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Regularity Criteria to the Axially Symmetric Tropical Climate Model without Swirl

Xian Chen[1,†] and Xinru Cheng[1]

Abstract In this paper, we consider the Cauchy problem of the axially symmetric tropical climate model with fractional dissipation. By using the energy method, we establish a new regularity criteria for the axisymmetric solutions of the 3D Tropical climate model without swirl.

Keywords Axisymmetric solution, Regularity criteria, Tropical climate model.

MSC(2010) 35B65, 35Q35, 76W05.

1. Introduction

In this paper, we consider the following 3D tropical climate model:

$$\partial_t u + (u \cdot \nabla)u - \mu\Lambda^{2\alpha} u + \nabla p + \text{div}\,(v \otimes v) = 0, \tag{1.1}$$
$$\partial_t v + (u \cdot \nabla)v - \nu\Lambda^{2\beta} v + \nabla\psi + (v \cdot \nabla)u = 0, \tag{1.2}$$
$$\partial_t \psi + (u \cdot \nabla)\psi - \eta\Lambda^{2\gamma}\psi + \text{div}\, v = 0, \tag{1.3}$$
$$\text{div}\, u = 0, \tag{1.4}$$
$$(u, v, \psi)(\mathbf{x}, 0) = (u_0, v_0, \psi_0), \tag{1.5}$$

where the vector fields $u(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), u_3(\mathbf{x},t))$ and $v(\mathbf{x},t) = (v_1(\mathbf{x},t), v_2(\mathbf{x},t), v_3(\mathbf{x},t))$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively. The scalar functions $p(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ represent the pressure and the temperature, respectively. The fractional Laplacian operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined by means of the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of f. In this paper, we set the constants $\mu = \nu = \eta = 1$.

In this paper, we study the axially symmetric solution of systems (1.1)-(1.5) without swirl ($u_\theta = 0$). Then, u, v and ψ can be rewritten as

$$u(\mathbf{x},t) = u_r(r,z,t)e_r + u_z(r,z,t)e_z, \tag{1.6}$$
$$v(\mathbf{x},t) = v_r(r,z,t)e_r + v_z(r,z,t)e_z, \tag{1.7}$$

[†]The corresponding author.
Email address: xianchen@zjnu.edu.cn (X. Chen), chengxr@zjnu.edu.cn (X. Cheng)
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

$$\psi(\mathbf{x}, t) = \psi(r, z, t). \tag{1.8}$$

Here,

$$\mathbf{x} = (x, y, z), \tag{1.9}$$

$$e_r = (\frac{x}{r}, \frac{y}{r}, 0),\ e_\theta = (-\frac{y}{r}, \frac{x}{r}, 0),\ e_z = (0, 0, 1), \tag{1.10}$$

$$r = \sqrt{x^2 + y^2},\ (x, y, z) = (r\cos\theta, r\sin\theta, z). \tag{1.11}$$

By direct calculation, we obtain

$$\omega(\mathbf{x}, t) = \nabla \times u = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z,$$

where

$$\omega_r = -\partial_z u_\theta,\ \omega_\theta = \partial_z u_r - \partial_r u_z,\ \omega_z = \frac{1}{r}\partial_r(r u_\theta).$$

Now, we review some related results about the tropical climate models (1.1)-(1.5). By presenting a new quantity and utilizing a logarithmic Gronwall inequality, Li and Titi [11] established the global existence of strong solutions for systems (1.1)-(1.5) without diffusion, when $\alpha = \beta = 1$ and $\mu > 0,\ \nu > 0,\ \eta = 0$. The global well-posedness of classical solutions for the tropical climate model was obtained by Wan [16] in terms of the dissipation of the first baroclinic model of the velocity and some damping terms at small initial data. By applying the "weakly nonlinear" energy estimates, the global regularity of a tropical climate model with greatly weak dissipation of the barotropic mode was proved by Ye in [20] ($\alpha > 0,\ \beta = \gamma = 1$ and $\mu,\ \nu,\ \eta > 0$). Recently, the global regularity for the 3D tropical climate model with fractional diffusion on barotropic mode has been established by Zhu [23], when $\alpha \geq \frac{5}{2}$ and $\mu > 0,\ \nu = \eta = 0$. Then, by using the spectral analysis, the global well-posedness of the 2D viscous tropical climate model with only one damping term was proved by Ma and Wan in [14], when $\mu = \nu = 1,\ \eta = 0$. The d-dimensional system (1.1) was studied by Ma [13], and he got the local smooth solution. More studies on tropical climate models are available in [4, 5, 19, 21].

When $\psi = 0$, the tropical climate models (1.1)-(1.5) become the axisymmetric MHD system. For the axisymmetric MHD system, the global well-posedness of classical solutions was established by Lei [7]. Then, the solutions of 3D axially symmetric incompressible MHD equation was studied by Wang and Wu in [17]. Also, they established a group of global smooth solutions by using the one-dimensional solutions. Lately, the regularity criteria for the axisymmetric solutions to MHD equation was established by Li and Yuan [10], as long as $\omega_\theta \in L^q(0, T; L^p(\mathbb{R}^3))$ and $n_\theta \in L^q(0, T; L^p(\mathbb{R}^3))$ satisfy

$$\int_0^T (\|\omega_\theta\|_{L^p}^q + \|n_\theta\|_{L^p}^q) dt < \infty,\ with\ \frac{3}{p} + \frac{2}{q} \leq 2,\ \frac{3}{2} < p \leq \infty,\ 0 < q < \infty.$$

For more studies about MHD system, we can refer to [12, 15].

Systems (1.1)-(1.5) reduce to the Navier-Stokes Equations, when $\psi = v = 0$. For more studies about the axisymmetric Navier-Stokes equation, we can refer to [2, 6–9, 17, 18]. Here, we only introduce some related results. First of all, some regularity criteria about the axisymmetric weak solutions of 3D Navier-Stokes equations were established by Chae-Lee in [2]. Then, Wei [18] obtained the global regularity for the solutions of the axially symmetric Navier-Stokes system, as long as $\|r u_\theta(r, z, t)\|_{L^\infty}$

or $\|ru_\theta(r,z,t)\|_{L^\infty(r\le r_0)}$ is smaller than some dimensionless quantity of the initial data. The result improves the one in Lei and Zhang [9]. In this paper, we will establish regularity criteria of the solutions to the axially symmetric tropical climate model without swirl.

Our main results are as follows.

