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On the Boundary of the Attraction Basin in a Class of Piecewise Linear Systems*

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Abstract In this paper, we investigate the boundary of the attraction basin of a class of piecewise linear systems arising from anti-stable linear systems with saturated linear state feedback. In three-dimensional cases, for this class of systems, we prove that the equilibrium points other than the origin lie on the boundary of the attraction basin of the origin. This gives strong evidence that the boundary of the attraction basin is homeomorphic to a sphere. Some examples are provided to illustrate the results.

Keywords Piecewise linear system, boundary of attraction basin, index, saturated state feedback

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1. Introduction

In piecewise smooth systems, saturation nonlinearities have attracted a lot of attention, because they are ubiquitous due to the inherent physical limitations of devices. For example, control systems with saturated feedback are such a class of piecewise smooth systems which have been widely studied in the literature [5, 8, 9, 12, 13, 16, 19–21, 25, 28–31, 33, 35–37]. It is easy to see that any stabilizing linear feedback would locally stabilize a linear system with saturated actuators in the presence of actuator saturation, and the attraction basin of the origin may be bounded [3, 11, 14, 15, 17, 26, 27].

The research on the properties of the boundary of the attraction basin is of great importance because of its practical significance and a theoretical challenge from the viewpoint of dynamical system. Clearly, the structure of the boundary of the attraction basin provides much information on the dynamical behaviour of

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trajectories before approaching the origin. It is a well-known fact that a fractal boundary usually gives rise to the transient chaotic behaviour for trajectories near the boundary [1,2,18,23].

The attraction basin problem of the anti-stable linear control system with saturated state feedback is a typical problem in control systems. For a planar anti-stable linear system with saturated stabilizing state feedback, Hu, Lin and Qiu [16] showed that the boundary of the attraction basin of the closed loop system is a convex limit cycle. In view of this result, it is reasonable to conjecture that the boundary of the attraction basin in a three-dimensional anti-stable linear system with saturated stabilizing state feedback is homeomorphic to a sphere. Our numerical simulations on a lot of specific systems support this conjecture. Suppose the boundary of the attraction basin of the origin is homeomorphic to a sphere, then according to the Poincaré–Hopf theorem and its corollary [10,22], there generically exist at least two equilibrium points on the boundary of the attraction basin. Therefore, in order to verify the above conjecture, it is natural to prove that the equilibrium points (if exist) other than the origin are indeed contained in the boundary of the attraction basin as a first step.

To the best of our knowledge, no study on the boundary of the attraction basin of three-dimensional anti-stable linear systems with saturated stabilizing state feedback or even higher dimensions has appeared in the literature. Since the existence and distribution of equilibrium points affect estimation of the attraction basin, we investigate the properties of equilibrium points in such a class of n-dimensional piecewise linear systems. We prove that such an n-dimensional system has a unique equilibrium point if n is an even number, and has three equilibrium points if n is an odd number. Furthermore, for three-dimensional cases, we prove that two equilibrium points other than the origin are contained in the boundary of the attraction basin of the origin.

The paper is organized as follows. Some preliminaries regarding control systems with saturated stabilizing state feedback are presented in Section 2. The main results are established in Section 3, and the proofs are given in Section 4 and Section 5. Section 6 offers some examples to illustrate the main results, while Subsection 6.3 gives the steps of finding the boundary of the attraction basin. A brief conclusion is given in Section 7.

2. Preliminaries

A matrix is called to be Hurwitz, if all of its eigenvalues have negative real parts. A matrix is called to be anti-stable, if all of its eigenvalues have positive real parts.

Consider the piecewise smooth system of the following form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n},\ \|\mathbf{u}\| = \max\{|u_i|\} \le M$, where $\mathbf{u} = (u_1, \cdots, u_m)^T \in \mathbb{R}^m,\ M > 0$. Define sat: $\mathbb{R} \to \mathbb{R}$ as

$$\text{sat}(s) = \text{sign}(s)\min\{M,\ |s|\},$$

and for $\mathbf{u} \in \mathbb{R}^m$,

$$\text{sat}(\mathbf{u}) = (\text{sat}(u_1),\ \cdots,\ \text{sat}(u_m))^T,$$

where $(\cdot)^T$ means the transpose. The feedback law $\mathbf{u} = \text{sat}(\mathbf{K}\mathbf{x})$ is said to be stabilizing, if $\mathbf{A} + \mathbf{B}\mathbf{K}$ is Hurwitz. Assume that the system $(\mathbf{A}, \mathbf{B})$ is controllable.

Then, according to eigenvalue placement theorem [31], system (2.1) is stabilizable by linear state feedback, which means that there exists $\mathbf{K} \in \mathbb{R}^{m\times n}$ such that $\mathbf{A}+\mathbf{BK}$ is Hurwitz.

It is easy to see that under stabilizing saturated state feedback $\mathbf{u} = \mathrm{sat}(\mathbf{Kx})$, and the closed loop system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}\,\mathrm{sat}(\mathbf{Kx}) \tag{2.2}$$

has the origin as its asymptotically stable equilibrium point $\mathbf{x}_{e0}$. This makes up a class of piecewise linear systems.

In this paper, what we are interested is the structure of the boundary of the attraction basin of the origin of system (2.2), when $\mathbf{A}$ is anti-stable. It is not difficult to obtain that the attraction basin of the origin of system (2.2) is bounded, if $\mathbf{A}$ is anti-stable (see [14] for more details). The attraction basin is defined as follows.

Definition 2.1. The attraction basin of the stable equilibrium point $\mathbf{x}_{e0}$ of system (2.2) is defined as

$$\mathcal{D}^{\mathbf{K}} = \{\mathbf{x}_0 \in \mathbb{R}^n \mid \lim_{t\to+\infty} \phi^{\mathbf{K}}(t,\mathbf{x}_0) = \mathbf{x}_{e0}\},$$

where $\phi^{\mathbf{K}}(t,\mathbf{x}_0)$ is the solution of system (2.2) with the initial condition $\phi^{\mathbf{K}}(0,\mathbf{x}_0) = \mathbf{x}_0$. The boundary of $\mathcal{D}^{\mathbf{K}}$ is denoted by $\partial\mathcal{D}^{\mathbf{K}}$.

It is easy to get that $\mathcal{D}^{\mathbf{K}}$ is an open set and $\partial\mathcal{D}^{\mathbf{K}}$ is an invariant set.

