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Traveling Epizootic Waves of a Fox Rabies Model with Small Spatial Diffusion*

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Abstract In this paper, to describe the spread of fox rabies, a degenerate SEI epidemic model with small spatial diffusion equipped by infectious foxes due to rabies is investigated. In particular, the existence of traveling waves is established by the geometric singular perturbation theory for the larger speeds, while the non-existence of traveling wave is still derived for the smaller speeds. Moreover, some numerical simulations are implemented to illustrate the propagation dynamics driven by traveling waves.

Keywords Fox rabies, traveling waves, geometric singular perturbation theory, small spatial diffusion

MSC(2010) 35K57, 35R35, 92D25, 92D40.

1. Introduction

This paper investigates the propagation dynamics in the following reaction-diffusion equation modeling the rabies among foxes:

$$\begin{cases} \dfrac{\partial S}{\partial \tau} = (a-b)S\left(1-\dfrac{N}{K}\right) - \beta IS, \\ \dfrac{\partial E}{\partial \tau} = \beta IS - \sigma E - \left[b + (a-b)\dfrac{N}{K}\right]E, \\ \dfrac{\partial I}{\partial \tau} = D_0 \dfrac{\partial^2 I}{\partial x^2} + \sigma E - \alpha I - \left[b + (a-b)\dfrac{N}{K}\right]I, \end{cases} \tag{1.1}$$

where

$$N = S + E + I$$

is the density of total fox population, while $S(x,\tau), E(x,\tau)$ and $I(x,\tau)$ are the densities of susceptible foxes, infected but non-infectious, i.e., exposed rabid foxes, and infectious foxes at location x and time τ respectively; σ is the average rate at which infected foxes become infectious; β is the transmission coefficient of fox

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rabies; α is the mortality of infectious foxes; a, b are the birth rate and the intrinsic death rate; K is the environmental carrying capacity. Thus, the term $(a-b)N/K$ describes the depletion of the food supply by all foxes. In particular, the diffusion coefficient D_0 is introduced on behalf of the additional spatial activity of infectious foxes due to rabies, which both the susceptible and exposed foxes do not have. Note that all parameters in the model are positive constants.

As is well-known, rabies is a serious disease threatening human health since it could be easily transmitted from infectious foxes to domestic animals, and then to humans. For example, despite of various efforts to stop it by hunting foxes and vaccinating them, the wave of epizootic in Europe was widely dispersed in northern France in 1980 [15]. Indeed, evidences suggest that epidemics such as fox rabies always spread spatially in a way like traveling waves [17]. Thus, there are natural reasons to comprehend the transmission behavior of rabies, in particular, the propagation dynamics of epidemics [12].

In fact, Anderson et al. [1] proposed earlier a kinetic model of fox rabies without the spatial diffusion to describe the dynamics of the spread of rabies. Furthermore, Murray et al. [17] adopted an asymptotic analysis method to study the first-order approximation traveling waves. Different from such studies, this paper accurately analyzes the existence of traveling waves by selecting the small diffusion coefficient D_0 as the perturbation parameter and combining with the geometric singular perturbation method, as employed by Szmolyan [20] and Gourley [8]. The readers are referred to Ruan and Xiao [19], Wang and Wang [21] and Pang and Xiao [18] for more existence or non-existence results of traveling waves established by such a way. As a result, we can therefore draw the conclusion of the dynamic propagation of the system in the space with more practical significance and more abundant phenomena.

Henceforth, in order to simplify the model, we rescale the previous system (1.1) to the following model with non-dimensional quantities by setting $s = S/K, e = E/K, i = I/K, n = N/K, D = D_0/\beta K, t = \beta K\tau, \epsilon = (a-b)/\beta K, \delta = b/\beta K, \mu = \sigma/\beta K$ and $d = (\alpha + b)/\beta K$:

$$\begin{cases} \dfrac{\partial s}{\partial t} = \epsilon(1-n)s - is, \\ \dfrac{\partial e}{\partial t} = is - (\mu + \delta + \epsilon n)e, \\ \dfrac{\partial i}{\partial t} = D\dfrac{\partial^2 i}{\partial x^2} + \mu e - (d + \epsilon n)i, \end{cases} \tag{1.2}$$

where $n = s + e + i$, and ϵ, μ, δ, d and D are positive.