Theorem 1.1. *For $\alpha \ge \frac{1}{2}$, $\beta \ge 1$, $\gamma \ge \frac{1}{2}$, assume $(u_0, v_0, \psi_0) \in H^1(\mathbb{R}^3)$, div $u_0 = 0$ and $(u,v,\psi)(x,t)$ is an axially symmetric solution of systems* (1.1)-(1.5). *If*

$$\omega_\theta \in L^q(0,T;L^p),\ \text{with}\ \frac{3}{p\min\{\alpha,\beta,\gamma\}} + \frac{2}{q} \le 2,\ \max\{\frac{3}{2\alpha},\frac{3}{\beta},\frac{3}{2\gamma},1\} \le p \le 3, \tag{1.12}$$

then the solution remains smooth in [0,T].

This paper is organized as follows. In Section 2, we will give some notations and lemmas which will be used in the proof of Theorem 1.1. We will give the proof of Theorem 1.1 in Section 3.

2. Notations and lemmas

First of all, let us recall the relation between cylindrical coordinates and rectangular Cartesian coordinates. The Laplacian operator Δ and the gradient operator ∇ in the cylindrical coordinate are

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2, \quad \nabla = e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z, \tag{2.1}$$

where u and v are axially symmetric vector fields. We set $\tilde{u} = u_r e_r + u_z e_z$, $\tilde{v} = v_r e_r + v_z e_z$ and the corresponding curl component ω. Also, we set $\tilde{\omega} = \omega_r e_r + \omega_z e_z$ and $\tilde{\nabla} = (\partial_r, \partial_z)$.

Lemma 2.1 (See [10])**.** *Let u and v be axially symmetric vector fields. Then we have*

$$|\nabla\tilde{u}|^2 = |\frac{u_r}{r}|^2 + |\tilde{\nabla}u_r|^2 + |\tilde{\nabla}u_z|^2, \tag{2.2}$$

$$|\nabla\tilde{v}|^2 = |\frac{v_r}{r}|^2 + |\tilde{\nabla}v_r|^2 + |\tilde{\nabla}v_z|^2. \tag{2.3}$$

Then, let us review the classical Biot-Savart law. For more details, please refer to [3, 22].

Lemma 2.2 (See [10])**.** *(Biot-Savart law) Let $u \in L^2(\mathbb{R}^3)$ be a smooth vector field with* div $u = 0$. *Then the corresponding curl component $\omega = curl\ u \in L^2(\mathbb{R}^3)$ vanishes sufficiently rapid as $x \to \infty$. Then,* div $\omega = 0$, *and the velocity*

$$u(x) = \int_{\mathbb{R}^3} M(x-y)\omega(y)\, dx. \tag{2.4}$$

Here, kernel M is a 3×3 matrix.

Then, the gradient of u can be denoted by ω and the singular integral

$$\nabla u = C\omega(x) + K * \omega(x), \tag{2.5}$$

where the kernel $K(x)$ is the matrix valued functions homogeneous of degree -3, defining a singular integral operator by convolution. C is a constant matric. Hence,

$$\|\nabla u\|_{L^p} \le C\,\|\omega\|_{L^p}\,, \quad 1 < p < \infty. \tag{2.6}$$

Lemma 2.3 (See [10])**.** *Let u be the axially symmetric vector field with* div $u = 0$. *Its corresponding vorticity $\omega = curl\ u$ vanishes sufficiently fast as $x \to \infty$ in $\mathbb{R}^3$. Then, ∇u can be represented as the singular integral form*

$$\nabla \tilde{u}(x) = C_1 \omega_\theta e_\theta(x) + [K_1 * (\omega_\theta e_\theta)](x), \tag{2.7}$$

where the kernel $K_1(x)$ and $K_2(x)$ are the matrix valued functions homogeneous of degree -3. C_1 and C_2 are the constant matrics, and $f * g(x) = \int_{\mathbb{R}^3} f(x-y)g(y)dy$ denote a standard convolution operator.

The proof is anology with the proof of Lemma 2 of [1]. It is easy to have $\tilde{u} = u_r e_r + u_z e_z$ and div $\tilde{u} = 0$. Therefore, we get

$$curl\ \tilde{u} = (\partial_z u_r - \partial_r u_z)e_\theta = \omega_\theta e_\theta. \tag{2.8}$$

This completes the proof of Lemma 2.3. By (2.5), (2.6) and Lemma 2.3, we have the lemma as follows.

Lemma 2.4 (See [10])**.** *Suppose $1 < p < \infty$. Then*

$$\|\nabla \tilde{u}\|_{L^p} \le C\,\|\omega_\theta\|_{L^p}\,. \tag{2.9}$$

3. Proof of the main theorem

In this section, we prove Theorem 1.1. First, we state a priori L^2 -estimates for systems (1.1)-(1.5). Multiplying (1.1)-(1.3) by (u, v, ψ) after integration by parts and using $\nabla \cdot u = 0$, we have the following energy estimate

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2 + 2\int_0^T (\|\Lambda^\alpha u(t)\|_{L^2}^2 + \left\|\Lambda^\beta v(t)\right\|_{L^2}^2 + \|\Lambda^\gamma \psi(t)\|_{L^2}^2)\,\mathrm{d}s\\ =& \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\psi_0\|_{L^2}^2\,.\end{aligned} \tag{3.1}$$

Multiplying (1.1), (1.2) and (1.3) by $-\Delta u$, $-\Delta v$ and $-\Delta\psi$ respectively, integrating them in $\mathbb{R}^3$ and adding the resulting equations, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2) + \left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 + \left\|\Lambda^{\beta+1}v\right\|_{L^2}^2 + \left\|\Lambda^{\gamma+1}\psi\right\|_{L^2}^2\\ =& \int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot\Delta u\,dx + \int_{\mathbb{R}^3} (v\cdot\nabla)v\cdot\Delta u\,dx + \int_{\mathbb{R}^3} (\nabla\cdot v)v\cdot\Delta u\,dx\\ &+ \int_{\mathbb{R}^3} (u\cdot\nabla)v\cdot\Delta v\,dx + \int_{\mathbb{R}^3} (v\cdot\nabla)u\cdot\Delta v\,dx + \int_{\mathbb{R}^3} (u\cdot\nabla)\psi\cdot\Delta\psi\,dx\\ :=& I_1 + I_2 + I_3 + I_4 + I_5 + I_6,\end{aligned} \tag{3.2}$$

where we have used the following fact

$$\int_{\mathbb{R}^3} \nabla\psi\cdot\Delta v\,dx + \int_{\mathbb{R}^3} (\nabla\cdot v)\Delta\psi\,dx = 0, \tag{3.3}$$

$$\int_{\mathbb{R}^3} div\ (v \otimes v) \cdot \Delta u\ dx = \int_{\mathbb{R}^3} (v \cdot \nabla) v \cdot \Delta u\ dx + \int_{\mathbb{R}^3} (\nabla \cdot v) v \cdot \Delta u\ dx. \tag{3.4}$$