3. Statements of main results

On one hand, it is interesting to study the boundary of the attraction basin from both theoretical and practical points of view. On the other hand, the locations of other (unstable) equilibrium points are also of some interest, because the "size" of the attraction basin cannot be large to contain the equilibrium points other than the origin. As noted in [14], system (2.2) may have 3^m "potentia" equilibrium points. However, only some of them are real equilibrium points.

The multiple inputs are general cases, but they are very complicated and difficult to study. There are no systematic theoretical results even in the single input case. Thus, before going further into the multiple inputs case, we consider the single input case first.

Definition 3.1. Consider the equation

$$\mathbf{Ax} + \mathbf{b}\mathrm{sat}(\mathbf{bx}) = 0. \tag{3.1}$$

A zero point of (3.1) is said to be in general position, if it is not on the plane $\mathbf{kx} = \pm M$.

Theorem 3.1. *For the following piecewise smooth system*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u,\ \boldsymbol{x} \in \mathbb{R}^n, \tag{3.2}$$

where $\boldsymbol{A}$ is anti-stable, let $u = sat(\boldsymbol{kx})$ be a stabilizing state feedback for (3.2). *Then generically the system*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\ sat(\boldsymbol{kx}) \tag{3.3}$$

has a unique equilibrium point, the origin, if n is an even number, and has three equilibrium points if n is an odd number.

Theorem 3.1 is proved in Section 4.

Since our concern with the equilibrium points of system (3.3) is how to characterize the boundary of the attraction basin of the origin, we will give a brief discussion on the boundary property in this section.

According to Theorem 3.1 and Poincaré–Hopf index theorem, the following is obvious.

Proposition 3.1. *For piecewise linear system* (3.3)*, if the attraction basin of the origin is homeomorphic to $S^{n-1}(1)$, then the equilibrium points other than the origin all lie on the boundary if n is odd, and no equilibrium point lies on the boundary, if n is even.*

It is known that the structure of the boundary of the attraction basin provides a measure of how a trajectory in the attraction basin approaches the origin. If the boundary of the attraction basin is homeomorphic to a sphere, then the equilibrium points other than the origin lie on the boundary of the attraction basin. In order to study the property of the boundary of the attraction basin of system (3.3), as the first step, we will prove that the equilibrium points other than the origin all lie on the boundary in three-dimensional cases. The main result is shown in the following.

Theorem 3.2. *Consider a class of three-dimensional piecewise linear system*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\, sat(\boldsymbol{k}\boldsymbol{x}), \tag{3.4}$$

where $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^3$. If $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ is anti-stable and $(\boldsymbol{A}, \boldsymbol{b})$ is controllable, then system (3.4) with any saturated stabilizing state feedback $u = sat(\boldsymbol{k}\boldsymbol{x})$ has three equilibrium points, and the two equilibrium points other than the origin must lie on the boundary of the attraction basin of the origin.

The proof of Theorem 3.2 will be given in Section 5.

4. Proof of Theorem 3.1

We will complete the proof of Theorem 3.1 using Proposition 4.1 and Theorem 4.1 given in this section (see [6,22] for the proofs of Proposition 4.1 and Theorem 4.1). The detailed proving process can be found in [34].

Consider a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there is an $r > 0$ such that $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in S^{n-1}(r)$, where

$$S^{n-1}(r) = \{\mathbf{x} \in \mathbb{R}^n |\, \|\mathbf{x}\| = r\},$$

where $\|\cdot\|$ means a vector norm. Define the sphere map $\bar{f} : S^{n-1}(r) \to S^{n-1}(1)$ as

$$\bar{f}(\mathbf{x}) = \frac{f(\mathbf{x})}{\|f(\mathbf{x})\|},\ \mathbf{x} \in S^{n-1}(r).$$

Definition 4.1. The index of f, denoted by $\operatorname{ind} f_{B(r)}$, and $B(r)$ is defined as the topological degree of $\bar{f}$, where

$$B(r) = \{\mathbf{x} \in \mathbb{R}^n |\, \|\mathbf{x}\| \leq r\}.$$

Proposition 4.1. *Let $\boldsymbol{A}$ be a nonsingular matrix. Suppose that $g : \mathbb{R}^n \to \mathbb{R}^n$ is bounded. Then there is an $r > 0$ such that the map $\boldsymbol{A}\boldsymbol{x} + g(\boldsymbol{x})$ restricted to $B(r)$, has index $(-1)^m$, where m is the number of the eigenvalues of $\boldsymbol{A}$ with negative real parts.*

This fact is obvious, and for the map $\bar{A} = \frac{Ax}{\|Ax\|}$, one has $\operatorname{ind} \bar{A}_{B(r)} = \operatorname{sign}(\det A)$ by arguments in [22].

Suppose that f has only isolated zero points. Letting $\bar{\mathbf{x}}$ be a zero point of f, the index of at $\bar{\mathbf{x}}$ is defined as

$$\operatorname{ind} f(\bar{\mathbf{x}}) = \text{degree of } \bar{f} : S^{n-1}(\bar{\mathbf{x}}, \delta) \to S^{n-1}(1),$$

where

$$S^{n-1}(\bar{\mathbf{x}}, \delta) = \{\mathbf{x} \in \mathbb{R}^n | \, \|\mathbf{x} - \bar{\mathbf{x}}\| = \delta\}$$

and

$$\bar{f}(\mathbf{x}) = \frac{f(\mathbf{x})}{\|f(\mathbf{x})\|}, \ \mathbf{x} \in S^{n-1}(\bar{\mathbf{x}}, \delta).$$

We have the following theorem.

Theorem 4.1. *Suppose that $f : B(r) \to \mathbb{R}^n$ has the following properties. Then,*

1) $\|f(\boldsymbol{x})\| \neq 0$, for $\forall \boldsymbol{x} \in S^{n-1}(r)$.

2) Every zero point of f is isolated.

Denoted by E, the set of zero points of f in $B(r)$, then

$$\sum_{\bar{\boldsymbol{x}} \in E} \operatorname{ind}_f (\bar{\boldsymbol{x}}) = \operatorname{ind} f_{B(r)}.$$

Now, we are going to prove Theorem 3.1.

Proof. In generic cases, every equilibrium point is not on the hyperplane $\mathbf{k}\mathbf{x} = \pm M$. Hence the index of the origin is $(-1)^n$, and the index of the other equilibrium point is 1, because $\mathbf{A}$ is anti-stable, and all these equilibrium points are located off the saturated region.

