Journal of Nonlinear Modeling and Analysis
Volume 5, Number 2, June 2023, 272–287
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.272

Conservation Laws and Exact Solutions to the Modified Hyperbolic Geometric Flow*

Fang Gao[1,†] and Zenggui Wang[1]

Abstract In this paper, we investigate Lie symmetry group, optimal system, exact solutions and conservation laws of modified hyperbolic geometric flow via Lie symmetry method. Then, conservation laws of modified hyperbolic geometric flow are obtained by applying Ibragimov method.

Keywords The modified hyperbolic geometric flow, exact solution, conservation law

MSC(2010) 53C44, 35L70, 35L45, 58J47.

1. Introduction

Kong and Liu [7] first put forward the hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}(g, \frac{\partial g}{\partial t}) = 0, \tag{1.1}$$

in which g_{ij} is the surface metric, $\mathscr{F}$ is the smooth function of $g, \dfrac{\partial g}{\partial t}$ and R_{ij} is Ricci curvature tensor. Liu [8] discussed the classical global solution to the Cauchy problem of dissipative hyperbolic geometric flow, and discussed that the solution blows up. On the Riemann place, Wang [10] studied the exact solutions, the existence and uniqueness of global solution and the blow up of the solution for the geometrical flows.

Gao and Zhang [2] discussed the group-invariant solutions of the evolution equation of a hyperbolic curve flow by applying the classical Lie symmetry method. They [3] also studied the group invariant solutions of the normal hyperbolic mean curvature flow with dissipation via Lie symmetry method. Gao and Wang [4, 5] studied two different hyperbolic geometry flow equation by Lie symmetry analysis and nonlinear self-adjointness.

A new theorem of conservation laws for arbitrary differential equations is proposed by Ibragimov [6]. Belevtsov and Lukashchuk [1] investigated symmetry group classification by Lie symmetry analysis and constructed the conservation laws of

†The corresponding author.
Email address: gaofang960813@163.com (F. Gao), wangzenggui@lcu.edu.cn (Z. Wang)
1School of Mathematical Sciences, Liaocheng University, Liaocheng, Shandong 252059, China
*The authors were supported by the Young Scientists Fund of China (Grant Nos. 11001115, 11201473), the Natural Science Foundation of Shandong Province (Grant No. ZR2021MA084) and the Natural Science Foundation of Liaocheng University (Grant No. 318012025).

the nonlinear fractional diffusion equation with the Riesz potential, which satisfied nonlinear self-adjointness. Zhang, Simbanefayi and Khalique [11] studied the traveling wave solutions and conservation laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt Equation.

Silva [9] studied the nonlinear self-adjointness and conservation laws for the quasilinear 2D second-order evolution equation

$$u_{tt} = Au_{xy}+Bu_xu_y+Cu_{xx}+Du_{yy}+Eu_y+Fu_x+Pu_x^2+Qu_y^2+G+Hu_t+Iu_t^2, \quad (1.2)$$

in which $A, B, C, D, E, F, G, H, I, P, Q$ and R are functions of x, y, t and $u = u(x, y, t)$. He discussed nonlinear self-adjointness and calculated conservation laws on Rieman surfaces for hyperbolic geometric flow equation.

Letting

$$A = \frac{1}{u},\ B = -\frac{1}{u^2},\ G = \lambda u,\ C = D = E = F = P = Q = H = I = 0 \quad (1.3)$$

be in equation (1.2), we obtain

$$u_{tt} = \frac{1}{u}u_{xy} - \frac{1}{u^2}u_xu_y + \lambda u, \quad (1.4)$$

in which λ is an arbitrary constant. Equation (1.3) is known as the modified hyperbolic geometric flow, and it is also given by equation (1.1) with $\mathscr{F} = -\alpha g_{ij}$.

In this paper, we will study the exact solutions and the conservation laws of equation (1.4). First, the Lie point symmetry group for the modified hyperbolic geometric flow is obtained by applying the Lie symmetry method. Second, the Optimal system and exact solutions are discussed. Finally, the conservation laws and nonlocal conservation laws of equation (1.4) are given by applying the Ibragimov method.

2. Lie symmetry group analysis of equation (1.4)

The Lie symmetry of equation (1.4) is generated by the infinitesimal generator

$$V = \rho(x,y,t,u)\frac{\partial}{\partial x} + \sigma(x,y,t,u)\frac{\partial}{\partial y} + \mu(x,y,t,u)\frac{\partial}{\partial t} + \omega(x,y,t,u)\frac{\partial}{\partial u}. \quad (2.1)$$

The second-order prolongation vector field is

$$pr^{(2)}V = \rho\frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial u} + \omega^x\frac{\partial}{\partial u_x} + \omega^y\frac{\partial}{\partial u_y} + \omega^t\frac{\partial}{\partial u_t} + \omega^{xx}\frac{\partial}{\partial u_{xx}} + \omega^{xy}\frac{\partial}{\partial u_{xy}} + \omega^{xt}\frac{\partial}{\partial u_{xt}} + \omega^{yt}\frac{\partial}{\partial u_{yt}} + \omega^{yy}\frac{\partial}{\partial u_{yy}} + \omega^{tt}\frac{\partial}{\partial u_{tt}}.$$