The rest of this paper is structured as follows: First, Section 2 is devoted to prove the non-existence of traveling waves for $R_0 > 1$ and $0 < c < c^*$, while obtaining the existence of traveling waves by the geometric singular perturbation method for $R_0 > 1$ and $c \geq c^*$. Finally, in Section 3, some numerical simulations are implemented to illustrate our main results about the existence of traveling wave solutions of system (1.2).

2. The non-existence and existence of traveling waves

In this section, we will consider traveling waves of system (1.2) on $\mathbb{R} = (-\infty, +\infty)$. Since the model in a homogenous habit with Neumann boundary condition defines the same basic reproduction number as its kinetic counterpart by [22, Theorem 3.4], the basic reproduction number $R_0 = \frac{\mu}{(d+\epsilon)(\mu+\epsilon+\delta)}$ can be given. When $R_0 > 1$, namely, $0 < d < \frac{1}{1+\frac{\epsilon+\delta}{\mu}} - \epsilon$, there are two non-negative steady-state solutions for system (1.2), $E_0 = (1, 0, 0)$ and $E_1 = (s^*, e^*, i^*)$, where $s^* = d + (\epsilon + \frac{\epsilon d+\delta}{\mu})d$, $e^* = \frac{\epsilon d(1-d)}{\mu}$ and $i^* = \epsilon(1-d)$ up to the first order in ϵ and δ.

We introduce the solutions in this form $s(x,t) = u(x+ct)$, $e(x,t) = v(x+ct)$ and $i(x,t) = w(x+ct)$, where u, v and w are functions of $\xi = x + ct$, and the wave speed c is positive. Then, system (1.2) can be rewritten as

$$\begin{cases} cu' = \epsilon(1-n)u - wu, \\ cv' = wu - (\mu + \delta + \epsilon n)v, \\ cw' = Dw'' + \mu v - (d + \epsilon n)w, \end{cases} \tag{2.1}$$

where the prime is the derivative with respect to the variable ξ. Considering the epidemic implication, the traveling waves u, v and w need to be non-negative and have the following conditions

$$\begin{cases} (u(-\infty), v(-\infty), w(-\infty)) = (1, 0, 0), \\ (u(+\infty), v(+\infty), w(+\infty)) = (u^*, v^*, w^*). \end{cases} \tag{2.2}$$

Here, $u^* = s^*, v^* = e^*, w^* = i^*$.

Denote $w' = g$. Then, system (2.1) becomes

$$\begin{cases} cu' = \epsilon(1-n)u - wu, \\ cv' = wu - (\mu + \delta + \epsilon n)v, \\ w' = g, \\ Dg' = cg - \mu v + (d + \epsilon n)w. \end{cases} \tag{2.3}$$

For system (2.3), there are two equilibria $\widehat{E}_0 = (1, 0, 0, 0)$ and $\widehat{E}_1 = (u^*, v^*, w^*, 0)$ when $R_0 > 1$. Since the half-space $\mathbb{R}^4_+ \triangleq \{(u, v, w, g) | u, v, w \geq 0\}$ is invariant, we note that the original system (1.2) possesses the traveling wave solutions satisfying (2.2), if and only if there is a heteroclinic orbit of system (2.3) connecting $\widehat{E}_0$ and $\widehat{E}_1$, and the readers are referred to Szmolyan [20] for more details about this point.

Linearizing system (2.3) at $\widehat{E}_0$, we get the characteristic equation

$$\left(\lambda + \frac{\epsilon}{c}\right)(\lambda^3 + A_1\lambda^2 - A_2\lambda + A_3) = 0, \tag{2.4}$$

where $A_1 = \frac{D\epsilon+D\delta+D\mu-c^2}{Dc}$, $A_2 = \frac{2\epsilon+\delta+d+\mu}{D}$, $A_3 = \frac{\mu-\epsilon\delta-\epsilon d-\delta d-\epsilon\mu-\mu d-\epsilon^2}{Dc}$.

