Journal of Nonlinear Modeling and Analysis
Volume 5, Number 1, March 2023, 1–23
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.1

Dynamical Property Analysis of a Delayed Diffusive Predator-prey Model with Fear Effect*

Xiao Zhao[1] and Ruizhi Yang[1,†]

Abstract In this paper, we study a delayed diffusive predator-prey model with fear effect and Holling II functional response. The stability of the positive equilibrium is investigated. We find that time delay can destabilize the stable equilibrium and induce Hopf bifurcation. Diffusion may lead to Turing instability and inhomogeneous periodic solutions. Through the theory of center manifold and normal form, some detailed formulas for determining the property of Hopf bifurcation are presented. Some numerical simulations are also provided.

Keywords Delay, Diffusion, Predator-prey, Turing instability, Hopf bifurcation.

MSC(2010) 35B32, 34C23.

1. Introduction

In recent years, reaction-diffusion models have been widely and profoundly applied in biomathematics. Many scholars have paid attention to them and studied their dynamics [13,14,16,25–28]. It was found that changes in population density depend not only on time, but also on space. The predator and prey are non-homogeneous in space. Thus, diffusion is a phenomenon that cannot be ignored. In [26], a cross-diffusive predator-prey model with pack predation-herd behavior was considered. Yang, Zhang and Yuan primarily investigated the Turing pattern caused by cross-diffusion. In [16], Peng, Li and Zhang studied a toxin producing phytoplankton-zooplankton system with prey-taxis. They mainly discussed prey-taxis induced Turing instability and the local existence of the nonconstant positive steady state. These papers show that diffusion may lead to Turing pattern and spatial inhomogeneous periodic oscillation, which are worth investigating. Motivated by them, we also introduce diffusion terms into our model.

The purpose of this article is to investigate the stability of the positive equilibrium, Turing instability and Hopf bifurcation of the new system. This paper is organized as follows. In Section 2, we give a detailed description about the formation of our model. In Section 3, we analyze the stability of the positive equilibrium, Turing instability and the existence of Hopf bifurcation. In Section 4, we study the

†The corresponding author.
Email address: 2504342344@qq.com (X. Zhao), yangruizhi529@163.com (R. Yang)
[1]Department of Mathematics, Northeast Forestry University, Harbin, Heilongjiang 150040, China
*The authors were supported by the Fundamental Research Funds for the Central Universities (Grant No. 2572019BC01) and the Postdoctoral Program of Heilongjiang Province (Grant No. LBH-Q21060).

property of Hopf bifurcation. In Section 5, some numerical simulations are shown. Finally, a conclusion is presented in Section 6.

2. Model formulation

In this section, we will introduce the process of model formulation. Let us lay the foundation for investigating the global dynamics of the system.

In the current study, a logistic equation is often used to model the growth of the prey population in the absence of a predator. Suppose that the functional response of predator-prey interaction is Holling type II. Then, the population densities of prey and predator at the time T are denoted as X and Y respectively. First, consider a two-dimensional Rosenzweig-MacArthur [18] predator-prey model with the form

$$\begin{cases} \dot{X} = R_0 X(1 - \frac{X}{K_0}) - \frac{CAXY}{B+X}, \\ \dot{Y} = \frac{AXY}{B+X} - DY. \end{cases} \tag{2.1}$$

All parameters are positive and their biological meanings are shown in Table 1.

Table 1. Biological description of parameters in the paper

Parameter	Definition	Parameter	Definition
X	Prey density	Y	Predator density
R_0	Prey intrinsic growth rate	K_0	Prey carrying capacity
A	Maximum predation rate of predator	B	Half-saturation coefficient of predator
C^{-1}	Conversion efficiency of predator	D	Natural mortality rate of predator
E	Density restriction of predator	K	Fear parameter

With the development of the biomathematics, various predator-prey models have been studied in [2, 8, 9, 11, 23, 24]. They found that the fear effect plays a crucial role in the ecosystem. The physiological feature or behaviors of the prey population may change due to the fear of predators, including the alteration of foraging [1], breeding [4], inhabiting [17] and so on, which further affect their growth rate. Then, consider the modified growth rate of the prey $\frac{R_0}{1+KY}$. Moreover, the prey needs some time to evaluate the predation risk for the perception of the dangers, and then it makes the above changes. Hence, the fear effect does not reduce the growth of the prey population instantaneously, but it needs time delay. Based on (2.1), Panday et al. [15] proposed the following model and studied the permanence, local and global stability, as well as Hopf bifurcation of this delayed differential equation

$$\begin{cases} \dot{X} = \frac{R_0}{1+KY(T-T_1)} X(1 - \frac{X}{K_0}) - \frac{CAXY}{B+X}, \\ \dot{Y} = \frac{AXY}{B+X} - DY. \end{cases} \tag{2.2}$$

As a matter of fact, many factors may affect the dynamics of a system such as time delay, diffusion terms, density restriction and competition [3,6,7,10,20,21]. It is well-known that when predator population becomes too large, a density restriction may exist due to the intraspecific competition denoted as E. After introducing diffusion terms, time delay and density restriction term to model (2.2), we obtain

$$\begin{cases} \frac{\partial X}{\partial T} = D_1\Delta X + \frac{R_0}{1+KY(T-T_1)}X(1-\frac{X}{K_0}) - \frac{CAXY}{B+X}, \\ \frac{\partial Y}{\partial T} = D_2\Delta Y + \frac{AXY}{B+X} - DY - EY^2. \end{cases} \tag{2.3}$$

Here, D_1 and D_2 are the diffusion coefficients of prey and predator respectively. This model is closer to reality and better reflects natural regulation. For simplicity, the non-dimensionalized model (2.3) is obtained by using the transformations: $u = \frac{X}{K_0}$, $v = \frac{CY}{K_0}$ and $t = R_0T$. Then, we obtain the following model with the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1\Delta u + u(\frac{1-u}{1+kv(t-\tau)} - \frac{\alpha v}{1+\beta u}), & x \in (0, l\pi),\ t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2\Delta v + v(\frac{\alpha u}{1+\beta u} - \gamma v - \delta), & x \in (0, l\pi),\ t > 0, \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi,t) = v_x(l\pi,t) = 0,\ t > 0, & \\ u(x,\theta) = u_0(x,\theta) \geq 0, v(x,\theta) = v_0(x,\theta) \geq 0, & x \in [0, l\pi], \theta \in [-\tau, 0], \end{cases} \tag{2.4}$$

where

$$\alpha = \frac{AK_0}{R_0B},\ \beta = \frac{K_0}{B},\ \gamma = \frac{EK_0}{CR_0},\ \delta = \frac{D}{R_0},\ k = \frac{KK_0}{C},\ d_1 = \frac{D_1}{R_0},\ d_2 = \frac{D_2}{CR_0},$$

and τ is the time delay. d_1 and d_2 are still the diffusion coefficients of prey and predator. All parameters involved in our model are non-negative.

3. Analysis of the characteristic equations

Here, we only consider the positive equilibria of system (2.4), which are the positive roots of

$$\begin{cases} u(\frac{1-u}{1+kv} - \frac{\alpha v}{1+\beta u}) = 0, \\ v(\frac{\alpha u}{1+\beta u} - \gamma v - \delta) = 0. \end{cases} \tag{3.1}$$

By direct calculation, it is easy to get

$$v = \frac{(\alpha - \beta\delta)u - \delta}{(1+\beta u)\gamma}.$$

Hence, when $\frac{\delta}{\alpha-\beta\delta} < u < 1$, $\alpha > \beta\delta$, $v > 0$ holds. From equation (3.1), we get

$$G(u) = s_1u^4 + s_2u^3 + s_3u^2 + s_4u + s_5 = 0, \tag{3.2}$$

where

$$\begin{aligned} s_1 &= -\beta^3\gamma^2, \\ s_2 &= \beta^2\gamma^2(\beta - 3), \\ s_3 &= \alpha^2\beta(2k\delta - \gamma) + \alpha\beta^2\delta(\gamma - k\delta) + 3\beta\gamma^2(\beta - 1) - \alpha^3k, \\ s_4 &= \alpha^2(2k\delta - \gamma) + \gamma^2(3\beta - 1) + 2\alpha\beta\delta(\gamma - k\delta), \\ s_5 &= \gamma^2 + \alpha\gamma\delta - k\alpha\delta^2. \end{aligned}$$

After that, $G(\frac{\delta}{\alpha-\beta\delta}) = \frac{\alpha^3\gamma^2(\alpha-\beta\delta-\delta)}{(\alpha-\beta\delta)^4}$ can be obtained. Then, we derive the following lemma.