Equation (1.4) remains invariant under an infinitesimal transformation, if and only if V satisfies

$$pr^{(2)}(V)(\Delta)|_{\Delta=0} = 0, \quad (2.2)$$

in which $\Delta = u_{tt} - \frac{1}{u}u_{xy} + \frac{1}{u^2}u_xu_y - \lambda u$, i.e.,

$$\omega(\frac{1}{u^2}u_{xy} - \frac{2}{u^3}u_xu_y - \lambda) + \omega^x\frac{u_y}{u^2} + \omega^y\frac{u_x}{u^2} - \omega^{xy}\frac{1}{u} + \omega^{tt} = 0, \quad (2.3)$$

in which

$$\begin{cases} \omega^x = D_x\omega - (D_x\rho)u_x - (D_x\sigma)u_y - (D_x\mu)u_t \\ \qquad = \omega_x + (\omega_u - \rho_x)u_x - \mu_u u_x u_t - \mu_x u_t - \rho_u u_x^2 - \sigma_x u_y - \sigma_u u_x u_y, \\ \omega^y = D_y\omega - (D_y\rho)u_x - (D_y\sigma)u_y - (D_y\mu)u_t \\ \qquad = \omega_y + (\omega_u - \sigma_y)u_y - \mu_u u_y u_t - \mu_y u_t - \sigma_u u_y^2 - \rho_y u_x - \rho_u u_x u_y, \\ \omega^{xy} = D_y\omega^x - (D_y\rho)u_{xx} - (D_y\sigma)u_{xy} - (D_y\mu)u_{xt} \\ \qquad = \omega_{xy} + (\omega_{uy} - \rho_{xy})u_x + (\omega_{xu} - \sigma_{xy})u_y + (\omega_u - \rho_x - \mu_u u_t - 2\rho_u u_x \\ \qquad - \sigma_y - 2\sigma_u u_y)u_{xy} + (\omega_{uu} - \rho_{xu} - \mu_{uu}u_t - \sigma_{uy})u_x u_y - (\rho_{uy} + \rho_{uu}u_y)u_x^2 \\ \qquad - (\sigma_{xu} + \sigma_{uu}u_x)u_y^2 - (\mu_y + \mu_u u_y)u_{xt} - (\rho_y + \rho_u u_y)u_{xx} - (\sigma_x + \sigma_u u_x)u_{yy} \\ \qquad - (\mu_u u_x + \mu_x)u_{ty} - \mu_{uy}u_x u_t - \mu_{xy}u_t - \mu_{xu}u_t u_y, \\ \omega^{tt} = D_t\omega^t - (D_t\rho)u_{tx} - (D_t\sigma)u_{ty} - (D_t\mu)u_{tt} \\ \qquad = \omega_{tt} + (2\omega_{tu} - \mu_{tt})u_t - \rho_{tt}u_x - \sigma_{tt}u_y + (\omega_{uu} - 2\mu_{tu})u_t^2 - 2\rho_{tu}u_t u_x \\ \qquad - 2\sigma_{tu}u_t u_y - \mu_{uu}u_t^3 - \rho_{uu}u_t^2 u_x - \sigma_{uu}u_t^2 u_y + (\omega_u - 2\mu_t - 3\mu_u u_t - \rho_u u_x \\ \qquad - \sigma_u u_y)u_{tt} - 2(\rho_t + \rho_u u_t)u_{tx} - 2(\sigma_t + \sigma_u u_t)u_{ty}. \end{cases}$$

Substituting equation (1.4) into ω^{tt}, we have

$$\begin{aligned} \omega^{tt} &= D_t\omega^t - (D_t\rho)u_{tx} - (D_t\sigma)u_{ty} - (D_t\mu)u_{tt} \\ &= \omega_{tt} + (2\omega_{tu} - \mu_{tt})u_t - \rho_{tt}u_x - \sigma_{tt}u_y + (\omega_{uu} - 2\mu_{tu})u_t^2 - 2\rho_{tu}u_t u_x \\ &- 2\sigma_{tu}u_t u_y - \mu_{uu}u_t^3 - \rho_{uu}u_t^2 u_x - \sigma_{uu}u_t^2 u_y - 2(\rho_t + \rho_u u_t)u_{tx} - 2(\sigma_t + \sigma_u u_t)u_{ty} \\ &+ (\omega_u - 2\mu_t - 3\mu_u u_t - \rho_u u_x - \sigma_u u_y)(\frac{1}{u}u_{xy} - \frac{1}{u^2}u_x u_y + \lambda u), \end{aligned} \tag{2.4}$$

in which the coefficients of $u_t u_{xy}$, $u_x u_{xy}$, $u_y u_{xy}$ in equation (2.3) and equation (2.4) are as follows

$$\begin{cases} u_t u_{xy} : -\frac{2}{u}\mu_u = 0, \\ u_y u_{xy} : \frac{1}{u}\rho_u = 0, \\ u_y u_{xy} : \frac{1}{u}\sigma_u. \end{cases}$$