Now, for r sufficiently large, following from Proposition 4.1, we have

$$\operatorname{ind}(\mathbf{A}\mathbf{x} + \mathbf{b}\operatorname{sat}(\mathbf{k}\mathbf{x}))_{B(r)} = \operatorname{ind}(\mathbf{A}\mathbf{x})_{B(r)} = 1.$$

Thus, by Theorem 4.1,

$$\sum_{\bar{\mathbf{x}} \in E} \operatorname{ind}_f (\bar{\mathbf{x}}) = 1, \ f = \mathbf{A}\mathbf{x} + \mathbf{b}\operatorname{sat}(\mathbf{k}\mathbf{x}).$$

It is easy to obtain that $\mathbf{A}\mathbf{x} + \mathbf{b}\operatorname{sat}(\mathbf{k}\mathbf{x})$ has a unique zero, if n is an even number, and has three zeros, if n is an odd number. Consequently, we complete the proof of Theorem 3.1. □

5. Proof of Theorem 3.2

To complete the proof of Theorem 3.2, we need several lemmas to prove the boundedness of the map $F(t, \mathbf{x}_h)$.

Lemma 5.1. *The function*

$$F_1(t) = \frac{e^{-t\gamma}\beta(\alpha^2+\beta^2) + e^{-t\alpha}\gamma[\beta(\gamma-2\alpha)\cos\beta t - (\alpha^2-\beta^2-\alpha\gamma)\sin\beta t]}{\beta[(\alpha-\gamma)^2+\beta^2]}$$

satisfies $|F_1(t)| < 1$, *for all* $t > 0$, *where* $\alpha, \beta, \gamma > 0$.

Proof. Differentiating $F_1(t)$ with respect to t gives

$$F_1'(t) = \frac{(\alpha^2+\beta^2)\gamma[-\beta e^{-t\gamma} + \beta e^{-t\alpha}\cos\beta t + (\alpha-\gamma)e^{-t\alpha}\sin\beta t]}{\beta[(\alpha-\gamma)^2+\beta^2]}.$$

Let

$$G(t) = -e^{-t(\gamma-\alpha)} + \cos\beta t + \frac{\alpha-\gamma}{\beta}\sin\beta t. \tag{5.1}$$

It is sufficient to study $G(t)$ instead of $F_1'(t)$ for considering the monotonicity of $F_1(t)$.

Letting $\alpha - \gamma = m\beta,\ \beta t = \tau$ and

$$\cos\theta = \frac{m}{\sqrt{1+m^2}},\ \sin\theta = \frac{1}{\sqrt{1+m^2}},$$

then $G(t) = \bar{G}(\tau)$, where

$$\bar{G}(\tau) = -e^{m\tau} + \sqrt{1+m^2}\sin(\tau+\theta)$$

and

$$\bar{G}'(\tau) = -me^{m\tau} + \sqrt{1+m^2}\cos(\tau+\theta).$$

First, we consider the case $m \geq 0$. On one hand, it is apparent to see that $\bar{G}'(\tau)$ monotonically decreases with respect to τ, for $\tau \in [0, \pi-\theta]$ and $\bar{G}'(0) = 0$. It follows that

$$\bar{G}'(\tau) < 0, \tau \in [0, \pi-\theta].$$

For $\bar{G}(0) = 0$, we have

$$\bar{G}(\tau) < 0, \tau \in [0, \pi-\theta].$$

On the other hand, it is easy to find

$$\sqrt{1+m^2}(\sin(\tau+\theta)) < 1, \tau \in (\pi-2\theta, 2\pi),$$

which means

$$\bar{G}(\tau) < 0, \tau \in (\pi-2\theta, 2\pi].$$

Thus, we get

$$\bar{G}(\tau) < 0, \tau \in (0, 2\pi].$$

In view of

$$\bar{G}(\tau+2n\pi) \leq \bar{G}(\tau),$$

where n is any positive integer, we obtain $\bar{G}(\tau) < 0$ for all $\tau > 0$, which implies that $F_1(t)$ monotonically decreases with respect to t for all $t \geq 0$. Since

$$\lim_{t\to+\infty} F_1(t) = 0,\ F_1(0) = 1,$$

we have

$$0 < F_1(t) < 1 \text{ for all } t \geq 0.$$

It remains to show $|F_1(t)| < 1$ for all $t > 0$, if $m < 0$. In this case, $\pi/2 < \theta < \pi$. Similar to the case of $m \geq 0$, we have

$$\bar{G}(\tau) < 0,\ \tau \in [0, \pi - \theta].$$

It is obvious to get

$$\bar{G}(\tau) < 0,\ \tau \in [\pi - \theta, 2\pi - \theta].$$

Since

$$\sqrt{1+m^2}\sin(\tau+\theta) \geq 1,\ \tau \in [3\pi - 2\theta, 2\pi],$$

we have

$$\bar{G}(\tau) > 0, \tau \in [3\pi - 2\theta, 2\pi].$$

It can be easily checked that $\bar{G}(\tau)$ monotonically increases with respect to τ for $\tau \in (2\pi-\theta, 3\pi-2\theta)$. Thus, $F_1(t)$ has a minimum when $t \in ((2\pi-\theta)/\beta, (3\pi-2\theta)/\beta)$. Thus, from $F_1(2\pi/\beta) < F_1(0)$, we have

$$F_1(t) < F_1(0) = 1,\ t \in (0, 2\pi/\beta].$$

It is easy to obtain

$$F_1(t + 2n\pi/\beta) < e^{-\frac{2n\alpha\pi}{\beta}} F_1(t),\ t \in [0, 2\pi/\beta], n \in \mathbb{N}.$$

Therefore, we have

$$F_1(t) < 1 \text{ for all } t > 0.$$

Next, we will show that $F_1(t)$ has a lower bound for $t > 0$. It is easy to see that $F_1(t)$ can be transformed into

$$\frac{e^{-t\alpha}(\alpha^2+\beta^2)}{(\alpha-\gamma)^2+\beta^2}\left(e^{-t(\gamma-\alpha)} - \cos\beta t - \frac{\gamma(\alpha^2-\beta^2-\alpha\gamma)}{\beta(\alpha^2+\beta^2)}\sin\beta t\right) + e^{-t\alpha}\cos\beta t. \quad (5.2)$$

Denote the minimum of $F_1(t)$ on $(\frac{(2n+1)\pi-\theta}{\beta}, \frac{(2n+3)\pi-2\theta}{\beta}), n \in \mathbb{N}$ by $\bar{t}_n$. Then, $\bar{t}_n$ satisfies

$$\sin\beta\bar{t}_n < 0,\ e^{-\bar{t}_n(\gamma-\alpha)} - \cos\beta\bar{t}_n = \frac{\alpha-\gamma}{\beta}\sin\beta\bar{t}_n. \quad (5.3)$$

By substituting (5.3) into (5.2), we have

$$F_1(\bar{t}_n) = e^{-\bar{t}_n\alpha}\left(\frac{\alpha}{\beta}\sin\beta\bar{t}_n + \cos\beta\bar{t}_n\right) \geq -e^{-\pi\alpha/\beta}\sqrt{\left(\frac{\alpha}{\beta}\right)^2 + 1} > -1.$$

Therefore, we conclude $|F_1(t)| < 1$, for all $t > 0$. □

Lemma 5.2. *Suppose $\lambda_3 < \lambda_2 < \lambda_1 < 0$. Then, the function*

$$F_2(t) = c_1 e^{\lambda_1 t} - c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t},$$

where

$$c_1 = \frac{\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},\quad c_2 = \frac{\lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)},\quad c_3 = \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$

satisfies $0 < F_2(t) < 1$, for all $t > 0$.