Lemma 3.1. *For system* (2.4)*, when* $u \in [\frac{\delta}{\alpha-\beta\delta}, 1]$*, the following statements are true.*

(i) *(0,0) and (1,0) are two boundary equilibria.*

(ii) *If* $\delta < \frac{\alpha}{\beta+1}$*, then there is at least one positive equilibrium.*

(iii) *If* $\beta \leq 1$ *and* $\delta < \frac{\alpha}{\beta+1}$*, then there exists one positive equilibrium.*

(iv) *If* $\beta > 1$ *and* $\frac{\alpha(\beta-1)}{\beta(\beta+1)} < \delta < \frac{\alpha}{\beta+1}$ *, then there exists one positive equilibrium.*

(v) *If* $\beta > 1$, $\delta < \frac{\alpha(\beta-1)}{\beta(\beta+1)}$ *and* $k > \frac{(\alpha-\beta\delta)^2(\alpha(\beta-1)-\beta(\beta+1)\delta)\big(\alpha(\beta-1+\beta\gamma+\beta^2\gamma)-\beta(\beta+1)\delta\big)}{\alpha^2\beta^2(1+\beta)^2\gamma^2(\alpha-\beta\delta-\delta)}$, *then system* (2.4) *has one positive equilibrium.*

Proof. Suppose $u \in [\frac{\delta}{\alpha-\beta\delta}, 1]$. Through the first equation of (3.1), we define

$$M_1(u) = v = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 - 4\alpha k(\beta u^2 + (1-\beta)u - 1)}\right), \tag{3.3}$$

$$M_2(u) = v = \frac{1}{2}\left(-\alpha - \sqrt{\alpha^2 - 4\alpha k(\beta u^2 + (1-\beta)u - 1)}\right). \tag{3.4}$$

Through the second equation of (3.1), we let

$$J(u) = v = \frac{(\alpha - \beta\delta)u - \delta}{\gamma(1+\beta u)}. \tag{3.5}$$

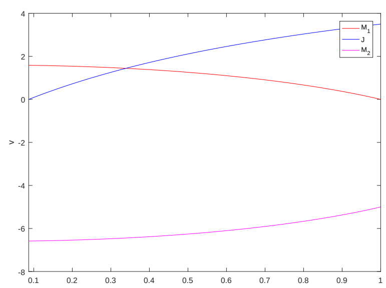

Figure 1. The graphs of $M_1(u)$, $M_2(u)$ and $J(u)$ for the values $\alpha = 5, \beta = 1, \gamma = 0.6, \delta = 0.4, k = 1$, and $\frac{\delta}{\alpha-\beta\delta} \approx 0.0869$, $u \in [0.0869, 1]$

Then, the positive equilibrium (u_0, v_0) is the intersection point between the curves $M_1(u)$ and $J(u)$, or $M_2(u)$ and $J(u)$ in $u-v$ plan. Let $\alpha > \beta\delta + \delta$. It is easy to show that

$$\begin{aligned}
&M_1\left(\frac{\delta}{\alpha-\beta\delta}\right) = \frac{1}{2}\left(-\alpha + \alpha\sqrt{1 + \frac{4k(\alpha-\beta\delta-\delta)}{(\alpha-\beta\delta)^2}}\right) > 0, \quad M_1(1) = 0,\\
&M_2\left(\frac{\delta}{\alpha-\beta\delta}\right) = \frac{1}{2}\left(-\alpha - \alpha\sqrt{1 + \frac{4k(\alpha-\beta\delta-\delta)}{(\alpha-\beta\delta)^2}}\right) < 0, \quad M_2(1) = -\alpha < 0,\\
&J\left(\frac{\delta}{\alpha-\beta\delta}\right) = 0, \quad J(1) = \frac{\alpha-\beta\delta-\delta}{\gamma(1+\beta)} > 0, \quad J\left(\frac{\beta-1}{2\beta}\right) = \frac{\alpha\beta - \alpha - \beta\delta - \beta^2\delta}{\beta\gamma(1+\beta)}.
\end{aligned} \tag{3.6}$$

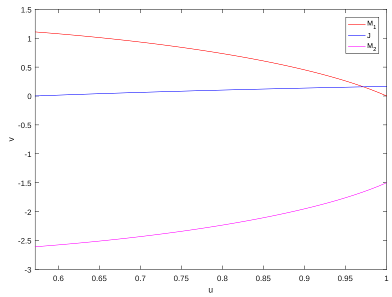

Figure 2. The graphs of $M_1(u)$, $M_2(u)$ and $J(u)$ for the values $\alpha = 1.5, \beta = 2, \gamma = 0.6, \delta = 0.4, k = 2.1$, and $\frac{\delta}{\alpha-\beta\delta} \approx 0.5714$, $u \in [0.5714, 1]$

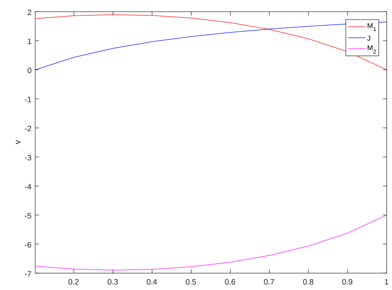

Figure 3. The graphs of $M_1(u)$, $M_2(u)$ and $J(u)$ for the values $\alpha = 5, \beta = 2.6, \gamma = 0.6, \delta = 0.4, k = 2.1$, and $\frac{\delta}{\alpha-\beta\delta} \approx 0.1010$, $u \in [0.1010, 1]$

Moreover,

$$
\begin{aligned}
M_1'(u) &= -\frac{k\alpha(1-\beta+2u\beta)}{\sqrt{\alpha^2-4k\alpha(\beta u^2+(1-\beta)u-1)}},\\
M_2'(u) &= \frac{k\alpha(1-\beta+2u\beta)}{\sqrt{\alpha^2-4k\alpha(\beta u^2+(1-\beta)u-1)}},\\
J'(u) &= \frac{\alpha}{\gamma(1+\beta u)^2} > 0.
\end{aligned}
\tag{3.7}
$$

Case 1. When $\beta \leq 1$, we have $M_1'(u) < 0$ and $M_2(u)' > 0$. Hence, $M_1(u)$ is monotonically decreasing, and $M_2(u)$ is monotonically increasing in $[\frac{\delta}{\alpha-\beta\delta}, 1]$. For $M_1(\frac{\delta}{\alpha-\beta\delta}) > J(\frac{\delta}{\alpha-\beta\delta})$, we know that the curves $M_1(u)$ and $J(u)$ will intersect at (u_0, v_0). Due to $M_2(1) < J(\frac{\delta}{\alpha-\beta\delta})$, then the curves $M_2(u)$ and $J(u)$ will not intersect in $[\frac{\delta}{\alpha-\beta\delta}, 1]$. Thus, there exists a unique positive equilibrium. The numerical simulation is shown in Figure 1, which completes the proof.

Case 2. When $\beta > 1$ and $\frac{\delta}{\alpha-\beta\delta} > \frac{\beta-1}{2\beta}$, from $M_1'(u) = 0$, we get $u_c = \frac{\beta-1}{2\beta}$. Then, $M_1(u)$ is monotonically decreasing in $[\frac{\delta}{\alpha-\beta\delta}, 1]$. Similar to Case 1, there exists a unique positive equilibrium (see Figure 2).

Case 3. When $\beta > 1$ and $\frac{\delta}{\alpha-\beta\delta} < \frac{\beta-1}{2\beta}$, then $M_1(u)$ is monotonically increasing in $[\frac{\delta}{\alpha-\beta\delta}, \frac{\beta-1}{2\beta}]$, and is monotonically decreasing in $[\frac{\beta-1}{2\beta}, 1]$, and the monotonicity of $M_2(u)$ is opposite to $M_1(u)$. $J(u)$ is monotonically increasing in $[\frac{\beta-1}{2\beta},1]$. Under the condition of Lemma 3.1(v), we have $J(\frac{\beta-1}{2\beta}) < M_1(\frac{\delta}{\alpha-\beta\delta})$. For $M_1(\frac{\delta}{\alpha-\beta\delta}) > J(\frac{\delta}{\alpha-\beta\delta})$, $M_1(1) < J_1(1)$ and the values of $M_2(u)$ are always less than $J(u)$. Then, there is only one intersection point between the curves $M_1(u)$ and $J(u)$ for $u \in [\frac{\beta-1}{2\beta},1]$. Thus, system (2.4) has only one unique positive equilibrium (see Figure 3). □

Next, we are going to discuss the stability of the positive equilibrium $P(u_0,v_0)$. Denote

$$u_1(t)=u(\cdot,t),\quad u_2(t)=v(\cdot,t),\quad U=(u_1,u_2)^T,$$
$$X=C([0,l\pi],\mathbb{R}^2),\quad \text{and}\quad \mathscr{C}_\tau:=C([-\tau,0],X).$$

Linearizing system (2.4) at $P=(u_0,v_0)$, we have

$$\dot{U}=\mathbb{D}\Delta U(t)+L(U_t). \tag{3.8}$$

Here,

$$\mathbb{D}=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},\quad \operatorname{dom}(\mathbb{D}\Delta)=\{(u,v)^T: u,v\in C^2([0,l\pi],\mathbb{R}^2), u_x,v_x=0, x=0,l\pi\},$$

and $L:\mathscr{C}_\tau\mapsto X$ is defined by

$$L(\phi_t)=L_1\phi(0)+L_2\phi(-\tau).$$

For $\phi=(\phi_1,\phi_2)^T\in\mathscr{C}_\tau$, with

$$L_1=\begin{pmatrix} a_1 & a_2\\ b_1 & b_2\end{pmatrix},\qquad L_2=\begin{pmatrix} 0 & c\\ 0 & 0\end{pmatrix},$$

$$\begin{aligned}
&\phi(t)=(\phi_1(t),\phi_2(t))^T,\qquad \phi(t)(\cdot)=(\phi_1(t+\cdot),\phi_2(t+\cdot))^T.\\
&a_1:=\frac{1-2u_0}{1+kv_0}-\frac{\alpha v_0}{(1+\beta u_0)^2},\quad a_2:=\frac{-\alpha u_0}{1+\beta u_0}<0,\\
&b_1:=\frac{\alpha v_0}{(1+\beta u_0)^2}>0,\quad b_2:=-\gamma v_0<0,\quad c:=\frac{ku_0(u_0-1)}{(1+kv_0)^2}.
\end{aligned} \tag{3.9}$$

From Wu [22], we know that the characteristic equation of linear system (3.8) is

$$\lambda y-d\Delta y-L(e^{\lambda}y)=0,\quad y\in\operatorname{dom}(d\Delta),\quad y\neq 0. \tag{3.10}$$