We obtain $\rho = \rho(x, y, t)$, $\sigma = \sigma(x, y, t)$, $\mu = \mu(x, y, t)$. Therefore, we further have

$$\begin{cases} \omega^x = \omega_x + (\omega_u - \rho_x)u_x - \mu_x u_t - \sigma_x u_y, \\ \omega^y = \omega_y + (\omega_u - \sigma_y)u_y - \mu_y u_t - \rho_y u_x, \\ \omega^{xy} = \omega_{xy} + (\omega_{uy} - \rho_{xy})u_x + (\omega_{xu} - \sigma_{xy})u_y + (\omega_u - \rho_x - \sigma_y)u_{xy} \\ \qquad + \omega_{uu}u_x u_y - \mu_y u_{xt} - \rho_y u_{xx} - \sigma_x u_{yy} - \mu_x u_{ty} - \mu_{xy}u_t, \\ \omega^{tt} = \omega_{tt} + (2\omega_{tu} - \mu_{tt})u_t - \rho_{tt}u_x - \sigma_{tt}u_y + \omega_{uu}u_t^2 - 2\rho_t u_{tx} - 2\sigma_t u_{ty} \\ \qquad + (\omega_u - 2\mu_t)(\frac{1}{u}u_{xy} - \frac{1}{u^2}u_x u_y + \lambda u). \end{cases}$$

Substituting the above equations into equation (2.3), we obtain

$$\omega(\frac{1}{u^2}u_{xy} - \frac{2}{u^3}u_x u_y - \lambda) + \frac{u_y}{u^2}[\omega_x + (\omega_u - \rho_x)u_x - \mu_x u_t - \sigma_x u_y] + \frac{u_x}{u^2}[\omega_y + (\omega_u - \sigma_y)u_y - \mu_y u_t - \rho_y u_x] - \frac{1}{u}[\omega_{xy} + (\omega_{uy} - \rho_{xy})u_x + (\omega_{xu} - \sigma_{xy})u_y + (\omega_u - \rho_x - \sigma_y)u_{xy} + \omega_{uu}u_x u_y - \mu_y u_{xt} - \rho_y u_{xx} - \sigma_x u_{yy} - \mu_x u_{ty} - \mu_{xy}u_t] + [\omega_{tt} + (2\omega_{tu} - \mu_{tt})u_t - \rho_{tt}u_x - \sigma_{tt}u_y + \omega_{uu}u_t^2 - 2\rho_t u_{tx} - 2\sigma_t u_{ty} + (\omega_u - 2\mu_t)(\frac{1}{u}u_{xy} - \frac{1}{u^2}u_x u_y + \lambda u)] = 0. \tag{2.5}$$

Comparing the coefficients of equation (2.5), the determining equations are given by

$$\begin{cases}
(1)u_x^2: -\frac{1}{u^2}\rho_y = 0,\\
(2)u_y^2: -\frac{1}{u^2}\sigma_x = 0,\\
(3)u_t^2: \omega_{uu} = 0,\\
(4)u_{xy}: \frac{1}{u^2}\omega + \frac{1}{u}(\rho_x + \sigma_y - 2\mu_t) = 0,\\
(5)u_{tx}: \frac{1}{u}\mu_y - 2\rho_t = 0,\\
(6)u_{ty}: \frac{1}{u}\mu_x - 2\sigma_t = 0,\\
(7)u_{xx}: \frac{1}{u}\rho_y = 0,\\
(8)u_{yy}: \frac{1}{u}\sigma_x = 0,\\
(9)u_x u_y: -\frac{2}{u^3}\omega + \frac{1}{u^2}(\omega_u - \rho_x - \sigma_y + 2\mu_t) - \frac{1}{u}\omega_{uu} = 0,\\
(10)u_x u_t: -\frac{1}{u^2}\mu_y = 0,\\
(11)u_y u_t: -\frac{1}{u^2}\mu_x = 0,\\
(12)u_x: \frac{1}{u^2}\omega_y - \frac{1}{u}(\omega_{uy} - \rho_{xy}) - \rho_{tt} = 0,\\
(13)u_y: \frac{1}{u^2}\omega_x - \frac{1}{u}(\omega_{xu} - \sigma_{xy}) - \sigma_{tt} = 0,\\
(14)u_t: \frac{1}{u}\mu_{xy} + 2\omega_{tu} - \mu_{tt} = 0,\\
(15)1: -\lambda\omega - \frac{1}{u}\omega_{xy} + \omega_{tt} + \lambda(\omega_u - 2\mu_t)u = 0.
\end{cases} \tag{2.6}$$

From equation (2.6), we know

$$\rho = \rho(x),\ \sigma = \sigma(y),\ \mu = c_1,\ \omega = -(\rho_x + \sigma_y)u. \tag{2.7}$$

Let

$$\rho = c_2 x + c_4,\ \sigma = c_3 y + c_5,\ \mu = c_1,\ \omega = -(c_2 + c_3)u.$$

Therefore, (2.1) becomes

$$V = (c_2 x + c_4)\frac{\partial}{\partial x} + (c_3 y + c_5)\frac{\partial}{\partial y} + c_1\frac{\partial}{\partial t} - (c_2 + c_3)u\frac{\partial}{\partial u}. \tag{2.8}$$

in which $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ are arbitrary constants. From (2.8), we have five subalgebras

$$\begin{cases} V_1 = \dfrac{\partial}{\partial x}, \\ V_2 = \dfrac{\partial}{\partial y}, \\ V_3 = \dfrac{\partial}{\partial t}, \\ V_4 = x\dfrac{\partial}{\partial x} - u\dfrac{\partial}{\partial u}, \\ V_5 = y\dfrac{\partial}{\partial y} - u\dfrac{\partial}{\partial u}. \end{cases} \tag{2.9}$$

3. Optimal system and exact solutions

3.1. Optimal system

Theorem 3.1. *Generators in (2.9) generate an optimal system M:*

$$\{V_1,\ V_2,\ V_3,\ V_4,\ V_5,\ V_1 \pm V_2,\ V_1 \pm V_3,\ V_2 \pm V_3,\ V_4 \pm V_5,\ V_4 \pm V_2,\ V_4 \pm V_3,$$
$$V_5 \pm V_1,\ V_5 \pm V_3,\ V_1 \pm V_2 \pm V_3,\ V_4 \pm V_5 \pm V_3,\ V_4 \pm V_2 \pm V_3,\ V_5 \pm V_1 \pm V_3\}.$$

Proof. From (2.9) and the formula $[V_i, V_j] = V_i V_j - V_j V_i$, we obtain the table of Lie brackets.

