|{}^C D_t^\alpha v(t) + A^\beta v(t) - f(t, v(t))\|_{\mathbb{H}^\beta(\Omega)} \leqslant \varphi(t), \quad t \in [0, T]; \tag{4.2}$$

$$\|{}^C D_t^\alpha v(t) + A^\beta v(t) - f(t, v(t))\|_{\mathbb{H}^\beta(\Omega)} \leqslant \epsilon\varphi(t), \quad t \in [0, T]. \tag{4.3}$$

Definition 4.1. Equation (3.1) is Mittag-Leffler-Ulam-Hyers stable with respect to E_α, if there exists a real number $c > 0$ such that for each $\epsilon > 0$, and for each solution $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ of inequality (4.1), there exists a mild solution $u \in C([0,T], \mathbb{H}^\beta(\Omega))$ of equation (3.1) with $\|v(t) - u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant c\epsilon E_\alpha[t]$, $t \in [0, T]$.

Definition 4.2. Equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers stablity with respect to E_α, if there exists $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\psi(0) = 0$ such that for each solution $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ of inequality (4.1), there exists a mild solution $u \in C([0,T], \mathbb{H}^\beta(\Omega))$ of equation (3.1) with $\|v(t) - u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \psi(\epsilon)E_\alpha[t]$, $t \in [0, T]$.

Definition 4.3. Equation (3.1) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to φE_α, if there exists $C_\varphi > 0$ such that for each $\epsilon > 0$, and for each solution $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ of inequality (4.3), there exists a mild solution $u \in C([0,T], \mathbb{H}^\beta(\Omega))$ of equation (3.1) with $\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant C_\varphi\epsilon\varphi(t)E_\alpha[t]$, $t \in [0, T]$.

Definition 4.4. Equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stablity with respect to φE_α, if there exists $C_\varphi > 0$ such that for each solution $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ of inequality (4.2), there exists a mild solution $u \in C([0,T], \mathbb{H}^\beta(\Omega))$ of equation (3.1) with $\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant C_\varphi\varphi(t)E_\alpha[t]$, $t \in [0, T]$.

Remark 4.1. It is clear that: (i) Definition 4.1 $\Rightarrow$ Definition 4.2; (ii) Definition 4.3 $\Rightarrow$ Definition 4.4.

Remark 4.2. A function $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ is a solution of inequality (4.1), if and only if there exists a function $g \in C([0,T], \mathbb{H}^\beta(\Omega))$ (which depend on v) such that
(i) $\|g(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \epsilon$, for all $t \in [0, T]$;
(ii) ${}^C D_t^\alpha u(t) + A^\beta u(t) = f(t, u(t)) + g(t)$, $t \in [0, T]$.

We have similar remarks for inequalities (4.2) and (4.3).

Remark 4.3. If $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ is a solution of inequation (4.1), then v satisfies the following inequality

$$\begin{aligned}
&\left\| v(t) - \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 - \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k - s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k - s)f(s, v(s))\,ds \right.\\
&\left. - \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s, v(s))\,ds \right\|_{\mathbb{H}^\beta(\Omega)}\\
\leqslant&\epsilon\int_0^t (t-s)^{\alpha-1}\|\mathscr{S}_{\alpha,\beta}(t-s)\|_{\mathbb{H}^\beta(\Omega)}\,ds.
\end{aligned}$$

We have similar remarks for the solutions of inequations (4.2) and (4.3).

Now, we give the proof of Mittag-Leffler-Ulam stabilities of FRDE (1.1).

Proof of Theorem 1.3. Let $v \in C^1([0,T], \mathbb{H}^\beta(\Omega))$ be a solution of the inequation (4.1), and denoted by $u \in C([0,T], \mathbb{H}^\beta(\Omega))$, the unique mild solution of the problem is as follows

$$\begin{cases} {}^C D_t^\alpha u(t) + A^\beta u(t) = f(t, u(t)), \quad t \in [0, T],\\ u(0) = v(0). \end{cases} \tag{4.4}$$

We have

$$u(t) =\mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 + \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds$$
$$+\int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds,$$

and by Remark 4.2, we get

$$\Big\| v(t) - \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 - \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,v(s))\,ds$$
$$-\int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,v(s))\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}$$
$$\leqslant \epsilon\int_0^t (t-s)^{\alpha-1}\|\mathscr{S}_{\alpha,\beta}(t-s)\|_{\mathbb{H}^\beta(\Omega)}\,ds \leqslant \frac{T^\alpha}{\Gamma(\alpha+1)}\epsilon.$$

From these relations, we have

$$\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)}$$
$$=\Big\| v(t) - \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 - \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,u(s))\,ds$$
$$-\int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}$$
$$\leqslant\Big\| v(t) - \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 - \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,v(s))\,ds$$
$$-\int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,v(s))\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}$$
$$+\Big\|\sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)[f(s,v(s))-f(s,u(s))]\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}$$
$$+\Big\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)[f(s,v(s))-f(s,u(s))]\,ds\Big\|_{\mathbb{H}^\beta(\Omega)}$$
$$\leqslant\frac{T^\alpha}{\Gamma(\alpha+1)}\epsilon + \frac{\sum_{k=1}^p c_k L}{(1-\sum_{k=1}^p c_k)\Gamma(\alpha)}\int_0^{t_k}(t_k-s)^{\alpha-1}\|v(s))-u(s)\|_{\mathbb{H}^\beta(\Omega)}\,ds$$
$$+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|v(s))-u(s)\|_{\mathbb{H}^\beta(\Omega)}\,ds. \tag{4.5}$$

Applying Remark 2.1 and Remark 2.2 to inequality (4.5), we get

$$\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(E_\alpha\Big[\frac{\sum_{k=1}^p c_k L}{1-\sum_{k=1}^p c_k}t_k^\alpha\Big] + E_\alpha[Lt^\alpha]\Big)\epsilon.$$

Hence, by Definition 4.1, the nonlocal problem (3.1) is Mittag-Leffler-Ulam-Hyers stable. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Let $v \in C^1([0,T],\mathbb{H}^\beta(\Omega))$ be a solution of inequation 4.2. By Remark 4.2, we get

$$\begin{aligned}
&\Big\|v(t) - \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 - \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,v(s))\,ds \\
&\quad - \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,v(s))\,ds\Big\|_{\mathbb{H}^\beta(\Omega)} \\
&\leqslant \int_0^t (t-s)^{\alpha-1}\|\mathscr{S}_{\alpha,\beta}(t-s)\|_{\mathbb{H}^\beta(\Omega)}\varphi(s)\,ds \leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)\,ds \leqslant \eta\varphi(t).
\end{aligned}$$

Letting us be denoted by $u \in C([0,T],\mathbb{H}^\beta(\Omega))$, the unique mild solution of problem (4.4), we have

$$\begin{aligned}
u(t) =& \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}v_0 + \sum_{k=1}^{p} c_k \mathscr{T}_{\alpha,\beta}(t)\mathscr{B}\int_0^{t_k}(t_k-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t_k-s)f(s,v(s))\,ds \\
&+ \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha,\beta}(t-s)f(s,v(s))\,ds.
\end{aligned}$$

Estimating $\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)}$ in the same manner as (4.5), from Remark 2.1 and Remark 2.2, we have

$$\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \eta\varphi(t)\left(E_\alpha\left[\frac{\sum_{k=1}^{p} c_k L}{1-\sum_{k=1}^{p} c_k} t_k^\alpha\right] + E_\alpha[Lt^\alpha]\right).$$

By Definition 4.4, then the nonlocal problem (3.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable. This completes the proof of Theorem 1.4. □

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