It is obvious that the eigenvalue problem

$$-\varphi''=\mu\varphi,\quad x\in(0,l\pi);\quad \varphi'(0)=\varphi'(l\pi)=0$$

has eigenvalues $\mu_n=n^2/l^2$ $(n=0,1,\cdots)$ with the corresponding eigenfunction

$$\varphi_n(x)=\cos\frac{n\pi}{l},\quad n=0,1,\cdots.$$

Substituting

$$y=\sum_{n=0}^{\infty}\begin{pmatrix} y_{1n}\\ y_{2n}\end{pmatrix}\cos\frac{n\pi}{l}$$

into the characteristic equation (3.10), it follows that

$$\begin{pmatrix} a_1 - \frac{d_1 n^2}{l^2} & a_2 + ce^{-\lambda\tau} \\ b_1 & b_2 - \frac{d_2 n^2}{l^2} \end{pmatrix} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} = \lambda \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}, \ n = 0, 1, \cdots .$$

Hence, the characteristic equation (3.10) is equivalent to

$$\Delta_n(\lambda, \tau) = \lambda^2 + \lambda A_n + B_n - b_1 ce^{-\lambda\tau} = 0, \tag{3.11}$$

where

$$A_n = (d_1+d_2)\frac{n^2}{l^2} - a_1 - b_2,\ B_n = d_1 d_2 \frac{n^4}{l^4} - (d_1 b_2 + d_2 a_1)\frac{n^2}{l^2} + a_1 b_2 - a_2 b_1,\ C_n = b_1 c.$$

Then, we make the following hypothesis

$$(\boldsymbol{H}) \quad a_1 + b_2 < 0, \qquad a_1 b_2 - a_2 b_1 - b_1 c > 0. \tag{3.12}$$

3.1. Non-delay model

When $\tau = 0$, the characteristic equation becomes

$$\lambda^2 - T_n \lambda + D_n = 0, n \in \mathcal{N}_0, \tag{3.13}$$

where

$$\begin{cases} T_n = a_1 + b_2 - (d_1 + d_2)\frac{n^2}{l^2}, \\ D_n = d_1 d_2 \frac{n^4}{l^4} - (d_1 b_2 + a_1 d_2)\frac{n^2}{l^2} + a_1 b_2 - a_2 b_1 - b_1 c, \end{cases}$$

and the eigenvalues are given by

$$\lambda_i^n(r) = \frac{T_n \pm \sqrt{T_n^2 - 4D_n}}{2}, \quad n \in \mathcal{N}_0. \tag{3.14}$$

Define the parameters

$$q = \frac{d_1 b_2 + d_2 a_1}{2 d_1 d_2}, m = (d_1 b_2 + d_2 a_1)^2 - 4 d_1 d_2 (a_1 b_2 - a_2 b_1 - b_1 c), p_\pm = q \pm \frac{\sqrt{m}}{2 d_1 d_2}.$$

Theorem 3.1. *Suppose that $d_1 = d_2 = 0$, $\tau = 0$ and ($\boldsymbol{H}$) hold. Then, the equilibrium (u_0, v_0) is locally asymptotically stable.*

Theorem 3.2. *Suppose that $d_1 > 0$, $d_2 > 0$, $\tau = 0$ and ($\boldsymbol{H}$) hold. Then, for model* (2.4)*, the following statements are true.*

(i) *If $q \le 0$, then the equilibrium (u_0, v_0) is locally asymptotically stable.*

(ii) *If $q > 0$, $m < 0$, then the equilibrium (u_0, v_0) is locally asymptotically stable.*

(iii) *If $q > 0$, $m > 0$, and there is no $k \in \mathcal{N}$ such that $\frac{n^2}{l^2} \in (p_-, p_+)$, then the equilibrium (u_0, v_0) is locally asymptotically stable.*

(iv) *If $q > 0$, $m > 0$, and there is a $k \in \mathcal{N}$ such that $\frac{n^2}{l^2} \in (p_-, p_+)$, then the equilibrium (u_0, v_0) is Turing unstable.*

Proof. By calculation, we have $T_n < 0$ and $D_n > 0$, for $q \leq 0$. Therefore, all eigenvalues have negative real parts. Then, the equilibrium (u_0, v_0) is locally asymptotically stable (statement (i) is true). Similarly, statements (i)-(iii) are also true. If the conditions in statement (iv) hold, then there exists at least one eigenvalue root with positive real part. Then, the equilibrium (u_0, v_0) is Turing unstable. □

From the above analysis, we have found that β is related to the stability of the positive equilibrium. Diffusion can induce Turing instability, which further affects the dynamics of the system. Thus, we choose β as the bifurcation parameter and find the propers d_1 and d_2 to investigate our system. Fix the following parameters

$$d_1 = 0.001, \ d_2 = 5, \ \alpha = 5, \ \beta = 2, \ \gamma = 0.6, \ \delta = 0.4, \ k = 2.1. \tag{3.15}$$

Then, the equilibrium is $P(u_0, v_0) \approx (0.12, 0.16)$, and $(\boldsymbol{H})$ is satisfied. If $\beta = 2$, we know that $P(u_0, v_0)$ is Turing unstable (shown in Figure 4). Biologically, our results indicate that the diffusion terms and half-saturation constant will break the equilibrium state and lead to a spatially inhomogeneous population distribution.

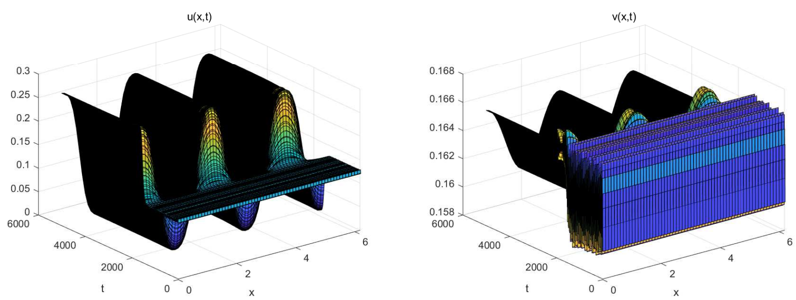

Figure 4. The numerical simulations of system (2.4) with $\tau = 0$ and the initial condition at $(0.12, 0.16)$. Left: component u (stable); Right: component v (stable)

3.2. Delay model

Assume that one of conditions (i-iii) in Theorem 3.2 and $(\boldsymbol{H})$ hold. Thus, we obtain $\Delta_n(0, \tau) = B_n - C_n = D_n > 0$. Then, we have the following conclusion.

Lemma 3.2. *Suppose that one of conditions (i-iii) in Theorem 3.2 and* $(\boldsymbol{H})$ *hold. Then,* $\lambda = 0$ *is not a root of equation* (3.11) *for any* $n \in \mathbb{N}_0$.

Our purpose is to find the critical value of τ, so that there is a pair of simple purely imaginary eigenvalues. Suppose that $i\omega(\omega > 0)$ is a root of (3.11), we have

$$-\omega^2 + i\omega A_n + B_n - C_n(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Then, we obtain

$$\begin{cases} -\omega^2 + B_n - C_n \cos\omega\tau = 0, \\ \omega A_n + C_n \sin\omega\tau = 0, \end{cases}$$

which leads to

$$\omega^4 + \omega^2(A_n^2 - 2B_n) + B_n^2 - C_n^2 = 0. \tag{3.16}$$

Letting $z = \omega^2$, then (3.16) becomes

$$z^2 + z(A_n^2 - 2B_n) + B_n^2 - C_n^2 = 0, \tag{3.17}$$

which has the roots

$$z_\pm = \frac{1}{2}\left[-(A_n^2 - 2B_n) \pm \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C_n^2)}\right].$$

Provided that one of conditions (i-iii) in Theorem 3.2 and ($\boldsymbol{H}$) hold, we have

$$A_n^2 - 2B_n = \left(d_1\frac{n^2}{l^2} - a_1\right)^2 + \left(d_2\frac{n^2}{l^2} - b_2\right)^2 + 2a_2b_1 > 0,$$

$$B_n - C_n = D_n > 0,$$

$$B_n + C_n = d_1d_2\frac{n^4}{l^4} - (d_1b_2 + d_2a_1)\frac{n^2}{l^2} + a_1b_2 - a_2b_1 + b_1c.$$

Define

$$\mathbb{S} = \{n \in \mathbb{N}_0 | \textit{equation } (3.17) \textit{ has positive roots}\}. \tag{3.18}$$

Then, the following lemma holds.

Lemma 3.3. *Suppose that one of conditions (i-iii) in Theorem 3.2, ($\boldsymbol{H}$) and $\mathbb{S} \neq \emptyset$ hold. Then, (3.11) has a pair of purely imaginary roots $\pm i\omega_n$ $(n \in \mathbb{S})$ at*

$$\tau_n^j = \tau_n^0 + \frac{2j\pi}{\omega_n},\ j = 0, 1, 2, \cdots, \tag{3.19}$$

where

$$\tau_n^0 = \begin{cases} \frac{1}{\omega_n^\pm}\arccos(V_{\cos}), & V_{\sin} \geq 0; \\ \frac{1}{\omega_n^\pm}[2\pi - \arccos(V_{\cos})], & V_{\sin} < 0. \end{cases} \tag{3.20}$$

$$V_{\cos} = \frac{B_n - \omega_n^2}{C_n}, \quad V_{\sin} = \frac{\sqrt{C_n^2 - (B_n - \omega^2)^2}}{C_n} \tag{3.21}$$

and

$$\omega_n = \sqrt{\frac{1}{2}\left[-(A_n^2 - 2B_n) + \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C_n^2)}\right]}. \tag{3.22}$$

Assume that $\lambda_n(\tau) = \alpha_n(\tau) + i\omega_n(\tau)$ is the root of (3.11), which satisfies $\alpha_n(\tau_n^j) = 0$ and $\omega_n(\tau_n^j) = \omega_n$, when τ is close to τ_n^j. Then, we calculate the transversality condition.