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	V_1	0
V_2	0	0	0	0	V_2
V_3	0	0	0	0	0
V_4	$-V_1$	0	0	0	0
V_5	0	$-V_2$	0	0	0

Table 1. The table of Lie brackets

Assume any vector

$$V = l_1 V_1 + l_2 V_2 + l_3 V_3 + l_4 V_4 + l_5 V_5. \tag{3.1}$$

Next, we will establish the linear transformation

$$\tilde{l} = (l_1, l_2, l_3, l_4, l_5). \tag{3.2}$$

Let

$$H_i = c_{ij}^k l_j \partial_{l_k},\ i = 1, 2, 3, 4, 5, \tag{3.3}$$

in which c_{ij}^k is from formula $[V_i, V_j] = c_{ij}^k V_k$. By (3.3) and Table 1, we can obtain $F_1,\ F_2,\ F_3,\ F_4,\ F_5$.

$$\begin{cases} F_1 = l_4\partial_{l_1}, \\ F_2 = l_5\partial_{l_2}, \\ F_3 = 0, \\ F_4 = -l_1\partial_{l_1}, \\ F_5 = -l_2\partial_{l_2}. \end{cases}$$

For $F_1,\ F_2,\ F_3,\ F_4,\ F_5$, the Lie equations in terms of parameter $b_1,\ b_2,\ b_3,\ b_4,\ b_5$ and initial conditions $\tilde{l}|_{b_i=0} = l,\ i = 1,2,3,4,5$ are as follows.

$$\begin{cases} \dfrac{d\tilde{l}_1}{db_1} = l_4,\ \dfrac{d\tilde{l}_2}{db_1} = 0,\ \dfrac{d\tilde{l}_3}{db_1} = 0,\ \dfrac{d\tilde{l}_4}{db_1} = 0,\ \dfrac{d\tilde{l}_5}{db_1} = 0, \\ \dfrac{d\tilde{l}_1}{db_2} = 0,\ \dfrac{d\tilde{l}_2}{db_2} = l_5,\ \dfrac{d\tilde{l}_3}{db_2} = 0,\ \dfrac{d\tilde{l}_4}{db_2} = 0,\ \dfrac{d\tilde{l}_5}{db_2} = 0, \\ \dfrac{d\tilde{l}_1}{db_3} = 0,\ \dfrac{d\tilde{l}_2}{db_3} = 0,\ \dfrac{d\tilde{l}_3}{db_3} = 0,\ \dfrac{d\tilde{l}_4}{db_3} = 0,\ \dfrac{d\tilde{l}_5}{db_3} = 0, \\ \dfrac{d\tilde{l}_1}{db_4} = -l_1,\ \dfrac{d\tilde{l}_2}{db_4} = 0,\ \dfrac{d\tilde{l}_3}{db_4} = 0,\ \dfrac{d\tilde{l}_4}{db_4} = 0,\ \dfrac{d\tilde{l}_5}{db_4} = 0, \\ \dfrac{d\tilde{l}_1}{db_5} = 0,\ \dfrac{d\tilde{l}_2}{db_5} = -l_2,\ \dfrac{d\tilde{l}_3}{db_5} = 0,\ \dfrac{d\tilde{l}_4}{db_5} = 0,\ \dfrac{d\tilde{l}_5}{db_5} = 0. \end{cases} \tag{3.4}$$

The solution of equation (3.4) constitute the transformation

$$\begin{cases} T_1 : \tilde{l}_1 = l_1 + b_1 l_4,\ \tilde{l}_2 = l_2,\ \tilde{l}_3 = l_3,\ \tilde{l}_4 = l_4,\ \tilde{l}_5 = l_5, \\ T_2 : \tilde{l}_1 = l_1,\ \tilde{l}_2 = l_2 + b_2 l_5,\ \tilde{l}_3 = l_3,\ \tilde{l}_4 = l_4,\ \tilde{l}_5 = l_5, \\ T_3 : \tilde{l}_1 = l_1,\ \tilde{l}_2 = l_2,\ \tilde{l}_3 = l_3,\ \tilde{l}_4 = l_4,\ \tilde{l}_5 = l_5, \\ T_4 : \tilde{l}_1 = l_1 e^{-b_4},\ \tilde{l}_2 = l_2,\ \tilde{l}_3 = l_3,\ \tilde{l}_4 = l_4,\ \tilde{l}_5 = l_5, \\ T_5 : \tilde{l}_1 = l_1,\ \tilde{l}_2 = l_2 e^{-b_5},\ \tilde{l}_3 = l_3,\ \tilde{l}_4 = l_4,\ \tilde{l}_5 = l_5. \end{cases}$$

To find an optimal system, we need to simplify vector (3.2) by applying $T_1 - T_5$.

Case 1. When $l_4 \neq 0$, let $b_1 = -\dfrac{l_1}{l_4}$ (T_1) and make $\tilde{l}_1 = 0$. Therefore, (3.2) can be reduced to $(0, l_2, l_3, l_4, l_5)$.