Proof. Multiplying $F_2(t)$ by $e^{-\lambda_3 t}$, we have

$$e^{-\lambda_3 t}F_2(t) = c_1 e^{(\lambda_1-\lambda_3)t} - c_2 e^{(\lambda_2-\lambda_3)t} + c_3. \tag{5.4}$$

Letting $g_1(t)$ denote the right hand side of (5.4), the derivative of $g_1(t)$ with respect to t is given as

$$g_1'(t) = e^{(\lambda_2-\lambda_3)t}\frac{\lambda_2\lambda_3 e^{(\lambda_1-\lambda_2)t} - \lambda_1\lambda_3}{\lambda_1-\lambda_2} > 0.$$

It is apparent to see

$$g_1(t) > g_1(0) = 1,\ t > 0.$$

Thus, we get $F_2(t) > 0$ for all $t > 0$.

The derivative of $F_2(t)$ is

$$F_2'(t) = c_1\lambda_1 e^{\lambda_1 t} - c_2\lambda_2 e^{\lambda_2 t} + c_3\lambda_3 e^{\lambda_3 t}.$$

Similarly, letting

$$g_2(t) = e^{-\lambda_3 t}F_2'(t) = c_1\lambda_1 e^{(\lambda_1-\lambda_3)t} - c_2\lambda_2 e^{(\lambda_2-\lambda_3)t} + c_3\lambda_3,$$

we have

$$g_2'(t) = \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1-\lambda_2}[e^{(\lambda_1-\lambda_3)t} - e^{(\lambda_2-\lambda_3)t}] < 0.$$

It follows that

$$g_2(t) < g_2(0) = 0.$$

Thus, $F_2(t) < F_2(0) = 1$, for all $t > 0$. □

Lemma 5.3. *Suppose that* $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 < 0$. *Then the function*

$$F_3(t) = e^{\lambda_1 t}(d_1 + d_2 t) + d_3 e^{\lambda_2 t},$$

where

$$d_1 = \frac{\lambda_2(\lambda_2 - 2\lambda_1)}{(\lambda_1-\lambda_2)^2}, \quad d_2 = \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}, \quad d_3 = \frac{\lambda_1^2}{(\lambda_1-\lambda_2)^2}$$

satisfies $0 < F_3(t) < 1$, *for all* $t > 0$.

Proof. Suppose $\lambda_2 < \lambda_1$. Then letting

$$g_3(t) = e^{-\lambda_2 t}F_3(t) = e^{(\lambda_1-\lambda_2)t}(d_1 + d_2 t) + d_3,$$

it is easy to find that the derivative of $g_3(t)$ with respect to t is

$$g_3'(t) = e^{(\lambda_1-\lambda_2)t}[d_2 + (\lambda_1-\lambda_2)(d_1+d_2 t)] = e^{(\lambda_1-\lambda_2)t}(\lambda_1\lambda_2 t - \lambda_2) > 0.$$

It follows that

$$g_3(t) > g_3(0) = 1,\ t > 0,$$

which implies $F_3(t) > 0$, for all $t > 0$.

If $\lambda_1 < \lambda_2$, then we have

$$e^{-\lambda_1 t}F_3(t) \geq (1 + (\lambda_2-\lambda_1)t)d_3 + d_1 + d_2 t = 1 + \frac{\lambda_1^2 - \lambda_1\lambda_2}{\lambda_2-\lambda_1}t > 0.$$

Thus, $F_3(t) > 0$, for all $t > 0$.

It remains to show that $F_3(t)$ has an upper bound of less than 1, for $t > 0$. It can be easily seen that

$$F_3'(t) = e^{\lambda_1 t}(d_2 + \lambda_1 d_1 + \lambda_1 d_2 t) + \lambda_2 d_3 e^{\lambda_2 t}.$$

If $\lambda_2 < \lambda_1$, let

$$g_4(t) = e^{-\lambda_2 t}F_3'(t) = e^{(\lambda_1-\lambda_2)t}(d_2 + \lambda_1 d_1 + \lambda_1 d_2 t) + \lambda_2 d_3,$$

and then the derivative of $g_4(t)$ with respect to t is given by

$$g_4'(t) = e^{(\lambda_1-\lambda_2)t}(\lambda_1 - \lambda_2)\lambda_1 d_2 t < 0.$$

It is easy to obtain

$$g_4(t) < g_4(0) = 0.$$

Thus, $F_3'(t) < 0$, for all $t > 0$.

If $\lambda_1 < \lambda_2$, then

$$e^{-\lambda_1 t}F_3'(t) < [1 + (\lambda_2 - \lambda_1)t]\lambda_2 d_3 + d_2 + \lambda_1 d_1 + \lambda_1 d_2 t = 0,$$

which implies $F_3'(t) < 0$, for all $t > 0$.

In conclusion, we have

$$F_3(t) < F_3(0) = 1, \text{ for all } t > 0.$$

□

We are now turning to the proof of Theorem 3.2. For simplicity of presentation, some notations are defined as:

$$\begin{aligned}\Sigma^{\mathbf{k}}_{\pm} =& \{\mathbf{x} \in \mathbb{R}^3 \mid \pm\mathbf{kx} = M\},\ \Sigma^{\mathbf{k}}_{\pm,out} = \{\mathbf{x} \in \mathbb{R}^3 \mid \pm\mathbf{kx} > M\},\\ \Sigma^{\mathbf{k}}_{in} =& \{\mathbf{x} \in \mathbb{R}^3 \mid -M \leq \mathbf{kx} \leq M\}.\end{aligned}$$

Proof. It is easy to find that if $(\mathbf{A}, \mathbf{b})$ is controllable, and $u = \mathrm{sat}(\mathbf{kx})$ is stabilizing, then system (3.4) has three equilibrium points: $\mathbf{x}_{e+} = -\mathbf{A}^{-1}\mathbf{b}$, $\mathbf{x}_{e-} = \mathbf{A}^{-1}\mathbf{b}$, and $\mathbf{x}_{e0} = \mathbf{0}$. Because all the eigenvalues of $\mathbf{A} + \mathbf{bk}$ have negative real parts, and all the eigenvalues of $\mathbf{A}$ have positive real parts, $\mathbf{x}_{e0}$ is asymptotically stable, and $\mathbf{x}_{e+}, \mathbf{x}_{e-}$ are repelling.