Lemma 3.4. *Suppose that one of conditions (i-iii) in Theorem 3.2 and ($\boldsymbol{H}$) hold. Then,*

$$\alpha_n'(\tau_n^j) = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\bigg|_{\tau=\tau_n^j} > 0 \quad \textit{for} \quad n \in \mathbb{S} \quad \textit{and} \quad j \in \mathbb{N}_0.$$

Proof. Differentiating two sides of (3.11) with respect τ, we obtain

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = -\frac{2\lambda + A_n + \tau C_n e^{-\lambda\tau}}{\lambda C_n e^{-\lambda\tau}}.$$

After that,

$$Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_n^j} = \frac{A_n^2 - 2B_n + 2\omega_n^2}{C_n^2} = \frac{\sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C_n^2)}}{C_n^2} > 0.$$

Thus, $\alpha_n'(\tau_n^j) > 0$. □

Find $\tau_m^j = \tau_n^k$, for some $m \neq n$ may occur. In this paper, we do not consider this case. Namely, we consider

$$\tau \in \mathcal{D} := \{\tau_n^j : \ \tau_m^j \neq \tau_n^k, \ m \neq n, \ m, n \in \mathbb{S}, \ j, k \in \mathbb{N}_0\}.$$

Define $\tau_* = \min\{\tau \in \mathcal{D}\}$.

According to the above analysis, we have the following theorem.

Theorem 3.3. *For system* (2.4), *suppose that one of conditions (i-iii) in Theorem 3.2 and* (***H***) *hold. Then, the following statements are true.*

(1) *If $\mathbb{S} = \emptyset$, then the equilibrium $P(u_0, v_0)$ is locally asymptotically stable for $\tau \geq 0$.*

(2) *If $\mathbb{S} \neq \emptyset$, $\tau \in [0, \tau_*)$, then the equilibrium $P(u_0, v_0)$ is locally asymptotically stable, and unstable for $\tau > \tau_*$.*

(3) *$\tau = \tau_0^j$ ($j \in \mathbb{N}_0$) are the Hopf bifurcation values of system* (2.4), *and the bifurcating periodic solutions are spatially homogeneous, which coincide with the periodic solutions of the corresponding FDE system. When $\tau \in \mathcal{D}/\{\tau_0^k : k \in \mathbb{N}_0\}$, system* (2.4) *also undergoes a Hopf bifurcation and the bifurcating periodic solutions are spatially non-homogeneous.*

4. Direction and stability of spatial Hopf bifurcation

Next, we will investigate the property of Hopf bifurcation by the theory of center manifold and normal form [12, 19, 22]. For fixed $j \in \mathbb{N}_0$ and $n \in \mathbb{S}$, make $\tilde{\tau} = \tau_n^j$. Let $\tilde{u}(x,t) = u(x, \tau t) - u_0$, $\tilde{v}(x,t) = v(x, \tau t) - v_0$. For convenience, we drop the tilde. Then, system (2.4) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = \tau \left[d_1 \Delta u + (u + u_0) \left(\frac{1-(u+u_0)}{1+k(v(t-1)+v_0)} - \frac{\alpha(v+v_0)}{1+\beta(u+u_0)} \right) \right], \\ \frac{\partial v}{\partial t} = \tau \left[d_2 \Delta v + (v + v_0) \left(\frac{\alpha(u+u_0)}{1+\beta(u+u_0)} - \gamma(v + v_0) - \delta \right) \right]. \end{cases} \tag{4.1}$$

For $x \in (0, l\pi)$, $t > 0$, suppose

$$\tau = \tilde{\tau} + \mu, \quad u_1(t) = u(\cdot, t), \quad u_2(t) = v(\cdot, t) \ \text{ and } \ U = (u_1, u_2)^T.$$

Under the phase space $\mathscr{C}_1 := C([-1, 0], X)$, (4.1) can be written as

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t, \mu), \tag{4.2}$$

where

$$L_\mu(\phi) = \mu \begin{pmatrix} a_1\phi_1(0) + a_2\phi_2(0) + c\phi_2(-1) \\ b_1\phi_1(0) + b_2\phi_2(0) \end{pmatrix}, \tag{4.3}$$

$$F(\phi, \mu) = \mu D\Delta\phi + L_\mu(\phi) + f(\phi, \mu), \tag{4.4}$$

with

$$\begin{aligned}
f(\phi, \mu) =& (\tilde{\tau} + \mu)(F_1(\phi, \mu), F_2(\phi, \mu))^T, \\
F_1(\phi, \mu) =& (\phi_1(0) + u_0)\left(\frac{1 - (\phi_1(0) + u_0)}{1 + k(\phi_2(-1) + v_0)} - \frac{\alpha(\phi_2(0) + v_0)}{1 + \beta(\phi_1(0) + u_0)}\right) \\
& - a_1\phi_1(0) - a_2\phi_2(0) - c\phi_2(-1), \\
F_2(\phi, \mu) =& (\phi_2(0) + v_0)\left(\frac{\alpha(\phi_1(0) + u_0)}{1 + \beta(\phi_1(0) + u_0)} - \gamma(\phi_2(0) + v_0) - \delta\right) \\
& - b_1\phi_1(0) - b_2\phi_2(0)
\end{aligned}$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathscr{C}_1$.

The linear equation is

$$\frac{dU(t)}{dt} = \tilde{\tau}D\Delta U(t) + L_{\tilde{\tau}}(U_t). \tag{4.5}$$

Based on [12, 19, 22], the solution of (4.2) on the center manifold is given, and the normal form of the system (2.4) is calculated. The detailed calculation process is shown in appendix. Finally, we obtain the following quantities, which determine the direction and stability of bifurcating periodic orbits with the form

$$\begin{cases} c_1(0) = \frac{i}{2\omega_n\tilde{\tau}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{1}{2}g_{21}, \; \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_n^j))}, \\ T_2 = -\frac{1}{\omega_n\tilde{\tau}}[Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_n^j))], \qquad \beta_2 = 2Re(c_1(0)). \end{cases} \tag{4.6}$$

Theorem 4.1. *For any critical value τ_n^j, we have that*

(i) *μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ (respectively < 0), then the Hopf bifurcation is forward (respectively backward). That is, the bifurcating periodic solutions exist for $\tau > \tau_n^j$ (respectively $\tau < \tau_n^j$).*

(ii) *β_2 determines the stability of the bifurcating periodic solutions on the center manifold: if $\beta_2 < 0$ (respectively > 0), then the bifurcating periodic solutions are orbitally asymptotically stable (respectively unstable).*

(iii) *T_2 determines the period of bifurcating periodic solutions: if $T_2 > 0$ (respectively $T_2 < 0$), then the period increases (respectively decreases).*

5. Numerical simulations

In order to explore the influence of time delay, we choose time delay as the bifurcation parameter, and some numerical simulations are presented by using Matlab. Referring to [15], for system (2.4), choose the suitable parameters

$$d_1 = 2,\; d_2 = 2,\; \alpha = 5,\; \beta = 1,\; \gamma = 0.6,\; \delta = 0.4,\; k = 2.1. \tag{5.1}$$

Then, we know that $u_0 \approx 0.11$, $v_0 \approx 0.15$, $\mathbb{S} = \{0\} \neq \emptyset$ and $(\boldsymbol{H})$ hold. From (3.20) and (3.22), we obtain $\tau_* = \tau_0^0 \approx 1.7455$ and $\omega_0 \approx 0.5839$. By Theorem 3.3, we find that the equilibrium $P(u_0, v_0)$ is locally asymptotically stable, when $\tau \in [0, \tau_*)$, which is shown in Figure 5. Then, we choose $\tau = 1$, and the initial condition is

$(0.11, 0.15)$. At this moment, the predator and prey can coexist. They tend to the interior equilibrium $P(0.11, 0.15)$ over time. Then, we increase the delay parameter to investigate the rich dynamics of model (2.4). By Theorem 3.3 (iii), we know that the equilibrium $P(u_0, v_0)$ will lose its stability, and Hopf bifurcation will occur, if τ crosses τ_0^0. Then, the predator and prey may show oscillatory behavior. By Theorem 4.1,

$$\mu_2 \approx 3237.47 > 0, \quad \beta_2 \approx -117.955 < 0 \quad \text{and} \quad T_2 \approx 121.168 > 0.$$

Therefore, the direction of the bifurcation is forward, and the bifurcating period solutions are locally asymptotically stable. Moreover, the period of bifurcating periodic solutions increases, which is shown in Figure 6. Then, we choose $\tau = 2$, and the initial condition is $(0.11, 0.15)$. In this case, the predator and prey can also coexist. However, they coexist in the way of periodic oscillation. Hence, we confirm that time delay has an impact on the stability of the positive equilibrium. A large delay can cause periodic oscillation in a delayed diffusive predator-prey system with fear effect.