(1.1) When $l_5 \neq 0$, let $b_2 = -\dfrac{l_2}{l_5}(T_2)$ and make $\tilde{l}_2 = 0$. Therefore, (3.2) can be reduced to $(0, 0, l_3, l_4, l_5)$. Then, V is equivalent to

$$V_4 \pm V_5,\ V_4 \pm V_5 \pm V_3.$$

(1.2) When $l_5 = 0$, (3.2) can be reduced to $(0, l_2, l_3, l_4, 0)$. Then, V is equivalent to

$$V_4,\ V_4 \pm V_2,\ V_4 \pm V_3,\ V_4 \pm V_2 \pm V_3.$$

Case 2. When $l_4 = 0$, (3.2) can be reduced to $(l_1, l_2, l_3, 0, l_5)$.

(2.1) When $l_5 \neq 0$, let $b_2 = -\dfrac{l_2}{l_5}$ and make $\tilde{l}_2 = 0$. Therefore, (3.2) can be reduced to $(l_1, 0, l_3, 0, l_5)$, and then V is equivalent to

$$V_5,\ V_5 \pm V_1,\ V_5 \pm V_3,\ V_5 \pm V_1 \pm V_3.$$

(2.2) When $l_5 = 0$, (3.2) can be reduced to $(l_1, l_2, l_3, 0, 0)$. Then, V is equivalent to

$$V_1,\ V_2,\ V_3,\ V_1 \pm V_2,\ V_1 \pm V_3,\ V_2 \pm V_3,\ V_1 \pm V_2 \pm V_3.$$

Therefore, we obtain the optimal system

$$\{V_1,\ V_2,\ V_3,\ V_4,\ V_5,\ V_1 \pm V_2,\ V_1 \pm V_3,\ V_2 \pm V_3,\ V_4 \pm V_5,\ V_4 \pm V_2,\ V_4 \pm V_3,$$
$$V_5 \pm V_1,\ V_5 \pm V_3,\ V_1 \pm V_2 \pm V_3,\ V_4 \pm V_5 \pm V_3,\ V_4 \pm V_2 \pm V_3,\ V_5 \pm V_1 \pm V_3\}.$$

□

3.2. Invariant solutions

(1)

$$V_4 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{-u},$$

the invariances are

$$y,\ t,\ h = xu,$$

and the group invariant solution is as follows

$$u = \frac{f(y,t)}{x}.$$

Then, equation (1.4) can be reduced to

$$f_{tt} - \lambda f = 0. \tag{3.5}$$

From equation (3.5), we have

$$u = \frac{1}{x}(c_1(y)e^{\sqrt{\lambda}t} + c_2(y)e^{-\sqrt{\lambda}t}),$$

in which $c_1(y),\ c_2(y)$ are arbitrary functions of y.

(2)

$$V_5 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{-u},$$

the invariances are

$$x,\ t,\ h = uy,$$

and the group invariant solution is as follows

$$u = \frac{f(x,t)}{y}.$$

Then, equation (1.4) can be reduced to

$$f''_{tt} - \lambda f = 0. \tag{3.6}$$

From equation (3.6), we have

$$u = \frac{1}{y}\big(c_3(x)e^{\sqrt{\lambda}t} + c_4(x)e^{-\sqrt{\lambda}t}\big),$$

in which $c_3(x),\ c_4(x)$ are arbitrary functions of x.

(3)

$$V = V_1 + V_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

The corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0},$$

the invariances are

$$t,\ z = x - y,\ h = u,$$

and the group invariant solution is as follows

$$u = f(t, z).$$

Then, equation (1.4) can be reduced to

$$f_{tt} + \frac{1}{f} f_{zz} - \frac{1}{f^2} f_z^2 - \lambda f = 0. \tag{3.7}$$

(4)

$$V = V_1 + V_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

The corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0},$$

the invariances are

$$y,\ z = x - t,\ h = u,$$

and the group invariant solution is as follows

$$u = f(y, z).$$

Then, equation (1.4) can be reduced to

$$f_{zz} - \frac{1}{f} f_{zy} + \frac{1}{f^2} f_z f_y - \lambda f = 0. \tag{3.8}$$

(5)

$$V = V_2 + V_3 = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}.$$

The corresponding characteristic equation is

$$\frac{dx}{0}=\frac{dy}{1}=\frac{dt}{1}=\frac{du}{0},$$

the invariances are

$$x,\ z=y-t,\ h=u,$$

and the group invariant solution is as follows

$$u=g(x,z).$$

Then, equation (1.4) can be reduced to

$$g_{zz}-\frac{1}{g}g_{xz}+\frac{1}{g^2}g_xg_z-\lambda g=0. \tag{3.9}$$

(6)

$$V=V_4+V_5=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-2u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{0}=\frac{du}{-2u},$$

the invariances are

$$t,\ z=\frac{x}{y},\ h=x^2u,$$

and the group invariant solution is as follows

$$u=\frac{f(z,t)}{x^2}.$$

Then, equation (1.4) can be reduced to

$$\frac{1}{x^2}f_{tt}+\frac{x}{y^3}\frac{1}{f}f_{zz}-\frac{x}{y^3}\frac{1}{f^2}f_z^2+\frac{1}{y^2}\frac{1}{f}f_z-\frac{\lambda}{x^2}f=0. \tag{3.10}$$