Suppose ρ_1, ρ_2, ρ_3, where Re $\rho_i > 0, i = 1, 2, 3$, are the eigenvalues of $\mathbf{A}$ and $\lambda_1, \lambda_2, \lambda_3$, where Re $\lambda_i < 0, i = 1, 2, 3$ are the eigenvalues of $\mathbf{A} + \mathbf{bk}$. Then, for different $\mathbf{k}$, the eigenvalues of $\mathbf{A} + \mathbf{bk}$ may be one of the following four cases.

i) two complex eigenvalues $\lambda_1 = \mu + i\nu, \lambda_2 = \mu - i\nu, \mu < 0, \nu > 0$;

ii) distinct negative eigenvalues $\lambda_i \neq \lambda_j, i \neq j,\ i, j = 1, 2, 3$;

iii) two different negative eigenvalues $\lambda_1 = \lambda_2 \neq \lambda_3$;

iv) multiple negative eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

Let $\mathcal{D}^{\mathbf{k}}$ be the attraction basin of $\mathbf{x}_{e0}$, and let $\partial\mathcal{D}^{\mathbf{k}}$ be the boundary of $\mathcal{D}^{\mathbf{k}}$. Since system (3.4) is symmetric about the origin, the boundary of the attraction basin of the origin is also symmetric about the origin. Therefore, we have that if $\mathbf{x}_{e+} \in \partial\mathcal{D}^{\mathbf{k}}$, then $\mathbf{x}_{e-} \in \partial\mathcal{D}^{\mathbf{k}}$. These two equilibrium points always come in pairs. To complete the proof of this theorem, it is sufficient to show that there exists a heteroclinic orbit between $\mathbf{x}_{e+}$ and $\mathbf{x}_{e0}$. In other words, we need to find a special point $\mathbf{x}$ such that

$$\lim_{t\to+\infty}\phi^{\mathbf{k}}(t,\mathbf{x})=\mathbf{x}_{e0},\ \lim_{t\to-\infty}\phi^{\mathbf{k}}(t,\mathbf{x})=\mathbf{x}_{e+},$$

where $\phi^{\mathbf{k}}(t,\mathbf{x})$ is the solution of system (3.4).

The special point we choose satisfies

$$\mathbf{kx}=M,\ \mathbf{k}(\mathbf{A}+\mathbf{bk})\mathbf{x}=0,\ \mathbf{k}(\mathbf{A}+\mathbf{bk})^2\mathbf{x}=0. \tag{5.5}$$

Denote this point by $\mathbf{x}_h$, which satisfies $\mathbf{x}_h\in\Sigma_+^{\mathbf{k}}$.

Consider the function

$$F(t,\mathbf{x}_h)=\mathbf{k}\phi^{\mathbf{k}}(t,\mathbf{x}_h).$$

If

$$|F(t,\mathbf{x}_h)|\le M,\quad\text{for all } t>0,$$

then the trajectory of $\phi^{\mathbf{k}}(t,\mathbf{x}_h)$ stays within $\Sigma_{in}^{\mathbf{k}}$, which implies

$$\lim_{t\to+\infty}\phi^{\mathbf{k}}(t,\mathbf{x}_h)=\mathbf{x}_{e0}.$$

If

$$F(t,\mathbf{x}_h)\ge M,\ \text{for all } t\le 0,$$

then the trajectory of $\phi^{\mathbf{k}}(t,\mathbf{x}_h)$ lies within $\Sigma_{+,out}^{\mathbf{k}}$ for all $t\le 0$. Since the backward trajectory of $\mathbf{x}_h$ cannot tent to infinity at $t\to-\infty$ and the periodic orbits of system (3.4) cannot lie within $\Sigma_{+,out}^{\mathbf{k}}$ entirely, we have

$$\phi^{\mathbf{k}}(t,\mathbf{x}_h)\to\mathbf{x}_{e+},\text{as } t\to-\infty.$$

Since $(\mathbf{A},\mathbf{b})$ is controllable, we get that system (3.4) can be transformed into the following companion system

$$\dot{\mathbf{z}}=\bar{\mathbf{A}}\mathbf{z}+\bar{\mathbf{b}}u=\begin{bmatrix}0&1&0\\0&0&1\\a_1&a_2&a_3\end{bmatrix}\mathbf{z}+\begin{bmatrix}0\\0\\1\end{bmatrix}\text{sat}(\bar{\mathbf{k}}\mathbf{z}), \tag{5.6}$$

where

$$a_1=\rho_1\rho_2\rho_3, a_2=-\rho_1\rho_2-\rho_2\rho_3-\rho_1\rho_3, a_3=\rho_1+\rho_2+\rho_3$$

and

$$\bar{\mathbf{k}}=\mathbf{kT}^{-1}=\begin{bmatrix}-a_1+\lambda_1\lambda_2\lambda_3\\-a_2-\lambda_1\lambda_2-\lambda_1\lambda_2-\lambda_2\lambda_3\\-a_3+\lambda_1+\lambda_2+\lambda_3\end{bmatrix}^T$$

by the nonsingular linear transformation $\mathbf{z}=\mathbf{Tx},\mathbf{T}=(\mathbf{b}\quad\mathbf{Ab}\quad\mathbf{A}^2\mathbf{b})$.