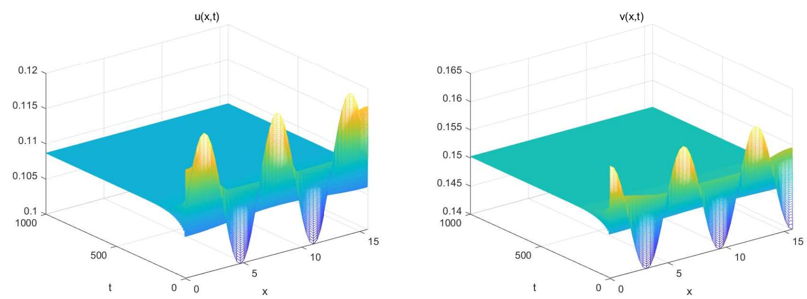

Figure 5. The numerical simulations of system (2.4) with $\tau = 1$ and the initial condition at $(0.11 - 0.01\sin(x), 0.15 - 0.01\cos(x))$. Left: component u (locally asymptotically stable); Right: component v (locally asymptotically stable)

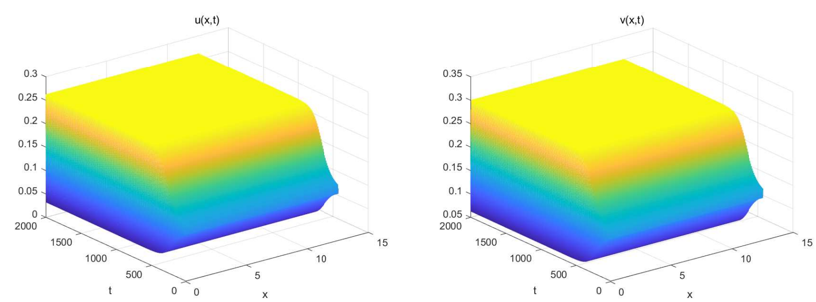

Figure 6. The numerical simulations of system (2.4) with $\tau = 2$ and the initial condition at $(0.11 - 0.01\sin(x), 0.15 - 0.01\cos(x))$. Left: component u (stable); Right: component v (stable)

6. Conclusion

Predator-prey model has always been a hot topic in biomathematics. Fear effect is widespread in nature. Rich dynamic behaviors appear among prey population due to the scarcity of predators. Diffusion may result in Turing instability and the non-homogeneous bifurcating periodic solutions. Hence, in this paper, we analyze a predator-prey model with diffusion terms, density restriction and fear induced time delay.

First, the existence and stability of the positive equilibrium can be known through the characteristic equation. The conditions which lead to the Turing instability are also derived. Second, by studying the delay model and regarding time delay as the bifurcation parameter, we get the existence conditions of Hopf bifurcation based on the Hopf bifurcation theory. Next, by the theory of center manifold and normal form, we obtain some formulas to decide the direction of bifurcation and the stability of the bifurcating periodic solutions. In fact, Hopf bifurcation occurs, when time delay is larger than the critical value. At this moment, predators and prey coexist in the way of periodic oscillations. Finally, some numerical simulations are given to illustrate the above conclusions.

7. Appendix

Here, we will investigate the property of Hopf bifurcation by the theory of center manifold and normal form [12,19,22]. For fixed $j \in \mathbb{N}_0$ and $n \in \mathbb{S}$, we denote $\tilde{\tau} = \tau_n^j$. Let $\tilde{u}(x,t) = u(x,\tau t) - u_0$, $\tilde{v}(x,t) = v(x,\tau t) - v_0$. For convenience, we drop the tilde. Then, the system (2.4) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau\left[d_1\Delta u + (u+u_0)\left(\dfrac{1-(u+u_0)}{1+k(v(t-1)+v_0)} - \dfrac{\alpha(v+v_0)}{1+\beta(u+u_0)}\right)\right], \\ \dfrac{\partial v}{\partial t} = \tau\left[d_2\Delta v + (v+v_0)\left(\dfrac{\alpha(u+u_0)}{1+\beta(u+u_0)} - \gamma(v+v_0) - \delta\right)\right], \end{cases} \tag{7.1}$$

for $x \in (0, l\pi)$, $t > 0$. Suppose

$$\tau = \tilde{\tau} + \mu, \quad u_1(t) = u(\cdot,t), \quad u_2(t) = v(\cdot,t) \quad \text{and} \quad U = (u_1, u_2)^T.$$

Under the phase space $\mathscr{C}_1 := C([-1,0],X)$, (7.1) can be written as

$$\frac{dU(t)}{dt} = \tilde{\tau}D\Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t,\mu), \tag{7.2}$$

where

$$L_\mu(\phi) = \mu \begin{pmatrix} a_1\phi_1(0) + a_2\phi_2(0) + c\phi_2(-1) \\ b_1\phi_1(0) + b_2\phi_2(0) \end{pmatrix}, \tag{7.3}$$

$$F(\phi,\mu) = \mu D\Delta\phi + L_\mu(\phi) + f(\phi,\mu), \tag{7.4}$$

with

$$\begin{aligned} f(\phi,\mu) =& (\tilde{\tau}+\mu)(F_1(\phi,\mu), F_2(\phi,\mu))^T, \\ F_1(\phi,\mu) =& (\phi_1(0)+u_0)\left(\frac{1-(\phi_1(0)+u_0)}{1+k(\phi_2(-1)+v_0)} - \frac{\alpha(\phi_2(0)+v_0)}{1+\beta(\phi_1(0)+u_0)}\right) \\ & - a_1\phi_1(0) - a_2\phi_2(0) - c\phi_2(-1), \\ F_2(\phi,\mu) =& (\phi_2(0)+v_0)\left(\frac{\alpha(\phi_1(0)+u_0)}{1+\beta(\phi_1(0)+u_0)} - \gamma(\phi_2(0)+v_0) - \delta\right) \\ & - b_1\phi_1(0) - b_2\phi_2(0) \end{aligned}$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathscr{C}_1$.

Discuss the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t). \tag{7.5}$$

From Section 3, we know that $\Lambda_n := \{i\omega_n \tilde{\tau}, -i\omega_n \tilde{\tau}\}$ are the characteristic values of

$$\frac{dz(t)}{dt} = -\tilde{\tau} D \frac{n^2}{l^2} z(t) + L_{\tilde{\tau}}(z_t). \tag{7.6}$$

By Riesz representation theorem, there is a 2×2 matrix function $\eta^n(\sigma, \tilde{\tau})$ $-1 \leq \sigma \leq 0$, whose elements are of bounded variation functions such that

$$-\tilde{\tau} D \frac{n^2}{l^2} \phi(0) + L_{\tilde{\tau}}(\phi) = \int_{-1}^{0} d\eta^n(\sigma, \tau)\phi(\sigma),$$

for $\phi \in C([-1,0], \mathbb{R}^2)$.

Choose

$$\eta^n(\sigma, \tau) = \begin{cases} \tau E, & \sigma = 0, \\ 0, & \sigma \in (-1, 0), \\ -\tau F, & \sigma = -1, \end{cases} \tag{7.7}$$

where

$$E = \begin{pmatrix} a_1 - d_1 \frac{n^2}{l^2} & a_2 \\ b_1 & b_2 - d_2 \frac{n^2}{l^2} \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}. \tag{7.8}$$

Suppose that $A(\tilde{\tau})$ is the infinitesimal generator of semigroup included by the solutions of equation (7.6). Then, A^* is the formal adjoint of $A(\tilde{\tau})$ under the bilinear pairing

$$\begin{aligned} (\psi, \phi) &= \psi(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\sigma} \psi(\xi - \sigma) d\eta^n(\sigma, \tilde{\tau}) \phi(\xi) d\xi \\ &= \psi(0)\phi(0) + \tilde{\tau} \int_{-1}^{0} \psi(\xi + 1) F \phi(\xi) d\xi, \end{aligned} \tag{7.9}$$

for $\phi \in C([-1,0], \mathbb{R}^2)$, $\psi \in C([-1,0], \mathbb{R}^2)$. $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_n \tilde{\tau}$, and they are also the eigenvalues of A^*. Let P and P^* be the center subspace. That is, the generalized eigenspaces of $A(\tilde{\tau})$ and A^* are associated with Λ_n respectively. Then, P^* is the adjoint space of P and $\dim P = \dim P^* = 2$.

We can prove that $p_1(\sigma) = (1, \xi)^T e^{i\omega_n \tilde{\tau} \sigma}$ $(\sigma \in [-1, 0])$, $p_2(\sigma) = \overline{p_1(\sigma)}$ are the basis of $A(\tilde{\tau})$ with Λ_n, and $q_1(r) = (1, \eta) e^{-i\omega_n \tilde{\tau} r}$ $(r \in [0, 1])$, $q_2(r) = \overline{q_1(r)}$ are the basis of A^* with Λ_n, where

$$\xi = \frac{b_1}{i\omega_n - b_2 + d_2 \frac{n^2}{l^2}}, \qquad \eta = \frac{-i\omega_n + d_1 \frac{n^2}{l^2} - a_1}{b_1}.$$