(7)

$$V=V_4+V_2=x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{x}=\frac{dy}{1}=\frac{dt}{0}=\frac{du}{-u},$$

the invariances are

$$t,\ z=xe^{-y},\ h=ue^{y},$$

and the group invariant solution is as follows

$$u=e^{-y}f(t,z).$$

Then, equation (1.4) can be reduced to

$$e^{-y}\big(f_{tt}+\frac{1}{f}f_z+xe^{-y}\frac{1}{f}f_{zz}-xe^{-y}\frac{1}{f^2}f_z^2-\lambda f\big)=0. \tag{3.11}$$

(8)

$$V = V_5 + V_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{-u},$$

the invariances are

$$t,\ z = ye^{-x},\ h = uy,$$

and the group invariant solution is as follows

$$u = \frac{f(z,t)}{y}.$$

Then, equation (1.4) can be reduced to

$$\frac{1}{y}f_{tt} + \frac{y}{f}e^{-2x}f_{zz} - \frac{y}{f^2}e^{-2x}f_z^2 + \frac{1}{f}e^{-x}f_z - \frac{\lambda}{y}f = 0. \tag{3.12}$$

(9)

$$V = V_5 + V_3 = \frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dt}{1} = \frac{du}{-u},$$

the invariances are

$$x,\ z = ye^{-t},\ h = uy,$$

and the group invariant solution is as follows

$$u = \frac{f(x,z)}{y}.$$

Then, equation (1.4) can be reduced to

$$e^{-t}f_z + ye^{-2t}f_{zz} - \frac{1}{f}e^{-t}f_z + \frac{1}{f^2}e^{-t}f_xf_z - \lambda f\frac{1}{y} = 0. \tag{3.13}$$

(10)

$$V = V_1 + V_2 + V_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial t}.$$

The corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dt}{1} = \frac{du}{0},$$

the invariances are

$$\xi = y - t,\ \eta = x - t,\ h = u,$$

and the group invariant solution is as follows

$$u = f(\xi, \eta).$$

Then, equation (1.4) can be reduced to

$$f_{\xi\xi} + 2f_{\xi\eta} + f_{\eta\eta} - \frac{1}{f} f_{\xi\eta} + \frac{1}{f^2} f_\xi f_\eta - \lambda f = 0. \tag{3.14}$$

(11)

$$V = V_4 + V_5 + V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{1} = \frac{du}{-2u},$$

the invariances are

$$z = \frac{y}{x}, \ h = ue^{2t},$$

and the group invariant solution is as follows

$$u = f(z)e^{-2t}.$$

Then, equation (1.4) can be reduced to

$$\frac{y}{x^3}\frac{1}{f}f'' + \frac{1}{x^2}\frac{1}{f}f' - \frac{y}{x^3}\frac{1}{f^2}f'^2 + (4-\lambda)e^{-2t}f = 0. \tag{3.15}$$

(12)

$$V = V_4 + V_2 + V_3 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{x} = \frac{dy}{1} = \frac{dt}{1} = \frac{du}{-u},$$

the invariances are

$$z = y - t, \ h = xu,$$

and the group invariant solution is as follows

$$u = \frac{f(z)}{x}.$$

Then, equation (1.4) can be reduced to

$$f'' - \lambda f = 0. \tag{3.16}$$

From equation (3.16), we have

$$u = \frac{1}{x}\left(c_5 e^{\sqrt{\lambda}(y-t)} + c_6 e^{-\sqrt{\lambda}(y-t)}\right),$$

in which $c_5, \ c_6$ are arbitrary constants.

(13)

$$V = V_5 + V_1 + V_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

The corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dy}{y} = \frac{dt}{1} = \frac{du}{-u},$$

the invariances are

$$z = x - t,\ h = uy,$$

and the group invariant solution is as follows

$$u = \frac{f(z)}{y}.$$

Then, equation (1.4) can be reduced to

$$f'' - \lambda f = 0. \tag{3.17}$$

From equation (3.16), we have

$$u = \frac{1}{y}\big(c_7 e^{\sqrt{\lambda}(x-t)} + c_8 e^{-\sqrt{\lambda}(x-t)}\big),$$

in which $c_7,\ c_8$ are arbitrary constants.

4. Conservation laws

Equation (1.4) can be written as

$$F = u_{tt} - \frac{1}{u}u_{xy} + \frac{1}{u^2}u_x u_y - \lambda u = 0. \tag{4.1}$$

First, we analyze equation (4.1) by applying the Ibragimov method. According to equation (1.3) and Theorem 3.1 in [9], we can calculate the adjoint variable ϕ as follows

$$\phi = 2(\alpha(x) + \beta(y))\cosh(\sqrt{\lambda}t). \tag{4.2}$$

Formal Lagrangian function is given by

$$L = \phi F = 2\big(\alpha(x) + \beta(y)\big)\big(u_{tt} - \frac{1}{u}u_{xy} + \frac{1}{u^2}u_x u_y - \lambda u\big)\cosh(\sqrt{\lambda}t).$$