Next, we will estimate $I_1 - I_6$ by applying (1.6)-(1.11) and the Sobolev inequality. For I_1, we have

$$\begin{aligned} I_1 = & \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u\ dx = \int_{\mathbb{R}^3} [(u_r \partial_r u_r + u_z \partial_z u_r)(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \partial_z^2 u_r) \\ & + (u_r \partial_r u_z + u_z \partial_z u_z)(\partial_r^2 u_z + \frac{1}{r}\partial_r u_z + \partial_z^2 u_z)]\ dx \\ = & \int_{\mathbb{R}^3} (u_r \partial_r u_r \partial_r^2 u_r + \frac{1}{r} u_r \partial_r u_r \partial_r u_r + u_r \partial_r u_r \partial_z^2 u_r + u_z \partial_z u_r \partial_r^2 u_r \\ & + \frac{1}{r} u_z \partial_z u_r \partial_r u_r + u_z \partial_z u_r \partial_z^2 u_r + u_r \partial_r u_z \partial_r^2 u_z + \frac{1}{r} u_r \partial_r u_z \partial_r u_z \\ & + u_r \partial_r u_z \partial_z^2 u_z + u_z \partial_z u_z \partial_r^2 u_z + \frac{1}{r} u_z \partial_z u_z \partial_r u_z + u_z \partial_z u_z \partial_z^2 u_z)\ dx. \end{aligned} \tag{3.5}$$

Then, further simplifying (3.5) by using integration by parts, we obtain

$$\begin{aligned} I_1 = & \int_{\mathbb{R}^3} [(\partial_r u_r)^2 \partial_z u_z - (\partial_z u_r)^2 \partial_r u_r - (\partial_z u_r)^2 \partial_z u_z - \partial_r u_r \partial_r u_z \partial_z u_r \\ & - \partial_z u_z \partial_z u_r \partial_r u_z - (\partial_z u_z)^2 \partial_z u_z + \frac{u_r}{r}(\partial_r u_r)^2 + \frac{u_r}{r}(\partial_r u_z)^2]\ dx \\ := & K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8, \end{aligned} \tag{3.6}$$

where we have used the fact

$$\nabla \cdot u = \frac{u_r}{r} + \partial_r u_r + \partial_z u_z = 0. \tag{3.7}$$

Now, we estimate the right-hand side of (3.6) one by one. By Lemma 2.1, we get

$$|\frac{u_r}{r}|, |\partial_r u_r|, |\partial_z u_r|, |\partial_r u_z|, |\partial_z u_z| \leq |\nabla u|. \tag{3.8}$$

First of all, we estimate K_1 as follows

$$\begin{aligned} |K_1| = & |\int_{\mathbb{R}^3} (\partial_r u_r)^2 \partial_z u_z\ dx| \\ \leq & \|\partial_z u_z\|_{L^p} \|\nabla u\|^2_{L^{\frac{2p}{p-1}}} \\ \leq & C \|\partial_z u_z\|_{L^p} \|\nabla u\|_{L^2}^{\frac{2p\alpha-3}{p\alpha}} \left\|\Lambda^{\alpha+1} u\right\|_{L^2}^{\frac{3}{p\alpha}} \\ \leq & \frac{1}{8} \left\|\Lambda^{\alpha+1} u\right\|^2_{L^2} + C \|\partial_z u_z\|_{L^p}^{\frac{2p\alpha}{2p\alpha-3}} \|\nabla u\|^2_{L^2}. \end{aligned} \tag{3.9}$$

Here, we have used the following Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^{\frac{2p}{p-1}}} \leq C \|\nabla u\|_{L^2}^{\frac{2p\alpha-3}{2p\alpha}} \left\|\Lambda^{1+\alpha} u\right\|_{L^2}^{\frac{3}{2p\alpha}}, \frac{3}{2\alpha} \leq p < \infty. \tag{3.10}$$

Similary, we have

$$\begin{aligned} |K_3| + |K_6| \leq & C \|\partial_z u_z\|_{L^p} \|\nabla u\|^2_{L^{\frac{2p}{p-1}}} \\ \leq & \frac{1}{8} \left\|\Lambda^{\alpha+1} u\right\|^2_{L^2} + C \|\partial_z u_z\|_{L^p}^{\frac{2p\alpha}{2p\alpha-3}} \|\nabla u\|^2_{L^2}, \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
|K_2|+|K_4|+|K_5| &\le C\left\|\partial_z u_r\right\|_{L^p}\left\|\nabla u\right\|^2_{L^{\frac{2p}{p-1}}} \\
&\le \frac{1}{8}\left\|\Lambda^{\alpha+1}u\right\|^2_{L^2}+C\left\|\partial_z u_r\right\|^{\frac{2p\alpha}{2p\alpha-3}}_{L^p}\left\|\nabla u\right\|^2_{L^2}
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
|K_7|+|K_8| &\le C\left\|\frac{u_r}{r}\right\|_{L^p}\left\|\nabla u\right\|^2_{L^{\frac{2p}{p-1}}} \\
&\le \frac{1}{8}\left\|\Lambda^{\alpha+1}u\right\|^2_{L^2}+C\left\|\frac{u_r}{r}\right\|^{\frac{2p\alpha}{2p\alpha-3}}_{L^p}\left\|\nabla u\right\|^2_{L^2}.
\end{aligned}
\tag{3.13}
$$

Hence, combining the above estimates (3.9), (3.11)-(3.13), we obtain

$$
|I_1| \le \frac{1}{2}\left\|\Lambda^{\alpha+1}u\right\|^2_{L^2}+C(\left\|\partial_z u_r\right\|^{\frac{2p\alpha}{2p\alpha-3}}_{L^p}+\left\|\partial_z u_z\right\|^{\frac{2p\alpha}{2p\alpha-3}}_{L^p}+\left\|\frac{u_r}{r}\right\|^{\frac{2p\alpha}{2p\alpha-3}}_{L^p})\left\|\nabla u\right\|^2_{L^2}.
\tag{3.14}
$$

We will estimate the I_2. Applying a method similar to I_1, we have

$$
\begin{aligned}
I_2 = \int_{\mathbb{R}^3}(v\cdot\nabla)v\cdot\Delta u\,dx = \int_{\mathbb{R}^3}&(v_r\partial_r v_r\partial_r^2 u_r+\frac{1}{r}v_r\partial_r v_r\partial_r u_r+v_r\partial_r v_r\partial_z^2 u_r \\
&+v_z\partial_z v_r\partial_r^2 u_r+\frac{1}{r}v_z\partial_z v_r\partial_r u_r+v_z\partial_z v_r\partial_z^2 u_r+v_r\partial_r v_z\partial_r^2 u_z+\frac{1}{r}v_r\partial_r v_z\partial_r u_z \\
&+v_r\partial_r v_z\partial_z^2 u_z+v_z\partial_z v_z\partial_r^2 u_z+\frac{1}{r}v_z\partial_z v_z\partial_r u_z+v_z\partial_z v_z\partial_z^2 u_z)\,dx.
\end{aligned}
\tag{3.15}
$$