Letting $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with

$$\Phi_1(\sigma) = \frac{p_1(\sigma) + p_2(\sigma)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{i\omega_n \tilde{\tau}\sigma}\right) \\ \mathrm{Re}\left(\xi \mathrm{e}^{i\omega_n \tilde{\tau}\sigma}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega_n \tilde{\tau}\sigma) \\ \frac{b_1}{(\frac{d_2 n^2}{l^2} - b_2)^2 + \omega_n^2}((\frac{d_2 n^2}{l^2} - b_2)\cos\sigma\tilde{\tau}\omega_n + \omega_n \sin\sigma\tilde{\tau}\omega_n) \end{pmatrix},$$

$$\Phi_2(\sigma) = \frac{p_1(\sigma) - p_2(\sigma)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{i\omega_n \tilde{\tau}\sigma}\right) \\ \mathrm{Im}\left(\xi \mathrm{e}^{i\omega_n \tilde{\tau}\sigma}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sin(\omega_n \tilde{\tau}\sigma) \\ \frac{b_1}{(\frac{d_2 n^2}{l^2} - b_2)^2 + \omega_n^2}(-\omega_n \cos\sigma\tilde{\tau}\omega_n + (\frac{d_2 n^2}{l^2} - b_2)\sin\sigma\tilde{\tau}\omega_n) \end{pmatrix},$$

for $\theta \in [-1, 0]$, and

$$\Psi_1^*(r) = \frac{q_1(r) + q_2(r)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{-i\omega_n \tilde{\tau}\mathrm{r}}\right) \\ \mathrm{Re}\left(\eta \mathrm{e}^{-i\omega_n \tilde{\tau}\mathrm{r}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega_n \tilde{\tau} r) \\ \frac{d_1 n^2 - a_1 l^2}{l^2 b_1} \cos r\tilde{\tau}\omega_n - \frac{\omega_n}{b_1} \sin r\tilde{\tau}\omega_n \end{pmatrix},$$

$$\Psi_2^*(r) = \frac{q_1(r) - q_2(r)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{-i\omega_n \tilde{\tau}\mathrm{r}}\right) \\ \mathrm{Im}\left(\eta \mathrm{e}^{-i\omega_n \tilde{\tau}\mathrm{r}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(\omega_n \tilde{\tau} r) \\ \frac{-\omega_n}{b_1} \cos r\tilde{\tau}\omega_n - \frac{d_1 n^2 - a_1 l^2}{l^2 b_1} \sin r\tilde{\tau}\omega_n \end{pmatrix},$$

for $r \in [0, 1]$, by (7.9), we can compute

$$D_1^* := (\Psi_1^*, \Phi_1),\ D_2^* := (\Psi_1^*, \Phi_2),\ D_3^* := (\Psi_2^*, \Phi_1),\ D_4^* := (\Psi_2^*, \Phi_2).$$

Let $(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}$ and

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1}\Psi^*.$$

Then, $(\Psi, \Phi) = I_2$. Furthermore, define $f_n := (\beta_n^1, \beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix} \cos\frac{n}{l}x \\ 0 \end{pmatrix}, \quad \beta_n^2 = \begin{pmatrix} 0 \\ \cos\frac{n}{l}x \end{pmatrix}.$$

We also define

$$c \cdot f_n = c_1\beta_n^1 + c_2\beta_n^2, \quad \text{for} \quad c = (c_1, c_2)^T \in \mathscr{C}_1.$$

Thus, the center subspace of linear equation (7.5) is given by $P_{CN}\mathscr{C}_1 \oplus P_S\mathscr{C}_1$, and $P_S\mathscr{C}_1$ represents the complement subspace of $P_{CN}\mathscr{C}_1$ in $\mathscr{C}_1$,

$$< u, v >:= \frac{1}{l\pi}\int_0^{l\pi} u_1\overline{v_1}dx + \frac{1}{l\pi}\int_0^{l\pi} u_2\overline{v_2}dx,$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u, v \in X$ and $< \phi, f_0 >= (< \phi, f_0^1 >, < \phi, f_0^2 >)^T$. Let $A_{\tilde{\tau}}$ be the infinitesimal generator of an analytic semigroup induced by the linear system (7.5). Equation (7.1) becomes

$$\frac{dU(t)}{dt} = A_{\tilde{\tau}}U_t + R(U_t, \mu), \tag{7.10}$$

where

$$R(U_t, \mu) = \begin{cases} 0, & \theta \in [-1, 0); \\ F(U_t, \mu), & \theta = 0. \end{cases} \tag{7.11}$$

Then, the solution is

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n + h(x_1, x_2, \mu), \tag{7.12}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, < U_t, f_n >)$$

and

$$h(x_1, x_2, \mu) \in P_S\mathscr{C}_1, \quad h(0, 0, 0) = 0, \quad Dh(0, 0, 0) = 0.$$

In fact, the solution of (7.2) on the center manifold is given by

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, 0). \tag{7.13}$$

Let $z = x_1 - ix_2$. Then,

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z+\overline{z}}{2} \\ \frac{i(z-\overline{z})}{2} \end{pmatrix} f_n = \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n$$

and

$$h(x_1, x_2, 0) = h\left(\frac{z + \overline{z}}{2}, \frac{i(z - \overline{z})}{2}, 0\right).$$

Hence, equation (7.13) can be written in the following form

$$\begin{aligned} U_t &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + h\left(\frac{z + \overline{z}}{2}, \frac{i(z - \overline{z})}{2}, 0\right) \\ &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + W(z, \overline{z}), \end{aligned} \tag{7.14}$$

where

$$W(z,\overline{z}) = h\left(\frac{z+\overline{z}}{2}, \frac{i(z-\overline{z})}{2}, 0\right).$$

From [22], z satisfies

$$\dot{z} = i\omega_n \tilde{\tau} z + g(z,\overline{z}), \tag{7.15}$$

where

$$g(z,\overline{z}) = (\Psi_1(0) - i\Psi_2(0)) < F(U_t, 0), f_n > . \tag{7.16}$$

Letting

$$W(z,\overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{7.17}$$

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{7.18}$$

then, we have

$$\begin{aligned}
u_t(0) =& \frac{1}{2}(z+\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
v_t(0) =& \frac{1}{2}(\xi z+\overline{\xi}\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
u_t(-1) =& \frac{1}{2}(ze^{-i\omega_n\tilde{\tau}} + \overline{z}e^{i\omega_n\tilde{\tau}})\cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} \\
&+ W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots, \\
v_t(-1) =& \frac{1}{2}(\xi ze^{-i\omega_n\tilde{\tau}} + \overline{\xi}\overline{z}e^{i\omega_n\tilde{\tau}})\cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} \\
&+ W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_1(U_t,0) = \frac{1}{\tilde{\tau}}F_1 =& \alpha_1 u_t^2(0) + \alpha_2 u_t(0)v_t(0) + \alpha_3 u_t(0)v_t(-1) + \alpha_4 v_t^2(-1) \\
&+ \alpha_5 u_t^3(0) + \alpha_6 u_t^2(0)v_t(0) + \alpha_7 u_t^2(0)v_t(-1) \\
&+ \alpha_8 u_t(0)v_t^2(-1) + \alpha_9 v_t^3(-1) + O(4),
\end{aligned} \tag{7.19}$$

$$\begin{aligned}
\overline{F}_2(U_t,0) = \frac{1}{\tilde{\tau}}F_2 =& \beta_1 u_t^2(0) + \beta_2 u_t(0)v_t(0) + \beta_3 v_t^2(0) + \beta_4 u_t^3(0) \\
&+ \beta_5 u_t^2(0)v_t(0) + O(4),
\end{aligned} \tag{7.20}$$

with

$$\begin{aligned}
&\alpha_1 = \frac{\alpha\beta v_0}{(1+\beta u_0)^3} - \frac{1}{1+kv_0}, \quad \alpha_2 = -\frac{\alpha}{(1+\beta u_0)^2}, \quad \alpha_3 = \frac{-k+2ku_0}{(1+kv_0)^2}, \\
&\alpha_4 = \frac{k^2(1-u_0)u_0}{(1+kv_0)^3}, \ \alpha_5 = -\frac{\alpha\beta^2 v_0}{(1+\beta u_0)^4}, \ \alpha_6 = \frac{\alpha\beta}{(1+\beta u_0)^3}, \ \alpha_7 = \frac{k}{(1+kv_0)^2}, \\
&\alpha_8 = \frac{k^2-2k^2u_0}{(1+kv_0)^3}, \ \alpha_9 = \frac{k^3(u_0-1)u_0}{(1+kv_0)^4}, \ \beta_1 = -\frac{\alpha\beta v_0}{(1+\beta u_0)^3}, \ \beta_2 = \frac{\alpha}{(1+\beta u_0)^2}, \\
&\beta_3 = -\gamma, \quad \beta_4 = \frac{\alpha\beta^2 v_0}{(1+\beta u_0)^4}, \quad \beta_5 = -\frac{\alpha\beta}{(1+\beta u_0)^3}.
\end{aligned}$$

Hence,

$$\begin{aligned}\overline{F}_1(U_t,0) =& \cos^2\left(\frac{nx}{l}\right)\left(\frac{z^2}{2}\chi_{20}+z\overline{z}\chi_{11}+\frac{\overline{z}^2}{2}\overline{\chi}_{20}\right)\\ &+\frac{z^2\overline{z}}{2}\cos\frac{nx}{l}\varsigma_{11}+\frac{z^2\overline{z}}{2}\cos^3\frac{nx}{l}\varsigma_{12}+\cdots,\end{aligned} \tag{7.21}$$

$$\begin{aligned}\overline{F}_2(U_t,0) =& \cos^2\left(\frac{nx}{l}\right)\left(\frac{z^2}{2}\varrho_{20}+z\overline{z}\varrho_{11}+\frac{\overline{z}^2}{2}\overline{\varrho}_{20}\right)\\ &+\frac{z^2\overline{z}}{2}\cos\frac{nx}{l}\varsigma_{21}+\frac{z^2\overline{z}}{2}\cos^3\frac{nx}{l}\varsigma_{22}+\cdots,\end{aligned} \tag{7.22}$$