The adjoint equation of equation (4.1) is as follows

$$F^* = \frac{\delta L}{\delta u} = 0,$$

in which $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator as follows

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x D_y\frac{\partial}{\partial u_{xy}} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} - \ldots,$$

in which D_i is the total derivative operator of x, y, t as follows

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + u_{ty}\frac{\partial}{\partial u_y} + ...,$$

$$D_x = \frac{\partial}{\partial x} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tx}\frac{\partial}{\partial u_t} + v_{tx}\frac{\partial}{\partial v_t} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xy}\frac{\partial}{\partial u_y} + ...,$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{ty}\frac{\partial}{\partial u_t} + v_{ty}\frac{\partial}{\partial v_t} + u_{xy}\frac{\partial}{\partial u_x} + v_{xy}\frac{\partial}{\partial v_x} + u_{yy}\frac{\partial}{\partial u_y} + ...$$

Definition 4.1. For the differential equation (4.1), if there exists a function

$$\phi = \phi(x, y, t, u) \neq 0, \tag{4.3}$$

and such that it satisfies

$$F^*|_{\phi=\phi(x,y,t,u)} = \lambda F, \tag{4.4}$$

in which $\lambda = \lambda(x, y, t, u, ...)$ is undetermined. Then, equation (4.1) is said to be nonlinearly self-adjoint.

Definition 4.2. A vector field $C(x, y, t, u, u_x, u_y, u_t, ...)$ has three components, i.e.,

$$C = C(C^1, C^2, C^3).$$

If each solution $u = u(x, y, t)$ of equation (4.1) satisfies the equation

$$D_i(C^i) = D_tC^1 + D_xC^2 + D_yC^3, \tag{4.5}$$

then the vector field $C(x, y, t, u, u_t, u_x, u_y, ...)$ is said to be a conserved vector, and (4.5) is said to be a conservation law of equation (4.1).

Next, we calculate the conservation laws according to Theorem 2.1 in [9].

$$C^1 = 2\big(\alpha(x)+\beta(y)\big)\Big(\sqrt{\lambda}\big((c_2+c_3)u+c_1u_t + (c_2x+c_4)u_x + (c_3y + c_5)u_y\big)\sinh(\sqrt{\lambda}t) - \big((c_2 + c_3)u_t + c_1u_{tt} + (c_2x + c_4)u_{xt} + (c_3y + c_5)u_{ty}\big)\cosh(\sqrt{\lambda}t)\Big),$$

$$C^2 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x) + \beta(y))(\frac{1}{u^2}u_y + \frac{1}{u_y}) + \frac{1}{u}\beta_y\big)\big((c_2 + c_3)u + c_1u_t + (c_2x + c_4)u_x + (c_3y + c_5)u_y\big) - \frac{1}{u}\big(\alpha(x) + \beta(y)\big)\big((c_2 + c_3)u_y + c_1u_{ty} + (c_2x + c_4)u_{xy} + c_3u_y + (c_3y + c_5)u_{yy}\big)\Big),$$

$$C^3 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x) + \beta(y))(\frac{1}{u^2}u_x + \frac{1}{u_x}) + \frac{1}{u}\alpha_x\big)\big((c_2 + c_3)u + c_1u_t + (c_2x + c_4)u_x + (c_3y + c_5)u_y\big) - \frac{1}{u}\big(\alpha(x) + \beta(y)\big)\big((c_2 + c_3)u_x + c_1u_{tx} + (c_2x + c_4)u_{xx} + c_2u_x + (c_3y + c_5)u_{xy}\big)\Big).$$

Corollary 4.1. *Assuming (2.8) is an infinitesimal generator of Lie symmetry for equation (1.4), and $\alpha = \alpha(x)$, $\beta = \beta(y)$ are both non-vanishing harmonic functions, then the corresponding nonlocal conserved vectors are as follows.*

(a) For $V_1 = \dfrac{\partial}{\partial x}$, *we can obtain the components of the conserved vector* $C = (C^1, C^2, C^3)$:

$$\begin{cases} C^1 = 2\big(\alpha(x) + \beta(y)\big)\big(\sqrt{\lambda}u_x \sinh(\sqrt{\lambda}t) - u_{xt}\cosh(\sqrt{\lambda}t)\big), \\ C^2 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_y+\frac{1}{u_y})+\frac{1}{u}\beta_y\big)u_x-\frac{1}{u}\big(\alpha(x)+\beta(y)\big)u_{xy}\Big), \\ C^3 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_x+\frac{1}{u_x})+\frac{1}{u}\alpha_x\big)u_x-\frac{1}{u}\big(\alpha(x)+\beta(y)\big)u_{xx}\Big). \end{cases}$$

(b) For $V_2 = \dfrac{\partial}{\partial y}$, *we can obtain the components of the conserved vector* $C = (C^1, C^2, C^3)$:

$$\begin{cases} C^1 = 2\big(\alpha(x) + \beta(y)\big)\big(\sqrt{\lambda}u_y \sinh(\sqrt{\lambda}t) - u_{ty}\cosh(\sqrt{\lambda}t)\big), \\ C^2 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_y+\frac{1}{u_y})+\frac{1}{u}\beta_y\big)u_y-\frac{1}{u}\big(\alpha(x)+\beta(y)\big)u_{yy}\Big), \\ C^3 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_x+\frac{1}{u_x})+\frac{1}{u}\alpha_x\big)u_y-\frac{1}{u}\big(\alpha(x)+\beta(y)\big)u_{xy}\Big). \end{cases}$$