Then, we can further simplify (3.15) by using integration by parts and (3.7). We obtain

$$
\begin{aligned}
I_2 = \int_{\mathbb{R}^3}&[\frac{u_r}{r}\partial_r v_z\partial_z v_r-\partial_r u_r(\partial_r v_r)^2-\partial_r u_r\partial_r^2 v_r v_r-\partial_z u_r\partial_z v_r\partial_r v_r \\
&-\partial_z u_r\partial_z\partial_r v_r v_r-\partial_r u_r\partial_r\partial_z v_r v_z-\partial_z u_r\partial_z v_z\partial_z v_r-\partial_z u_r\partial_z^2 v_r v_z \\
&-\partial_r u_z\partial_r v_z\partial_r v_r-\partial_r u_z\partial_r^2 v_z v_r-\partial_z u_z\partial_z\partial_r v_z v_r-\partial_r u_z\partial_z v_z\partial_r v_z \\
&-\partial_r u_z\partial_r\partial_z v_z v_z-\partial_z u_z(\partial_z v_z)^2-\partial_z u_z\partial_z^2 v_z v_z]\,dx \\
:=&J_1+J_2+J_3+J_4+J_5+\ldots+J_{15}.
\end{aligned}
\tag{3.16}
$$

Now, we estimate the right-hand side of (3.16) one by one. First, from Lemma 2.1, we get

$$
|\frac{u_r}{r}|,|\partial_r u_r|,|\partial_z u_r|,|\partial_r u_z|,|\partial_z u_z| \le |\nabla u|
\tag{3.17}
$$

and

$$
|\frac{v_r}{r}|,|\partial_r v_r|,|\partial_z v_r|,|\partial_r v_z|,|\partial_z v_z| \le |\nabla v|.
\tag{3.18}
$$

We estimate J_1 and J_2 as follows

$$
\begin{aligned}
|J_1| &= |\int_{\mathbb{R}^3}\frac{u_r}{r}\partial_r v_z\partial_z v_r\,dx| \\
&\le \left\|\frac{u_r}{r}\right\|_{L^p}\left\|\nabla v\right\|^2_{L^{\frac{2p}{p-1}}} \\
&\le C\left\|\frac{u_r}{r}\right\|_{L^p}\left\|\nabla v\right\|^{\frac{2p\beta-3}{p\beta}}_{L^2}\left\|\Lambda^{\beta+1}v\right\|^{\frac{3}{p\beta}}_{L^2} \\
&\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C\left\|\frac{u_r}{r}\right\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}\left\|\nabla v\right\|^2_{L^2},
\end{aligned}
\tag{3.19}
$$

$$\begin{aligned}|J_2| &= |\int_{\mathbb{R}^3}(\partial_r v_r)^2\partial_r u_r\, dx| \\ &\le \|\partial_r u_r\|_{L^p}\|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ &\le C\|\partial_r u_r\|_{L^p}\|\nabla v\|_{L^2}^{\frac{2p\beta-3}{p\beta}}\left\|\Lambda^{\beta+1}v\right\|_{L^2}^{\frac{3}{p\beta}} \\ &\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + C\|\partial_r u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|^2_{L^2},\end{aligned} \tag{3.20}$$

where we have used the following Gagliardo-Nirenberg inequality

$$\|\nabla v\|_{L^{\frac{2p}{p-1}}} \le C\|\nabla v\|_{L^2}^{\frac{2p\beta-3}{2p\beta}}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^{\frac{3}{2p\beta}}, \frac{3}{2\beta} \le p < \infty. \tag{3.21}$$

Similary, we get

$$\begin{aligned}|J_4| + |J_7| &\le C\|\partial_z u_r\|_{L^p}\|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ &\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + C\|\partial_z u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|^2_{L^2},\end{aligned} \tag{3.22}$$

$$\begin{aligned}|J_9| + |J_{12}| &\le C\|\partial_r u_z\|_{L^p}\|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ &\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + C\|\partial_r u_z\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|^2_{L^2}\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}|J_{11}| + |J_{14}| &\le C\|\partial_z u_z\|_{L^p}\|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ &\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + C\|\partial_z u_z\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|^2_{L^2}.\end{aligned} \tag{3.24}$$

It is different to estimate J_3 and J_6. By using the Young inequality and the *Hölder* inequality, we have

$$\begin{aligned}|J_3| + |J_6| &= |\int_{\mathbb{R}^3}\partial_r u_r\partial_r^2 v_r v_r\, dx| + |\int_{\mathbb{R}^3}\partial_r u_r\partial_r\partial_z v_r v_z\, dx| \\ &\le C\|\partial_r u_r\|_{L^p}\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ &\le C\|\partial_r u_r\|_{L^p}\|\nabla v\|_{L^2}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ &\le C\|\partial_r u_r\|_{L^p}\|\nabla v\|_{L^2}^{\frac{2p\beta-3}{p\beta}}\left\|\Lambda^{\beta+1}v\right\|_{L^2}^{\frac{3}{p\beta}} \\ &\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + C\|\partial_r u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|^2_{L^2}.\end{aligned} \tag{3.25}$$

Here, we have also used the following Sobolev inequality

$$\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \le C\|\nabla v\|_{L^2}^{1-\frac{3}{p\beta}}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^{\frac{3}{p\beta}}, \frac{3}{\beta} \le p \le 3. \tag{3.26}$$

Similary, we have

$$\begin{aligned}|J_5| + |J_8| \le& C\|\partial_z u_r\|_{L^p}\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ \le& \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + C\|\partial_z u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|^2_{L^2},\end{aligned} \tag{3.27}$$

$$\begin{aligned}|J_{10}|+|J_{13}| \leq& C\left\|\partial_r u_z\right\|_{L^p}\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\ \leq&\frac{1}{48}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_r u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}|J_{15}| \leq& C\left\|\partial_z u_z\right\|_{L^p}\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\ \leq&\frac{1}{48}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_z u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2 .\end{aligned} \tag{3.29}$$

Therefore, combining (3.19)-(3.29) with (3.16), we have

$$\begin{aligned}|I_2| \leq&\frac{1}{8}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C(\left\|\frac{u_r}{r}\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}+\left\|\partial_r u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\\ &+\left\|\partial_z u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}+\left\|\partial_r u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}+\left\|\partial_z u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}})\|\nabla v\|_{L^2}^2 .\end{aligned} \tag{3.30}$$