$$\begin{aligned}< F(U_t,0), f_n > =& \tilde{\tau}(\overline{F}_1(U_t,0)f_n^1+\overline{F}_2(U_t,0)f_n^2)\\ =& \frac{z^2}{2}\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varrho_{20}\end{pmatrix}\Gamma+z\overline{z}\tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varrho_{11}\end{pmatrix}\Gamma\\ &+\frac{\overline{z}^2}{2}\tilde{\tau}\begin{pmatrix}\overline{\chi}_{20}\\ \overline{\varrho}_{20}\end{pmatrix}\Gamma+\frac{z^2\overline{z}}{2}\tilde{\tau}\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix}+\cdots\end{aligned} \tag{7.23}$$

with

$$\begin{aligned}\Gamma &= \frac{1}{l\pi}\int_0^{l\pi}\cos^3\left(\frac{nx}{l}\right)dx,\\ \chi_{20} &= \frac{1}{2}(\alpha_1+\alpha_2\xi+\alpha_3\xi e^{-i\tilde{\tau}\omega_n}+\alpha_4\xi^2e^{-2i\tilde{\tau}\omega_n}),\\ \chi_{11} &= \frac{1}{4}\left(2\alpha_1+\alpha_2(\xi+\bar{\xi})+2\alpha_4\xi\bar{\xi}+e^{-i\tilde{\tau}\omega_n}\alpha_3(\bar{\xi}e^{2i\tilde{\tau}\omega_n}+\xi)\right),\\ \varsigma_{11} &= W_{11}^1(0)\left(2\alpha_1+\alpha_2\xi+\alpha_3\xi e^{-i\tilde{\tau}\omega_n}\right)+W_{11}^2(-1)\left(\alpha_3+2\alpha_4\xi e^{-i\tilde{\tau}\omega_n}\right)\\ &\quad+\frac{1}{2}W_{20}^1(0)\left(2\alpha_1+\alpha_2\bar{\xi}+\alpha_3\bar{\xi}e^{i\tilde{\tau}\omega_n}\right)+\frac{1}{2}W_{20}^2(0)\alpha_2\\ &\quad+\frac{1}{2}W_{20}^2(-1)\left(\alpha_3+2\alpha_4\bar{\xi}e^{i\tilde{\tau}\omega_n}\right)+W_{11}^2(0)\alpha_2,\\ \varsigma_{12} &= \frac{1}{4}(3\alpha_5+\alpha_6(2\xi+\bar{\xi})+\alpha_7(\bar{\xi}e^{i\tilde{\tau}\omega_n}+2\xi e^{-i\tilde{\tau}\omega_n})\\ &\quad+\alpha_8(2\xi\bar{\xi}+\xi^2e^{-2i\tilde{\tau}\omega_n})+3\alpha_9\xi^2\bar{\xi}e^{-i\tilde{\tau}\omega_n}),\\ \varrho_{20} &= \frac{1}{2}(\beta_1+\beta_2\xi+\beta_3\xi^2),\\ \varrho_{11} &= \frac{1}{4}(2\beta_1+\beta_2(\xi+\bar{\xi})+2\beta_3\xi\bar{\xi}),\\ \varsigma_{21} &= W_{11}^1(0)\left(2\beta_1+\beta_2\xi\right)+W_{11}^2(0)\left(\beta_2+2\beta_3\xi\right)\\ &\quad+\frac{1}{2}W_{20}^1(0)\left(2\beta_1+\beta_2\bar{\xi}\right)+\frac{1}{2}W_{20}^2(0)\left(\beta_2+2\beta_3\bar{\xi}\right),\\ \varsigma_{22} &= \frac{1}{4}(3\beta_4+\beta_5(2\xi+\bar{\xi})),\\ \kappa_1 &= \varsigma_{11}\frac{1}{l\pi}\int_0^{l\pi}\cos^2\left(\frac{nx}{l}\right)dx+\varsigma_{12}\frac{1}{l\pi}\int_0^{l\pi}\cos^4\left(\frac{nx}{l}\right)dx,\\ \kappa_2 &= \varsigma_{21}\frac{1}{l\pi}\int_0^{l\pi}\cos^2\left(\frac{nx}{l}\right)dx+\varsigma_{22}\frac{1}{l\pi}\int_0^{l\pi}\cos^4\left(\frac{nx}{l}\right)dx.\end{aligned} \tag{7.24}$$

Denote

$$\Psi_1(0) - i\Psi_2(0) := (\gamma_1 \quad \gamma_2).$$

Finding

$$\frac{1}{l\pi}\int_0^{l\pi} \cos^3\left(\frac{nx}{l}\right) dx = 0, \quad n = 1, 2, 3, \cdots,$$

we have

$$\begin{aligned}
&(\Psi_1(0) - i\Psi_2(0)) < F(U_t, 0), f_n >= \\
&\quad \frac{z^2}{2}(\gamma_1\chi_{20} + \gamma_2\varrho_{20})\Gamma\tilde{\tau} + z\overline{z}(\gamma_1\chi_{11} + \gamma_2\varrho_{11})\Gamma\tilde{\tau} + \frac{\overline{z}^2}{2}(\gamma_1\overline{\chi}_{20} + \gamma_2\overline{\varrho}_{20})\Gamma\tilde{\tau} \\
&\quad + \frac{z^2\overline{z}}{2}\tilde{\tau}(\gamma_1\kappa_1 + \gamma_2\kappa_2) + \cdots.
\end{aligned} \tag{7.25}$$

Then, by (7.16), (7.18) and (7.25), we have $g_{20} = g_{11} = g_{02} = 0$, for $n = 1, 2, 3, \cdots$. If $n = 0$, we have

$$g_{20} = \gamma_1\tilde{\tau}\chi_{20} + \gamma_2\tilde{\tau}\varrho_{20}, \qquad g_{11} = \gamma_1\tilde{\tau}\chi_{11} + \gamma_2\tilde{\tau}\varrho_{11}, \qquad g_{02} = \gamma_1\tilde{\tau}\overline{\chi}_{20} + \gamma_2\tilde{\tau}\overline{\varrho}_{20},$$

and for $n \in \mathbb{N}_0$, $g_{21} = \tilde{\tau}(\gamma_1\kappa_1 + \gamma_2\kappa_2)$. Now, a complete description for g_{21} depends on the algorithm for $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0]$, which we shall compute. From [22], we have

$$\dot{W}(z, \overline{z}) = W_{20}z\dot{z} + W_{11}\dot{z}\overline{z} + W_{11}z\dot{\overline{z}} + W_{02}\overline{z}\dot{\overline{z}} + \cdots,$$

$$A_{\tilde{\tau}}W(z, \overline{z}) = A_{\tilde{\tau}}W_{20}\frac{z^2}{2} + A_{\tilde{\tau}}W_{11}z\overline{z} + A_{\tilde{\tau}}W_{02}\frac{\overline{z}^2}{2} + \cdots$$

and

$$\dot{W}(z, \overline{z}) = A_{\tilde{\tau}}W + H(z, \overline{z}),$$

where

$$\begin{aligned}
H(z, \overline{z}) &= H_{20}\frac{z^2}{2} + W_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots \\
&= X_0F(U_t, 0) - \Phi(\Psi, < X_0F(U_t, 0), f_n > \cdot f_n).
\end{aligned} \tag{7.26}$$

Thus, we have

$$(2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{20} = H_{20}, \ \ -A_{\tilde{\tau}}W_{11} = H_{11}, \ (-2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{02} = H_{02}. \tag{7.27}$$

That is,

$$W_{20} = (2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{20}, \ W_{11} = -A_{\tilde{\tau}}^{-1}H_{11}, \ W_{02} = (-2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{02}. \tag{7.28}$$

By (7.25), for $\theta \in [-1, 0)$, we have

$$\begin{aligned}
H(z,\overline{z}) =& -\Phi(0)\Psi(0) < F(U_t,0), f_n > \cdot f_n \\
=& -\left(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2i}\right)\begin{pmatrix}\Phi_1(0)\\ \Phi_2(0)\end{pmatrix} < F(U_t,0), f_n > \cdot f_n \\
=& -\frac{1}{2}[p_1(\theta)(\Phi_1(0)-i\Phi_2(0)) + p_2(\theta)(\Phi_1(0)+i\Phi_2(0))] < F(U_t,0), f_n > \cdot f_n \\
=& -\frac{1}{2}\left[(p_1(\theta)g_{20}+p_2(\theta)\overline{g}_{02})\frac{z^2}{2} + (p_1(\theta)g_{11}+p_2(\theta)\overline{g}_{11})z\overline{z}\right. \\
& \left. +(p_1(\theta)g_{02}+p_2(\theta)\overline{g}_{20})\frac{\overline{z}^2}{2}\right] + \cdots .
\end{aligned}$$

Therefore, by (7.26), for $\theta \in [-1, 0)$,

$$H_{20}(\theta) = \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{20}+p_2(\theta)\overline{g}_{02})\cdot f_0, & n = 0, \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{11}+p_2(\theta)\overline{g}_{11})\cdot f_0, & n = 0, \end{cases}$$

$$H_{02}(\theta) = \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{02}+p_2(\theta)\overline{g}_{20})\cdot f_0, & n = 0, \end{cases}$$

and

$$H(z,\overline{z})(0) = F(U_t,0) - \Phi(\Psi, < F(U_t,0), f_n >)\cdot f_n,$$

where

$$H_{20}(0) = \begin{cases} \tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varrho_{20}\end{pmatrix}\cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varrho_{20}\end{pmatrix} - \frac{1}{2}(p_1(0)g_{20}+p_2(0)\overline{g}_{02})\cdot f_0, & n = 0. \end{cases} \tag{7.29}$$

$$H_{11}(0) = \begin{cases} \tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varrho_{11}\end{pmatrix}\cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varrho_{11}\end{pmatrix} - \frac{1}{2}(p_1(0)g_{11}+p_2(0)\overline{g}_{11})\cdot f_0, & n = 0. \end{cases} \tag{7.30}$$