Let $\psi^{\bar{\mathbf{k}}}(t,\mathbf{z}_0)$ denote the solution of system (5.6) with the initial condition $\psi^{\bar{\mathbf{k}}}(0,\mathbf{z}_0)=\mathbf{z}_0=\mathbf{Tx}_0$. It is easy to see

$$F(t,\mathbf{x}_0)=\mathbf{k}\phi^{\mathbf{k}}(t,\mathbf{x}_0)=\bar{\mathbf{k}}\psi^{\bar{\mathbf{k}}}(t,\mathbf{z}_0),$$

since

$$\bar{\mathbf{A}} = \mathbf{TAT}^{-1}, \quad \bar{\mathbf{b}} = \mathbf{Tb}, \quad \bar{\mathbf{k}} = \mathbf{kT}^{-1}.$$

Therefore, we have

$$F(t, \mathbf{x}_h) = \bar{\mathbf{k}}\psi^{\bar{\mathbf{k}}}(t, \mathbf{z}_h), \tag{5.7}$$

where $\mathbf{z}_h = \mathbf{T}\mathbf{x}_h$ satisfies

$$\bar{\mathbf{k}}\mathbf{z}_h = M,\ \bar{\mathbf{k}}(\bar{\mathbf{A}} + \bar{\mathbf{b}}\bar{\mathbf{k}})\mathbf{z}_h = 0,\ \bar{\mathbf{k}}(\bar{\mathbf{A}} + \bar{\mathbf{b}}\bar{\mathbf{k}})^2\mathbf{z}_h = 0.$$

We will show that the point $\mathbf{x}_h$ determined by (5.5) satisfies

$$\lim_{t\to+\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e0}, \lim_{t\to-\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e+}$$

for each case in virtue of the companion form (5.6) and function (5.7).

Without loss of generality, let $M = 1$. Our first goal is to show

$$-1 < F(t, \mathbf{x}_h) < 1, \quad \text{for all } t > 0.$$

It is easy to see that there exists a $t_1 > 0$ such that

$$\phi^{\mathbf{k}}(t, \mathbf{x}_h) \in \Sigma^{\mathbf{k}}_{in}, \quad 0 < t < t_1.$$

Case (i). A routine computation gives rise to

$$F(t, \mathbf{x}_h) = \frac{e^{t\lambda_3}\nu(\mu^2 + \nu^2) - e^{t\mu}\lambda_3[\nu(2\mu - \lambda_3)\cos\nu t + (\nu^2 + \mu\lambda_3 - \mu^2)\sin\nu t]}{\nu[(\mu - \lambda_3)^2 + \nu^2]}. \tag{5.8}$$

By comparing (5.8) with the function in Lemma 5.1, we obtain

$$|F(t, \mathbf{x}_h)| < 1, \text{ for } \quad 0 < t < t_1,$$

which implies

$$\phi^{\mathbf{k}}(t_1, \mathbf{x}_h) \in \Sigma^{\mathbf{k}}_{in}.$$

Hence, it is easy to get that t_1 can be extended to infinity and

$$|F(t, \mathbf{x}_h)| < 1, \text{ for all } t > 0.$$

Therefore, we have

$$\lim_{t\to+\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e0}.$$

Case (ii). A simple manipulation leads to

$$F(t, \mathbf{x}_h) = c_1 e^{\lambda_1 t} - c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}, 0 < t < t_1,$$

where

$$c_1 = \frac{\lambda_2\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad c_2 = \frac{\lambda_1\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}, \quad c_3 = \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}.$$

Similar to Case (i), we get from Lemma 5.2 that t_1 can be extended to infinity and

$$0 < F(t, \mathbf{x}_h) < 1 \text{ for all } t > 0,$$

which implies

$$\lim_{t\to+\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e0}.$$

Case (iii). Some computation gives

$$F(t, \mathbf{x}_h) = e^{\lambda_1 t}(d_1 + d_2 t) + d_3 e^{\lambda_3 t}, \quad 0 < t < t_1,$$

where

$$d_1 = \frac{\lambda_3(\lambda_3 - 2\lambda_1)}{(\lambda_1 - \lambda_3)^2}, \quad d_2 = \frac{\lambda_1\lambda_3}{\lambda_1 - \lambda_3}, \quad d_3 = \frac{\lambda_1^2}{(\lambda_1 - \lambda_3)^2}.$$

Then, similarly, according to Lemma 5.3, we have that t_1 can be extended to infinity and

$$0 < F(t, \mathbf{x}_h) < 1 \text{ for all } t > 0.$$

Thus, we have

$$\lim_{t\to+\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e0}.$$

Case (iv). It is easy to obtain

$$F(t, \mathbf{x}_h) = \frac{1}{2} e^{\lambda t}\left[(\lambda t - 1)^2 + 1\right], \quad 0 < t < t_1 \tag{5.9}$$

and

$$\frac{\partial F}{\partial t}(t, \mathbf{x}_h) = \frac{\lambda^3}{2} t^2 e^{\lambda t}, \quad 0 < t < t_1. \tag{5.10}$$

It follows from (5.9) and (5.10) that t_1 can be extended to infinity and

$$0 < F(t, \mathbf{x}_h) < 1 \text{ for all } t > 0.$$

Therefore, we have

$$\lim_{t\to+\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e0}.$$

The next thing is to prove $F(t, \mathbf{x}_h) \geq 1$ for all $t < 0$. It is easy to find from the above process that if we let

$$\mathbf{y}_h = \mathbf{x}_h - \mathbf{x}_{e+},\ c = \mathbf{k}(\mathbf{x}_h - \mathbf{x}_{e+}),$$

then

$$\mathbf{k}\left[\mathbf{y}_h\ \mathbf{A}\mathbf{y}_h\ \mathbf{A}^2\mathbf{y}_h\right] = \left[c\ 0\ 0\right],$$

where $c < 0$.
Hence, we obtain

$$|\mathbf{k}e^{-\mathbf{A}t}\mathbf{y}_h| < -c, \quad t > 0,$$

which implies

$$F(t, \mathbf{x}_h) > 1, \text{ for all } t < 0.$$

Therefore, we have

$$\lim_{t\to-\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e+}.$$

In conclusion, we have

$$\lim_{t\to+\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e0}, \lim_{t\to-\infty} \phi^{\mathbf{k}}(t, \mathbf{x}_h) = \mathbf{x}_{e+}.$$

Thus, the proof is completed. □

6. Examples and simulation

To illustrate the results in Section 3, we give some examples.