For I_3, we have

$$\begin{aligned}I_3 =& \int_{\mathbb{R}^3}(\nabla\cdot v)v\cdot\Delta u\, dx=\int_{\mathbb{R}^3}[(\partial_r v_r+\frac{v_r}{r}+\partial_z v_z)(v_r\partial_r^2 u_r+\frac{v_r}{r}\partial_r u_r+\\ & v_r\partial_z^2 u_r+v_z\partial_r^2 u_z+\frac{v_z}{r}\partial_r u_z+v_z\partial_z^2 u_z)]\, dx\\ =&\int_{\mathbb{R}^3}(v_r\partial_r v_r\partial_r^2 u_r+\frac{v_r}{r}\partial_r v_r\partial_r u_r+v_r\partial_r v_r\partial_z^2 u_r+v_z\partial_r v_r\partial_r^2 u_z+\\ &\frac{v_z}{r}\partial_r v_r\partial_r u_z+v_z\partial_r v_r\partial_z^2 u_z+\frac{v_r^2}{r}\partial_r^2 u_r+(\frac{v_r}{r})^2\partial_r u_r+\frac{v_r^2}{r}\partial_z^2 u_r+\\ &\frac{v_r}{r}v_z\partial_r^2 u_z+\frac{v_z v_r}{r^2}\partial_r u_z+\frac{v_r v_z}{r}\partial_z^2 u_z+v_r\partial_z v_z\partial_r^2 u_r+\frac{v_r}{r}\partial_z v_z\partial_r u_r+\\ &v_r\partial_z v_z\partial_z^2 u_r+v_z\partial_z v_z\partial_r^2 u_z+\frac{v_z}{r}\partial_r u_z\partial_z v_z+v_z\partial_z v_z\partial_z^2 u_z)\, dx.\end{aligned} \tag{3.31}$$

Further simplifying (3.31) by using integration by parts and (3.7), we obtain

$$\begin{aligned}I_3 =& \int_{\mathbb{R}^3}[(\frac{v_r}{r})^2\partial_r u_r+\frac{v_r v_z}{r^2}\partial_r u_z+\frac{u_r}{r}\partial_z v_z\partial_r v_r-(\partial_r v_r)^2\partial_r u_r-\partial_r u_r\partial_r^2 v_r v_r\\ &-\partial_z u_r\partial_z v_r\partial_r v_r-v_r\partial_z u_r\partial_z\partial_r v_r-\partial_r u_z\partial_r v_r\partial_r v_z-\partial_r u_z\partial_r^2 v_r v_z-\partial_z u_z\partial_z\partial_r v_r v_z\\ &-2\partial_r u_r\partial_r v_r\frac{v_r}{r}-2\frac{v_r}{r}\partial_z u_r\partial_z v_r-\partial_r u_z\partial_r v_r\frac{v_z}{r}-\partial_r u_z\partial_r v_z\frac{v_r}{r}-\partial_z u_z\partial_z v_r\frac{v_z}{r}\\ &-\partial_z u_z\partial_z v_z\frac{v_r}{r}-v_r\partial_r\partial_z v_z\partial_r u_r-\partial_z u_r\partial_z v_r\partial_z v_z-\partial_z u_r\partial_z^2 v_z v_r-\partial_r u_z\partial_z v_z\partial_r v_z\\ &-\partial_r u_z\partial_r\partial_z v_z v_z-\partial_z u_z\partial_z v_z\partial_z v_z-\partial_z u_z\partial_z^2 v_z v_z]\, dx\\ =&M_1+M_2+\ldots+M_{23}.\end{aligned} \tag{3.32}$$

We will estimate the right-hand side of (3.32) one by one. First, we estimate M_1 as follows

$$\begin{aligned}|M_1|+|M_4|+|M_{11}| =& |\int_{\mathbb{R}^3}(\frac{v_r}{r})^2\partial_r u_r\, dx|+|\int_{\mathbb{R}^3}(\partial_r v_r)^2\partial_r u_r\, dx|+|2\int_{\mathbb{R}^3}\partial_r u_r\partial_r v_r\frac{v_r}{r}\, dx|\\ \leq& C\left\|\partial_r u_r\right\|_{L^p}\|\nabla v\|_{L^{\frac{2p}{p-1}}}^2\\ \leq& C\left\|\partial_r u_r\right\|_{L^p}\|\nabla v\|_{L^2}^{\frac{2p\beta-3}{p\beta}}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^{\frac{3}{p\alpha}}\\ \leq& \frac{1}{72}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_r u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2 ,\end{aligned} \tag{3.33}$$

where we used the following Gagliardo-Nirenberg inequality

$$\|\nabla v\|_{L^{\frac{2p}{p-1}}} \leq C \|\nabla v\|_{L^2}^{\frac{2p\beta-3}{2p\beta}} \left\|\Lambda^{1+\beta} v\right\|_{L^2}^{\frac{3}{2p\beta}}, \ \frac{3}{2\beta} \leq p < \infty. \tag{3.34}$$

Similary, we get

$$\begin{aligned} |M_6| + |M_{12}| + |M_{18}| \leq & C \|\partial_z u_r\|_{L^p} \|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \|\partial_z u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} |M_2| + |M_8| + |M_{13}| + |M_{14}| + |M_{20}| \leq & C \|\partial_r u_z\|_{L^p} \|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \|\partial_r u_z\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} |M_3| \leq & C \left\|\frac{u_r}{r}\right\|_{L^p} \|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \left\|\frac{u_r}{r}\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2} \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} |M_{15}| + |M_{16}| + |M_{22}| \leq & C \|\partial_z u_z\|_{L^p} \|\nabla v\|^2_{L^{\frac{2p}{p-1}}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \|\partial_z u_z\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2}. \end{aligned} \tag{3.38}$$

It is different to estimate M_5 and M_{17}. By applying the Young inequality and the *Hölder* inequality, we have

$$\begin{aligned} |M_5| + |M_{17}| = & |\int_{\mathbb{R}^3} \partial_r u_r \partial_r^2 v_r v_r \, dx| + |\int_{\mathbb{R}^3} v_r \partial_r \partial_z v_z \partial_r u_r \, dx| \\ \leq & C \|\partial_r u_r\|_{L^p} \|v\|_{L^6} \left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ \leq & C \|\partial_r u_r\|_{L^p} \|\nabla v\|_{L^2} \left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ \leq & C \|\partial_r u_r\|_{L^p} \|\nabla v\|_{L^2}^{\frac{2p\beta-3}{p\beta}} \left\|\Lambda^{\beta+1} v\right\|_{L^2}^{\frac{3}{p\beta}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \|\partial_r u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2}. \end{aligned} \tag{3.39}$$