(c) For $V_3 = \dfrac{\partial}{\partial t}$, *we can obtain the components of the conserved vector* $C = (C^1, C^2, C^3)$:

$$\begin{cases} C^1 = 2\big(\alpha(x) + \beta(y)\big)\big(\sqrt{\lambda}u_t \sinh(\sqrt{\lambda}t) - u_{tt}\cosh(\sqrt{\lambda}t)\big), \\ C^2 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_y+\frac{1}{u_y})+\frac{1}{u}\beta_y\big)u_t-\frac{1}{u}\big(\alpha(x)+\beta(y)\big)u_{ty}\Big), \\ C^3 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_x+\frac{1}{u_x})+\frac{1}{u}\alpha_x\big)u_t-\frac{1}{u}\big(\alpha(x)+\beta(y)\big)u_{tx}\Big). \end{cases}$$

(d) For $V_4 = x\dfrac{\partial}{\partial x} - u\dfrac{\partial}{\partial u}$, *we can obtain the components of the conserved vector* $C = (C^1, C^2, C^3)$:

$$\begin{cases} C^1 = 2\big(\alpha(x) + \beta(y)\big)\big(\sqrt{\lambda}(u + xu_x)\sinh(\sqrt{\lambda}t) - (u_t + xu_{xt})\cosh(\sqrt{\lambda}t)\big), \\ C^2 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x) + \beta(y))(\frac{1}{u^2}u_y + \frac{1}{u_y}) + \frac{1}{u}\beta_y\big)(u + xu_x) - \frac{1}{u}\big(\alpha(x) \\ \quad + \beta(y)\big)(u_y + xu_{xy})\Big), \\ C^3 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x) + \beta(y))(\frac{1}{u^2}u_x + \frac{1}{u_x}) + \frac{1}{u}\alpha_x\big)(u + xu_x) - \frac{1}{u}\big(\alpha(x) \\ \quad + \beta(y)\big)(2u_x + xu_{xx})\Big). \end{cases}$$

(e) For $V_5 = y\dfrac{\partial}{\partial y} - u\dfrac{\partial}{\partial u}$, *we can obtain the components of the conserved vector*

$C = (C^1, C^2, C^3)$:

$$\begin{cases} C^1 = 2\big(\alpha(x)+\beta(y)\big)\big(\sqrt{\lambda}(u+yu_y)\sinh(\sqrt{\lambda}t)-(u_t+yu_{ty})\cosh(\sqrt{\lambda}t)\big), \\ C^2 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_y+\frac{1}{u_y})+\frac{1}{u}\beta_y\big)(u+yu_y)-\frac{1}{u}\big(\alpha(x) \\ \quad +\beta(y)\big)(2u_y+yu_{yy})\Big), \\ C^3 = -2\cosh(\sqrt{\lambda}t)\Big(\big((\alpha(x)+\beta(y))(\frac{1}{u^2}u_x+\frac{1}{u_x})+\frac{1}{u}\alpha_x\big)(u+yu_y)-\frac{1}{u}\big(\alpha(x) \\ \quad +\beta(y)\big)(u_x+yu_{xy})\Big). \end{cases}$$

5. Conclusions

This paper is divided into four parts. First, the background of writing this paper is introduced. Second, based on the Lie symmetry method, the Lie point symmetry group for the modified hyperbolic geometric flow is obtained. Third, the Optimal system and exact solutions are discussed. Finally, on the basis of the Ibragimov method, the conservation laws and nonlocal conservation laws of the modified hyperbolic geometric flow are given.

Acknowledgements

The authors are grateful to the reviewers and editors for their valuable suggestions that have helped us improve our paper.

References

[1] N. S. Belevtsov and S. Yu. Lukashchuk, *Symmetry Group Classification and Conservation Laws of the Nonlinear Fractional Diffusion Equation with the Riesz Potential*, Symmetry, 2020, 12(1), 178, 16 pages.

[2] B. Gao and S. Zhang, *Analysis of the evolution equation of a hyperbolic curve flow via Lie symmetry method*, Pramana–Journal of Physics, 2020, 94, 55, 8 pages.

[3] B. Gao and S. Zhang, *Invariant solutions of the normal hyperbolic mean curvature flow with dissipation*, Archiv der Mathematik, 2020, 114, 227–239.

[4] F. Gao, Z. Niu and Z. Wang, *Group-invariant solutions to SL'(2)-motion equation*, Pramana–Journal of Physics, 2021, 95, 118, 7 pages.

[5] F. Gao and Z. Wang, *Nonlinear self-adjointness and conservation laws for the modified dissipative hyperbolic geometric flow equation*, Journal of Geometry and Physics, 2021, 167, Article ID 104304, 9 pages.

[6] N. H. Ibragimov, *A new conservation theorem*, Journal of Mathematical Analysis and Applications, 2007, 333(1), 311–328.

[7] D. Kong and K. Liu, *Wave character of metrics and hyperbolic geometric flow*, Journal of Mathematical Physics, 2007, 48(10), 157–162.

[8] F. Liu, *Global Classical Solutions to the Dissipative Hyperbolic Geometric Flow on Riemann Surfaces*, Chinese Annals of Mathematics, Series A, 2009, 30A(5), 717–726.

[9] K. A. A. Silva, *Nonlinear Self-Adjointness and Conservation Laws for the Hyperbolic Geometric Flow Equation*, Journal of Nonlinear Mathematical Physics, 2013, 20(1), 28–43.

[10] J. Wang, *Symmetries and solutions to geometrical flows*, Science China Mathematics, 2013, 56(8), 1689–1704.

[11] L. Zhang, I. Simbanefayi and C. Khalique, *Travelling Wave Solutions and Conservation Laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt Equation*, Journal of Nonlinear Modeling and Analysis, 2021, 3(3), 421–430.