6.1. Example 1

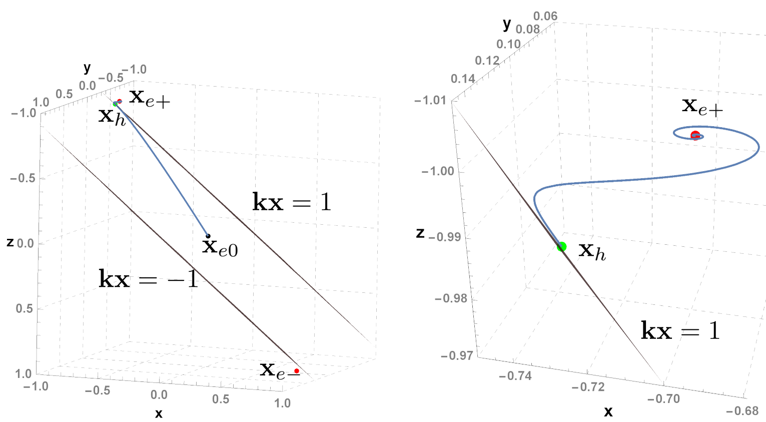
Consider the following closed loop system

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}\mathbf{x} + \mathbf{b} & \text{for } \mathbf{k}\mathbf{x} > 1, \\ (\mathbf{A} + \mathbf{b}\mathbf{k})\mathbf{x} & \text{for } |\mathbf{k}\mathbf{x}| \leq 1\,, \\ \mathbf{A}\mathbf{x} - \mathbf{b} & \text{for } \mathbf{k}\mathbf{x} < -1, \end{cases} \tag{6.1}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}. \tag{6.2}$$

If $\mathbf{k} = [\frac{7}{3}, \frac{4}{3}, -\frac{35}{12}]$, then the eigenvalues of $\mathbf{A}+\mathbf{b}\mathbf{k}$ are $-1, -2$ and -3. Hence, the origin is asymptotically stable. In this case, the special point is $\mathbf{x}_h = \{-\frac{97}{130}, \frac{9}{130}, -\frac{34}{35}\}$. The backward trajectory starting from this point will tend to $\mathbf{x}_{e+}$, and the forward trajectory will tend to the origin, as shown in Figure 1. The attraction basin $\mathcal{D}^{\mathbf{k}}$ can be obtained by numerical simulation, as shown in Figure 2. The boundary of the attraction basin $\partial\mathcal{D}^{\mathbf{k}}$ is divided into two parts by a periodic orbit Γ, one of which is colored and the other is transparent. These two parts are symmetric about the origin.



(a) (b)

Figure 1. A special trajectory of system (6.1) with parameters (6.2): (a) The trajectory of the special point $\mathbf{x}_h$; (b) A part of the trajectory through the special point $\mathbf{x}_h$

If $\mathbf{k} = [\frac{7}{3}, -\frac{2}{3}, -\frac{41}{12}]$, then the eigenvalues of $\mathbf{A} + \mathbf{b}\mathbf{k}$ are $-1 \pm 4i$ and -2. In this case, the special point is $\mathbf{x}_h = \{-\frac{81}{106}, -\frac{77}{318}, -\frac{106}{123}\}$. It is easy to find that the solution passing through $\mathbf{x}_h$ approaches the origin as $t \to +\infty$ and approaches the repelling equilibrium point $\mathbf{x}_{e+}$ as $t \to -\infty$, as shown in Figure 3.

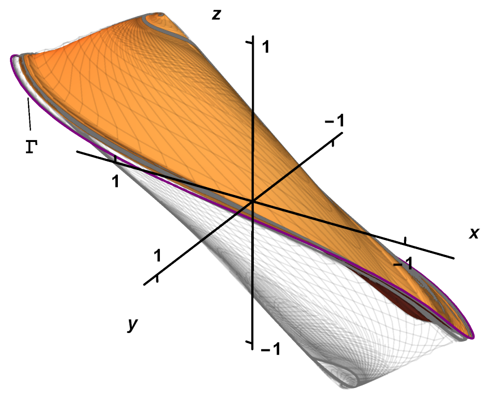


Figure 2. The boundary of the attraction basin of the origin of system (6.1) with parameters (6.2)

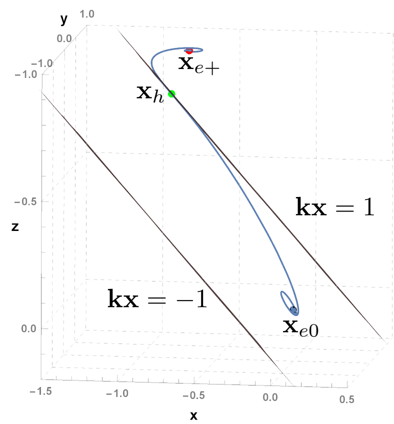


Figure 3. The trajectory of system (6.1) with parameters (6.2) through the special point $\mathbf{x}_h$

6.2. Example 2

Consider the following linear system with saturated input

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 78 & -32 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} sat(\mathbf{kx}). \tag{6.3}$$

Choosing $\mathbf{k} = (-142, -16, -17)$, it is easy to get that the two repelling equilibrium points are $\mathbf{x}_{e\pm} = (\mp 1/78, 0, 0)$. By numerical calculation, we observe the following facts, as shown in Figure 4 and 5.

(i) The boundary of the attraction basin of the origin of system (6.3) contains three periodic orbits $\Gamma_r, \Gamma_s, \Gamma_s^-$, where the information about $\Gamma_r, \Gamma_s, \Gamma_s^-$ is given in Table 1. The steps of finding the boundary of attraction basin are given in Subsection 6.3. According to the Floquet multipliers, Γ_s, Γ_s^- are saddle, and Γ_r is repelling.

Table 1. Three periodic orbits of system (6.3)

	Initial points	Periods	Floquet multipliers
Γ_r	(0.006444, 0.070062, -0.227774)	1.0807	8.6717, 4.1651, 1
Γ_s	(-0.012955, -0.016369, 0.066214)	1.1596	13.3414, 1, 0.0337
Γ_s^-	(0.012955, 0.016369, -0.066214)	1.1596	13.3414, 1, 0.0337

(ii) The boundary of the attraction basin of the origin of system is composed of the two equilibrium points $\mathbf{x}_{e+}, \mathbf{x}_{e-}$, the repelling periodic orbit Γ_r and the stable manifolds of Γ_s, Γ_s^-. This verifies that the boundary of the attraction basin of the origin of system (6.3) is homeomorphic to a sphere.

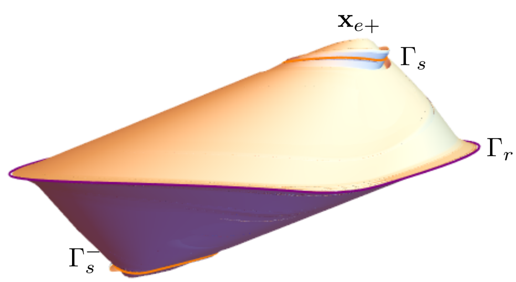


Figure 4. Structure of the boundary of the attraction basin of the origin of system (6.3) with $\mathbf{k} = (-142, -16, -17)$. The purple orbit on the boundary of the attraction basin is periodic orbit Γ_r, and the orange orbit nearest to $\mathbf{x}_{e+}$ is the periodic orbit Γ_s

.