Here, we have also used the following Gagliardo-Nirenberg inequality

$$\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \leq C \|\nabla v\|_{L^2}^{1-\frac{3}{p\beta}} \left\|\Lambda^{1+\beta} v\right\|_{L^2}^{\frac{3}{p\beta}}, \ \frac{3}{\beta} \leq p \leq 3. \tag{3.40}$$

Similary, we have

$$\begin{aligned} |M_7| + |M_{19}| \leq & C \|\partial_z u_r\|_{L^p} \|v\|_{L^6} \left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \|\partial_z u_r\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} |M_9| + |M_{21}| \leq & C \|\partial_r u_z\|_{L^p} \|v\|_{L^6} \left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}} \\ \leq & \frac{1}{72} \left\|\Lambda^{\beta+1} v\right\|^2_{L^2} + C \|\partial_r u_z\|_{L^p}^{\frac{2p\beta}{2p\beta-3}} \|\nabla v\|^2_{L^2} \end{aligned} \tag{3.42}$$

and

$$\begin{aligned}|M_{10}|+|M_{23}| \leq& C\left\|\partial_z u_z\right\|_{L^p}\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\ \leq& \frac{1}{72}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_z u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2 .\end{aligned} \tag{3.43}$$

Hence, combining the above estimates (3.33)-(3.43), we obtain

$$\begin{aligned}|I_3| \leq& \frac{1}{8}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C(\left\|\frac{u_r}{r}\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}+\left\|\partial_r u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\\ &+\left\|\partial_z u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}+\left\|\partial_r u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}+\left\|\partial_z u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}})\|\nabla v\|_{L^2}^2 .\end{aligned} \tag{3.44}$$

For I_4, applying a derivation similar to I_2, we have

$$\begin{aligned}I_4=&\int_{\mathbb{R}^3}(u \cdot \nabla) v \cdot \Delta v\, dx=\int_{\mathbb{R}^3}(u_r \partial_r v_r \partial_r^2 v_r+\frac{1}{r} u_r \partial_r v_r \partial_r v_r+u_r \partial_r v_r \partial_z^2 v_r\\ &+u_z \partial_z v_r \partial_r^2 v_r+\frac{1}{r} u_z \partial_z v_r \partial_r v_r+u_z \partial_z v_r \partial_z^2 v_r+u_r \partial_r v_z \partial_r^2 v_z+\frac{1}{r} u_r \partial_r v_z \partial_r v_z\\ &+u_r \partial_r v_z \partial_z^2 v_z+u_z \partial_z v_z \partial_r^2 v_z+\frac{1}{r} u_z \partial_z v_z \partial_r v_z+u_z \partial_z v_z \partial_z^2 v_z)\, dx.\end{aligned} \tag{3.45}$$

Further simplifying (3.45) by using integration by parts and (3.7), we obtain

$$\begin{aligned}I_4=&\int_{\mathbb{R}^3}[-\partial_r u_r(\partial_r v_r)^2-\partial_r u_r(\partial_r v_z)^2-\partial_z v_r \partial_z u_r \partial_r v_r-\partial_r v_r \partial_z v_r \partial_r u_z\\ &-(\partial_z v_r)^2 \partial_z u_z-(\partial_z v_z)^2 \partial_z u_z-\partial_z v_z \partial_z u_r \partial_r v_z-\partial_r v_z \partial_z v_z \partial_r u_z]\, dx\\ :=&N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8.\end{aligned} \tag{3.46}$$

We will estimate the right-hand side of (3.46) one by one. First, we estimate N_1 and N_2 as follows

$$\begin{aligned}|N_1|+|N_2| &= |\int_{\mathbb{R}^3}(\partial_r v_r)^2 \partial_r u_r\, dx|+|\int_{\mathbb{R}^3}(\partial_r v_z)^2 \partial_r u_r\, dx|\\ &\leq\left\|\partial_r u_r\right\|_{L^p}\|\nabla v\|_{L^{\frac{2p}{p-1}}}^2\\ &\leq C\left\|\partial_r u_r\right\|_{L^p}\|\nabla v\|_{L^2}^{\frac{2p\beta-3}{p\beta}}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^{\frac{3}{p\beta}}\\ &\leq \frac{1}{32}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_r u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2,\end{aligned} \tag{3.47}$$

where we have used the following Gagliardo-Nirenberg inequality

$$\|\nabla v\|_{L^{\frac{2p}{p-1}}} \leq C\|\nabla v\|_{L^2}^{\frac{2p\beta-3}{2p\beta}}\left\|\Lambda^{1+\beta} v\right\|_{L^2}^{\frac{3}{2p\beta}},\ \frac{3}{2\beta} \leq p<\infty. \tag{3.48}$$

Similary, we have

$$\begin{aligned}|N_4|+|N_8| \leq& C\left\|\partial_r u_z\right\|_{L^p}\|\nabla v\|_{L^{\frac{2p}{p-1}}}^2\\ \leq& \frac{1}{32}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_r u_z\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2,\end{aligned} \tag{3.49}$$

$$\begin{aligned}|N_3|+|N_7| \leq& C\left\|\partial_z u_r\right\|_{L^p}\|\nabla v\|_{L^{\frac{2p}{p-1}}}^2\\ \leq& \frac{1}{32}\left\|\Lambda^{\beta+1} v\right\|_{L^2}^2+C\left\|\partial_z u_r\right\|_{L^p}^{\frac{2p\beta}{2p\beta-3}}\|\nabla v\|_{L^2}^2\end{aligned} \tag{3.50}$$

and

$$\begin{aligned}|N_5|+|N_6| \leq & C\|\partial_z u_z\|_{L^p}\|\nabla v\|^2_{L^{\frac{2p}{p-1}}}\\ \leq & \frac{1}{32}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C\|\partial_z u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}\|\nabla v\|^2_{L^2}.\end{aligned} \tag{3.51}$$

Therefore, putting (3.47)-(3.51) to (3.46), we have

$$\begin{aligned}|I_4| \leq & \frac{1}{8}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C(\|\partial_r u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_z u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_r u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}\\ &+\|\partial_z u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p})\|\nabla v\|^2_{L^2}.\end{aligned} \tag{3.52}$$