By the definitions of $A_{\tilde{\tau}}$ and (7.27), we have

$$\dot{W}_{20} = A_{\tilde{\tau}}W_{20} = 2i\omega_n\tilde{\tau}W_{20} + \frac{1}{2}(p_1(\theta)g_{20}+p_2(\theta)\overline{g}_{02})\cdot f_n, \quad -1 \le \theta < 0.$$

That is,

$$W_{20}(\theta) = \frac{i}{2i\omega_n\tilde{\tau}}\left(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)\right)\cdot f_n + E_1 e^{2i\omega_n\tilde{\tau}\theta},$$

where

$$E_1 = \begin{cases} W_{20}(0), & n = 1, 2, 3, \cdots, \\ W_{20}(0) - \frac{i}{2i\omega_n\tilde{\tau}}\left(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)\right) \cdot f_0, & n = 0. \end{cases}$$

By the definitions of $A_{\tilde{\tau}}$ and (7.27), for $-1 \leq \theta < 0$, we have

$$\begin{aligned} &-\left(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)\right) \cdot f_0 + 2i\omega_n\tilde{\tau}E_1 - A_{\tilde{\tau}}\left(\frac{i}{2\omega_n\tilde{\tau}}\left(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)\right) \cdot f_0\right) \\ &- A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}\left(\frac{i}{2\omega_n\tilde{\tau}}\left(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)\right) \cdot f_n + E_1e^{2i\omega_n\tilde{\tau}\theta}\right) \\ &= \tilde{\tau}\begin{pmatrix} \chi_{20} \\ \varrho_{20} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\overline{g}_{02}) \cdot f_0. \end{aligned}$$

As

$$A_{\tilde{\tau}}p_1(0) + L_{\tilde{\tau}}(p_1 \cdot f_0) = i\omega_0p_1(0) \cdot f_0$$

and

$$A_{\tilde{\tau}}p_2(0) + L_{\tilde{\tau}}(p_2 \cdot f_0) = -i\omega_0p_2(0) \cdot f_0,$$

we have

$$2i\omega_nE_1 - A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}E_1e^{2i\omega_n} = \tilde{\tau}\begin{pmatrix} \chi_{20} \\ \varrho_{20} \end{pmatrix}\cos^2\left(\frac{nx}{l}\right), \quad n = 0, 1, 2, \cdots.$$

That is,

$$E_1 = \tilde{\tau}E\begin{pmatrix} \chi_{20} \\ \varrho_{20} \end{pmatrix}\cos^2\left(\frac{nx}{l}\right),$$

where

$$E = \begin{pmatrix} 2i\omega_n\tilde{\tau} + d_1\frac{n^2}{l^2} - a_1 & -a_2 - ce^{-2i\omega_n\tilde{\tau}} \\ -b_1 & 2i\omega_n\tilde{\tau} + d_2\frac{n^2}{l^2} - b_2 \end{pmatrix}^{-1}.$$

Similarly, from (7.28), we have

$$-\dot{W}_{11} = \frac{i}{2\omega_n\tilde{\tau}}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_n, \quad -1 \leq \theta < 0.$$

That is,

$$W_{11}(\theta) = \frac{i}{2i\omega_n\tilde{\tau}}(p_1(\theta)\overline{g}_{11} - p_1(\theta)g_{11}) + E_2.$$

Similar to the procedure of computing W_{20}, we have

$$E_2 = \tilde{\tau}E^*\begin{pmatrix} \chi_{11} \\ \varrho_{11} \end{pmatrix}\cos^2\left(\frac{nx}{l}\right),$$

where

$$E^* = \begin{pmatrix} d_1\frac{n^2}{l^2} - a_1 & -a_2 - c \\ -b_1 & d_2\frac{n^2}{l^2} - b_2 \end{pmatrix}^{-1}.$$

Therefore, u_2, β_2 and T_2 can be calculated.

Acknowledgements

The authors thank the two anonymous reviewers and the editors for their careful reading and helpful comments, which have significantly contributed to improving the quality of our manuscript.

References

[1] K. B. Altendorf, J. W. Laundré, C. A. L. González and J. S. Brown, *Assessing effects of predation risk on foraging behavior of mule deer*, Journal of Mammalogy, 2001, 82(2), 430–439.

[2] X. Cao and W. Jiang, *Interactions of Turing and Hopf Bifurcations in an Additional Food Provided Diffusive Predator-prey Model*, Journal of Applied Analysis and Computation, 2019, 9(4), 1277–1304.

[3] X. Chang and J. Wei, *Hopf bifurcation and optimal control in a diffusive predator-prey system with time delay and prey harvesting*, Nonlinear Analysis: Modelling and Control, 2012, 4(4), 379–409.

[4] S. Creel, D. Christianson, S. Liley and J. A. Winnie, *Predation Risk Affects Reproductive Physiology and Demography of Elk*, Science, 2007, 315(5814), Article ID 1135918, Page 960.

[5] R. Descartes, *La Géométrie*, Paris, 1637.

[6] Y. Du and J. Shi, *Some recent results on diffusive predator-prey models in spatially heterogeneous environment*, Nonlinear Dynamics and Evolution Equations, 2006, 48, 95–135.

[7] C. Duque and M. Lizana, *On the dynamics of a predator–prey model with nonconstant death rate and diffusion*, Nonlinear Analysis: Real World Applications, 2011, 12(4), 2198–2210.

[8] Q. Gan, R. Xu and P. Yang, *Bifurcation and chaos in a ratio-dependent predator–prey system with time delay*, Chaos, Solitons & Fractals, 2009, 39(4), 1883–1895.

[9] G. Gilioli, S. Pasquali and F. Ruggeri, *Nonlinear functional response parameter estimation in a stochastic predator-prey model*, Mathematical Biosciences and Engineering, 2012, 9(1), 75–96.

[10] S. A. Gourley, *Instability in a predator-prey system with delay and spatial averaging*, IMA Journal of Applied Mathematics, 1996, 56(2), 121–132.

[11] G. Guo and Y. Li, *Global Bifurcation and Stability for the Predator-Prey Model with B-D Functional Response*, Acta Mathematicae Applicatae Sinica, 2008, 31(2), 220–230.

[12] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.

[13] D. Jia, T. Zhang and S. Yuan, *Pattern Dynamics of a Diffusive Toxin Producing Phytoplankton–Zooplankton Model with Three-Dimensional Patch*, International Journal of Bifurcation and Chaos, 2019, 29(4), Article ID 1930011, 19 pages.

[14] H. Liu, R. Wu and B. Liu, *Spatiotemporal Dynamic Analysis in a Time-space Discrete Brusselator Model*, Journal of Nonlinear Modeling and Analysis, 2022, 4(3), 539–561.

[15] P. Panday, S. Samanta, N. Pal and J. Chattopadhyay, *Delay induced multiple stability switch and chaos in a predator–prey model with fear effect*, Mathematics and Computers in Simulation, 2019, 172, 134–158.

[16] Y. Peng, Y. Li and T. Zhang, *Global bifurcation in a toxin producing phytoplankton–zooplankton system with prey-taxis*, Nonlinear Analysis: Real World Applications, 2021, 61, Article ID 103326, 15 pages.

[17] W. J. Ripple and R. L. Beschta, *Wolves and the Ecology of Fear: Can Predation Risk Structure Ecosystems?*, BioScience, 2004, 54(8), 755–766.

[18] M. L. Rosenzweig and R. H. MacArthur, *Graphical Representation and Stability Conditions of Predator-prey Interactions*, The American Naturalist, 1963, 97(895), 209–223.

[19] S. Ruan and J. Wei, *On the zeros of transcendental functions with applications to stability of delay differential equations with two delays*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 2003, 10(6), 863–874.

[20] Y. Song, Y. Peng and T. Zhang, *The spatially inhomogeneous Hopf bifurcation induced by memory delay in a memory-based diffusion system*, Journal of Differential Equations, 2021, 300(5), 597–624.

[21] Y. Tang and Z. Li, *Stability switch and Hopf bifurcation for a diffusive prey–predator system with delay*, Journal of Mathematical Analysis and Applications, 2007, 334(2), 1290–1307.

[22] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Springer, New York, 1996.

[23] Y. Xiao and L. Chen, *Modeling and analysis of a predator–prey model with disease in the prey*, Mathematical Biosciences, 2001, 171(1), 59–82.

[24] B. Xie, Z. Wang, Y. Xue and Z. Zhang, *The Dynamics of a Delayed Predator-Prey Model with Double Allee Effect*, Discrete Dynamics in Nature and Society, 2015, Article ID 102597, 8 pages.

[25] S. Yan, D. Jia, T. Zhang and S. Yuan, *Pattern dynamics in a diffusive predator-prey model with hunting cooperations*, Chaos, Solitons & Fractals, 2020, 130, Article ID 109428, 12 pages.

[26] J. Yang, T. Zhang and S. Yuan, *Turing Pattern Induced by Cross-Diffusion in a Predator–Prey Model with Pack Predation-Herd Behavior*, International Journal of Bifurcation and Chaos, 2020, 30(7), Article ID 2050103, 22 pages.

[27] T. Zhang, X. Liu, X. Meng and T. Zhang, *Spatio-temporal dynamics near the steady state of a planktonic system*, Computers & Mathematics with Applications, 2018, 75(12), 4490–4504.

[28] N. Zhu and S. Yuan, *Spatial Dynamics of a Diffusive Prey-predator Model with Stage Structure and Fear Effect*, Journal of Nonlinear Modeling and Analysis, 2022, 4(2), 392–408.