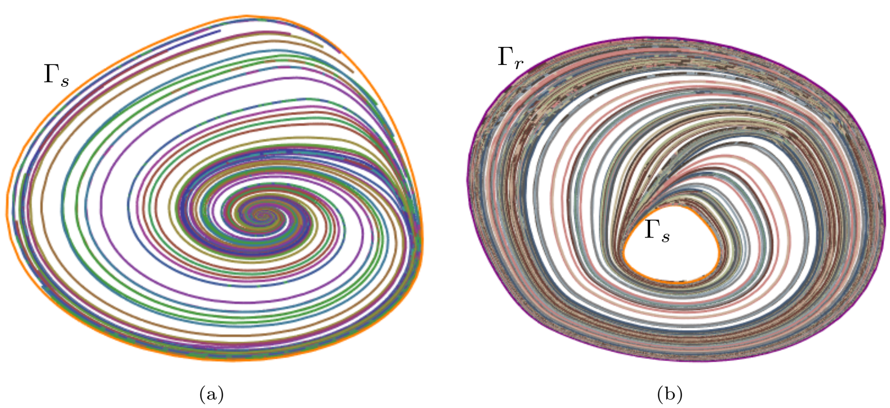


Figure 5. Figure (a) shows some trajectories on the stable manifold of Γ_s, and Figure (b) shows some trajectories on the stable manifold of Γ_s bounded by Γ_s and Γ_r.

6.3. The steps of finding the boundary of attraction basin

The method that we find the boundary of attraction basin of the origin in three-dimensional control systems with linear state feedback is briefly described as follows.

Step 1. (Search a repelling periodic orbit) Pick an initial point $\mathbf{p}_1$ very close to the origin (e.g., $(0.01, 0, 0)$ or $(0, 0.01, 0)$) and consider the reversed-time evolution trajectory $\phi(t, \mathbf{p}_1)$ of $\mathbf{p}_1$ under the system. If $\phi(t, \mathbf{p}_1)$ tends to a periodic orbit as $t \to -\infty$, then a repelling periodic orbit Γ_r is discovered. According to the results of numerical calculation, the period $\bar{T}_r$ and a initial value $\bar{\mathbf{p}}_r$ of Γ_r can be estimated. Then, proceed to Step 2. If $\phi(t, \mathbf{p}_1)$ tends to a equilibrium point as $t \to -\infty$, the system may not have a repelling periodic orbit, then it directly proceeds to Step 3.

Step 2. (Obtain a high-precision location of the repelling periodic orbit) Construct a cross-section to the vector field at $\bar{\mathbf{p}}_r$ and consider the Poincaré map P_r and $\bar{P}_r = \mathbf{x} - P_r(\mathbf{x})$. By using the interval Newton method [4,32], the high-precision zero $\mathbf{p}_r$ of $\bar{P}_r$ can be calculated numerically, which is a good initial value of Γ_r. The high-precision period T_r is obtained at the same time. The repellingness of the periodic orbit Γ_r can be verified by calculating the corresponding Floquet multipliers [7] or Lyapunov exponents [24]. Then, proceed to Step 3.

Step 3. (Search a saddle periodic orbit) Take a line $\mathbf{L}$ (e.g., x-axis, y-axis or z-axis) that goes through the origin. Consider the dynamic behaviors of the trajectories starting from the line. If the initial point $\mathbf{p}_l \in \mathbf{L}$ is near the origin, $\phi(t, \mathbf{p}_l)$ will tend to the origin as $t \to \infty$. If $\mathbf{p}_l \in \mathbf{L}$ is very far away from the origin, $\phi(t, \mathbf{p}_l)$ will tend to infinity as $t \to \infty$. By using the bisection method, the point $\mathbf{p}_l \in \mathbf{L}$ that is approximately on the boundary of the attraction basin of the origin can be found. If $\phi(t, \mathbf{p}_l)$ tends to a periodic orbit as $t \to \infty$, a saddle periodic orbit Γ_s is discovered. According to the results of numerical calculation, the period $\bar{T}_s$ and a initial value $\bar{\mathbf{p}}_2$ of Γ_s can be estimated. Then, proceed to Step 4.

Step 4. (Obtain a high-precision location of the saddle periodic orbit) It is exactly the same as Step 2. The high-precision initial value $\mathbf{p}_2$ and the high-precision period T_s of Γ_s can be obtained by numerical calculation. The fact that Γ_s is indeed saddle can be verified by calculating the corresponding Floquet multipliers [7] or Lyapunov exponents [24]. Then, proceed to Step 5.

Step 5. (Compute a two-dimensional stable manifold of the saddle periodic orbit) Take points evenly on the saddle periodic orbit Γ_s (To be more precise, curvature should be considered. More points need to be taken in the position where the curvature is large.) and denote the set of the points by $\mathcal{Q}$. Suppose $\mathbf{p}_2 \in \mathcal{Q}$. Construct a cross-section to the vector field at $\mathbf{p}_2$ and consider the Poincaré map P_s. By numerical calculation, the derivative of P_s at $\mathbf{p}_2$ and the eigenvector corresponding to the eigenvalue whose modulus is less than 1 can be obtained. Complete the calculation for every point in $\mathcal{Q}$. Next, the orientations of the eigenvectors are adjusted to ensure that the inner product of the eigenvectors of the adjacent two points is positive. According to these vectors, the two-dimensional stable manifold of Γ_s can be computed approximately. Then, proceed to Step 6.

Step 6. (Find the boundary of attraction basin of the origin) If the system has no repelling periodic orbit, then the boundary of attraction basin of the origin may only consist of the two-dimensional stable manifold of Γ_s and the two repelling equilibrium points. If the system has a repelling periodic orbit Γ_r, then there exist another saddle periodic orbits Γ_s^-. Γ_s and Γ_s^- which are symmetric about the origin. In this situation, the boundary of the attraction basin of the origin may only consist of the two-dimensional stable manifolds of Γ_s and Γ_s^-, the repelling periodic orbit Γ_r and the two repelling equilibrium points.

7. Conclusions

In this paper, we have studied a class of piecewise linear systems and obtained several results on the boundary of the attraction basin. We have proved that the equilibrium points except for the origin of the three-dimensional anti-stable linear system with saturated stabilizing linear state feedback lie on the boundary of the attraction basin of the origin. This supports the conjecture that the boundary of the attraction basin is homeomorphic to a sphere.

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