For I_5, applying a derivation similar to I_4, we have

$$\begin{aligned}I_5=&\int_{\mathbb{R}^3}(v\cdot\nabla)u\cdot\Delta v\,dx=\int_{\mathbb{R}^3}[(v_r\partial_r u_r+v_z\partial_z u_r)(\partial_r^2 v_r+\frac{1}{r}\partial_r v_r+\partial_z^2 v_r)\\ &+(v_r\partial_r u_z+v_z\partial_z u_z)(\partial_r^2 v_z+\frac{1}{r}\partial_r v_z+\partial_z^2 v_z)]\\ =&\int_{\mathbb{R}^3}(v_r\partial_r u_r\partial_r^2 v_r+\frac{v_r}{r}\partial_r u_r\partial_r v_r+v_r\partial_r u_r\partial_z^2 v_r+v_z\partial_z u_r\partial_r^2 v_r\\ &+\frac{v_z}{r}\partial_z u_r\partial_r v_r+v_z\partial_z u_r\partial_z^2 v_r+v_r\partial_r u_z\partial_r^2 v_z+\frac{v_r}{r}\partial_r u_z\partial_r v_z\\ &+v_r\partial_r u_z\partial_z^2 v_z+v_z\partial_z u_z\partial_r^2 v_z+\frac{v_z}{r}\partial_z u_z\partial_r v_z+v_z\partial_z u_z\partial_z^2 v_z)\,dx.\end{aligned} \tag{3.53}$$

Then, further simplifying (3.53) by using integration by parts and (3.7), we obtain

$$\begin{aligned}I_5=&\int_{\mathbb{R}^3}(v_r\partial_r u_r\partial_r^2 v_r+\frac{v_r}{r}\partial_r u_r\partial_r v_r+v_r\partial_r u_r\partial_z^2 v_r-\partial_r v_r\partial_z u_r\partial_r v_z+\partial_r u_r\partial_r v_r\partial_z v_z\\ &+\partial_r u_r\partial_z\partial_r v_r v_z+v_z\partial_z u_r\partial_z^2 v_r+v_r\partial_r u_z\partial_r^2 v_z+\frac{v_r}{r}\partial_r u_z\partial_r v_z+v_r\partial_r u_z\partial_z^2 v_z\\ &-\partial_r v_z\partial_z u_z\partial_r v_z+\partial_r u_z\partial_r v_z\partial_z v_z+\partial_r u_z\partial_z\partial_r v_z v_z+v_z\partial_z u_z\partial_z^2 v_z)\,dx\\ :=&P_1+P_2+\ldots+P_{14}.\end{aligned} \tag{3.54}$$

We will estimate the right-hand side of (3.54) one by one. First, we estimate P_2 as follows

$$\begin{aligned}|P_2|=&|\int_{\mathbb{R}^3}\frac{v_r}{r}\partial_r u_r\partial_r v_r\,dx|\\ \leq&\|\partial_r u_r\|_{L^p}\|\nabla v\|^2_{L^{\frac{2p}{p-1}}}\\ \leq& C\|\partial_r u_r\|_{L^p}\|\nabla v\|^{\frac{2p\beta-3}{p\beta}}_{L^2}\left\|\Lambda^{\beta+1}v\right\|^{\frac{3}{p\beta}}_{L^2}\\ \leq&\frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C\|\partial_r u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}\|\nabla v\|^2_{L^2},\end{aligned} \tag{3.55}$$

where we have used the following Gagliardo-Nirenberg inequality

$$\|\nabla v\|_{L^{\frac{2p}{p-1}}}\leq C\|\nabla v\|^{\frac{2p\beta-3}{2p\beta}}_{L^2}\left\|\Lambda^{1+\beta}v\right\|^{\frac{3}{2p\beta}}_{L^2},\ \frac{3}{2\beta}\leq p<\infty. \tag{3.56}$$

Similary, we get

$$\begin{aligned}|P_4|+|P_5|\leq&C(\|\partial_r u_r\|_{L^p}+\|\partial_z u_r\|_{L^p})\|\nabla v\|^2_{L^{\frac{2p}{p-1}}}\\ \leq&\frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C(\|\partial_r u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_z u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p})\|\nabla v\|^2_{L^2},\end{aligned} \tag{3.57}$$

$$\begin{aligned}|P_9|+|P_{11}|+|P_{12}| &\le C(\|\partial_r u_z\|_{L^p}+\|\partial_z u_z\|_{L^p})\|\nabla v\|^2_{L^{\frac{2p}{p-1}}}\\&\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C(\|\partial_r u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_z u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p})\|\nabla v\|^2_{L^2}.\end{aligned}\tag{3.58}$$

It is different to estimate P_1 and P_3. By using the Sobolev inequality, we have

$$\begin{aligned}|P_1|+|P_3| &= |\int_{\mathbb{R}^3}\partial_r u_r\partial_r^2 v_r v_r\,dx|+|\int_{\mathbb{R}^3}\partial_r u_r\partial_z^2 v_r v_r\,dx|\\&\le C\|\partial_r u_r\|_{L^p}\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\&\le C\|\partial_r u_r\|_{L^p}\|\nabla v\|_{L^2}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\&\le C\|\partial_r u_r\|_{L^p}\|\nabla v\|^{\frac{2p\beta-3}{p\beta}}_{L^2}\left\|\Lambda^{\beta+1}v\right\|^{\frac{3}{p\beta}}_{L^2}\\&\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C\|\partial_r u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}\|\nabla v\|^2_{L^2}.\end{aligned}\tag{3.59}$$

Here, we have also used the following Gagliardo-Nirenberg inequality

$$\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\le C\|\nabla v\|^{1-\frac{3}{p\beta}}_{L^2}\left\|\Lambda^{1+\beta}v\right\|^{\frac{3}{p\beta}}_{L^2},\ \frac{3}{\beta}\le p\le 3.\tag{3.60}$$

Similary, we have

$$\begin{aligned}|P_6|+|P_8|+|P_{10}|+|P_{13}| &\le C(\|\partial_r u_r\|_{L^p}+\|\partial_r u_z\|_{L^p})\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\&\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C(\|\partial_r u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_r u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p})\|\nabla v\|^2_{L^2},\end{aligned}\tag{3.61}$$

$$\begin{aligned}|P_7|+|P_{14}| &\le C(\|\partial_z u_r\|_{L^p}+\|\partial_z u_z\|_{L^p})\|v\|_{L^6}\left\|\nabla^2 v\right\|_{L^{\frac{6p}{5p-6}}}\\&\le \frac{1}{48}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C(\|\partial_z u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_z u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p})\|\nabla v\|^2_{L^2}.\end{aligned}\tag{3.62}$$

Hence, combining the above estimates (3.55)-(3.62), we obtain

$$\begin{aligned}|I_5| \le& \frac{1}{8}\left\|\Lambda^{\beta+1}v\right\|^2_{L^2}+C(\|\partial_r u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_r u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}+\|\partial_z u_r\|^{\frac{2p\beta}{2p\beta-3}}_{L^p}\\&+\|\partial_z u_z\|^{\frac{2p\beta}{2p\beta-3}}_{L^p})\|\nabla v\|^2_{L^2}.\end{aligned}\tag{3.63}$$

We will estimate I_6. For I_6, applying a derivation similar to I_5, we have

$$\begin{aligned}I_6 &= \int_{\mathbb{R}^3}(u\cdot\nabla)\psi\cdot\Delta\psi\,dx=\int_{\mathbb{R}^3}(u_r\partial_r\psi+u_z\partial_z\psi)(\partial_r^2\psi+\frac{1}{r}\partial_r\psi+\partial_z^2\psi)\\&=\int_{\mathbb{R}^3}(u_r\partial_r\psi\partial_r^2\psi+\frac{u_r}{r}(\partial_r\psi)^2+u_r\partial_r\psi\partial_z^2\psi+u_z\partial_z\psi\partial_r^2\psi+\frac{u_z}{r}\partial_r\psi\partial_z\psi\\&\quad+u_z\partial_z\psi\partial_z^2\psi)\,dx.\end{aligned}\tag{3.64}$$

Further simplifying (3.64) by using integration by parts and (3.7), we obtain

$$\begin{aligned}I_6 &= \int_{\mathbb{R}^3}[-(\partial_r\psi)^2\partial_r u_r-\partial_z\psi\partial_z u_r\partial_r\psi-(\partial_z\psi)^2\partial_z u_z-\partial_r\psi\partial_z\psi\partial_r u_z]\,dx\\&:=Q_1+Q_2+Q_3+Q_4.\end{aligned}\tag{3.65}$$

We estimate the right-hand side of (3.65) one by one. First, we estimate Q_1 as follows

$$
\begin{aligned}
|Q_1| &= |\int_{\mathbb{R}^3} (\partial_r \psi)^2 \partial_r u_r \, dx| \\
&\le \|\partial_r u_r\|_{L^p} \|\nabla \psi\|^2_{L^{\frac{2p}{p-1}}} \\
&\le C \|\partial_r u_r\|_{L^p} \|\nabla \psi\|_{L^2}^{\frac{2p\gamma-3}{p\gamma}} \left\|\Lambda^{\gamma+1}\psi\right\|_{L^2}^{\frac{3}{p\gamma}} \\
&\le \frac{1}{8} \left\|\Lambda^{\gamma+1}\psi\right\|^2_{L^2} + C \|\partial_r u_r\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} \|\nabla\psi\|^2_{L^2},
\end{aligned} \tag{3.66}
$$

where we have used the following Gagliardo-Nirenberg inequality

$$
\|\nabla \psi\|_{L^{\frac{2p}{p-1}}} \le C \|\nabla\psi\|_{L^2}^{\frac{2p\gamma-3}{2p\gamma}} \left\|\Lambda^{1+\gamma}\psi\right\|_{L^2}^{\frac{3}{2p\gamma}}, \ \frac{3}{2\gamma} \le p < \infty. \tag{3.67}
$$

Similary, we have

$$
\begin{aligned}
|Q_2| \le& C \|\partial_z u_r\|_{L^p} \|\nabla \psi\|^2_{L^{\frac{2p}{p-1}}} \\
\le& \frac{1}{8} \left\|\Lambda^{\gamma+1}\psi\right\|^2_{L^2} + C \|\partial_z u_r\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} \|\nabla\psi\|^2_{L^2},
\end{aligned} \tag{3.68}
$$

$$
\begin{aligned}
|Q_3| \le& C \|\partial_z u_z\|_{L^p} \|\nabla \psi\|^2_{L^{\frac{2p}{p-1}}} \\
\le& \frac{1}{8} \left\|\Lambda^{\gamma+1}\psi\right\|^2_{L^2} + C \|\partial_z u_z\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} \|\nabla\psi\|^2_{L^2}
\end{aligned} \tag{3.69}
$$

and

$$
\begin{aligned}
|Q_4| \le& C \|\partial_r u_z\|_{L^p} \|\nabla \psi\|^2_{L^{\frac{2p}{p-1}}} \\
\le& \frac{1}{8} \left\|\Lambda^{\gamma+1}\psi\right\|^2_{L^2} + C \|\partial_r u_z\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} \|\nabla\psi\|^2_{L^2}.
\end{aligned} \tag{3.70}
$$

Therefore, putting (3.66)-(3.70) to (3.65), we have

$$
\begin{aligned}
|I_6| \le& \frac{1}{2} \left\|\Lambda^{\gamma+1}\psi\right\|^2_{L^2} + C(\|\partial_r u_r\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} + \|\partial_r u_z\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} + \|\partial_z u_r\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}} \\
&+ \|\partial_z u_z\|_{L^p}^{\frac{2p\gamma}{2p\gamma-3}}) \|\nabla\psi\|^2_{L^2}.
\end{aligned} \tag{3.71}
$$

Combining the above estimates (3.14), (3.30), (3.44), (3.52), (3.63), (3.71) with (3.2), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2} + \|\nabla \psi\|^2_{L^2}) + \left\|\Lambda^{\alpha+1}u\right\|^2_{L^2} + \left\|\Lambda^{\beta+1}v\right\|^2_{L^2} + \left\|\Lambda^{\gamma+1}\psi\right\|^2_{L^2} \\
\le& C(\left\|\frac{u_r}{r}\right\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}} + \|\partial_r u_r\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}} + \|\partial_r u_z\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}} \\
&+ \|\partial_z u_r\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}} + \|\partial_z u_z\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}})(\|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2} + \|\nabla\psi\|^2_{L^2}).
\end{aligned} \tag{3.72}
$$

From Lemma 2.1, we get

$$
|\frac{u_r}{r}|, |\partial_r u_r|, |\partial_z u_r|, |\partial_r u_z|, |\partial_z u_z| \le |\nabla \tilde{u}|. \tag{3.73}
$$

By using (3.73), Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \psi\|_{L^2}^2)+\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+\left\|\Lambda^{\beta+1}v\right\|_{L^2}^2+\left\|\Lambda^{\gamma+1}\psi\right\|_{L^2}^2\\
\le& C(\left\|\frac{u_r}{r}\right\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}}+\|\partial_r u_r\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}}+\|\partial_r u_z\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}}\\
&+\|\partial_z u_r\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}}+\|\partial_z u_z\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}})(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \psi\|_{L^2}^2)\\
\le& C\|\nabla \tilde{u}\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}}(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \psi\|_{L^2}^2)\\
\le& C\|\omega_\theta\|_{L^p}^{\frac{2p\min\{\alpha,\beta,\gamma\}}{2p\min\{\alpha,\beta,\gamma\}-3}}(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \psi\|_{L^2}^2).
\end{aligned}\tag{3.74}$$

This completes the proof of Theorem 1.1 by (3.1) and Gronwall's inequality.

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