Denote

$$P(\lambda) = \lambda^3 + A_1\lambda^2 - A_2\lambda + A_3 = 0. \tag{2.5}$$

It is obvious that (2.5) has a negative root when $R_0 > 1$, or equivalently, when $A_3 > 0$. According to the Routh-Hurwitz formula, two roots of (2.5) have positive real parts, and given the conditions which they are positive, we investigate

$$P_1(\lambda) = \frac{P'(\lambda)}{3} = \lambda^2 + \frac{2A_1}{3}\lambda - \frac{A_2}{3}. \tag{2.6}$$

It is clear that $P'(\lambda) = 0$ has a unique positive root

$$\begin{aligned}\lambda^* =&\frac{1}{3Dc}\Big(c^2 - D\epsilon - D\delta - D\mu \\ &+ \sqrt{(c^2 - D\epsilon - D\delta - D\mu)^2 + 3Dc(2\epsilon c + \delta c + dc + \mu c)}\Big).\end{aligned}$$

Since $P(0) > 0$, in order to satisfy (2.5) with two different positive real roots, we only need $P(\lambda^*) < 0$. In addition, the equation (2.5) has two complex roots with positive real parts, if $P(\lambda^*) > 0$.

Next, we express the above condition $P(\lambda^*) < 0$ by the parameter c. For such an aim, we need to find conditions that $P(\lambda^*) = 0$ and $P'(\lambda^*) = 0$. Set

$$\begin{aligned}P(\lambda) &= P_1(\lambda)Q_1(\lambda) + R_1(\lambda), \\ P_1(\lambda) &= R_1(\lambda)Q_2(\lambda) + R_2(c).\end{aligned}$$

Here, the quotient and remainder of $P(\lambda)$ divided by $P_1(\lambda)$ are $Q_1(\lambda)$ and $R_1(\lambda)$, and the quotient and remainder of $P_1(\lambda)$ divided by $R_1(\lambda)$ are $Q_2(\lambda)$ and R_2. By a series of calculations, it is easy to get

$$\begin{aligned}R_2(c) =&\frac{3}{4}\frac{1}{\left(\frac{((\mu+\epsilon+\delta)D-c^2)^2}{c^2D^2} - \frac{3(2\epsilon+\delta+d+\mu)}{D}\right)^2} \\ &\Bigg(\frac{4\left((\mu+\epsilon+\delta)D-c^2\right)^3(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)}{c^4D^4} \\ &-\frac{\left((\mu+\epsilon+\delta)D-c^2\right)^2(2\epsilon+\delta+d+\mu)^2}{c^2D^4} \\ &-\frac{18\left((\mu+\epsilon+\delta)D-c^2\right)(2\epsilon+\delta+d+\mu)(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)}{c^2D^3} \\ &+\frac{4(2\epsilon+\delta+d+\mu)^3}{D^3} + \frac{27\left((R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)\right)^2}{c^2D^2}\Bigg),\end{aligned}$$

and we can see that the sign of $-R_2(c)$ is the same as

$$P_2(c) = b_0c^6 + b_1c^4 + b_2c^2 + b_3,$$

where

$$\begin{aligned}
b_0 =&(2\epsilon+\delta+d+\mu)^2+4(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)>0,\\
b_1 =&2D[2(2\epsilon+\delta+d+\mu)^3+(\epsilon+\delta+\mu)(2\epsilon+\delta+d+\mu)^2\\
&+9(2\epsilon+\delta+d+\mu)(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)\\
&+6(\mu+\epsilon+\delta)(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)],\\
b_2 =&(\epsilon+\delta+\mu)^2D^2(2\epsilon+\delta+d+\mu)^2\\
&+18(\mu+\delta+\epsilon)D^2(2\epsilon+\delta+d+\mu)(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)\\
&+12(\mu+\delta+\epsilon)^2D^2(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)\\
&-27D^2[(R_0-1)(d+\epsilon)(\mu+\epsilon+\delta)]^2,\\
b_3 =&4D^3(\mu+\delta+\epsilon)^3(d+\epsilon)(\mu+\epsilon+\delta)(1-R_0)<0.
\end{aligned}$$

It is obvious that we must get $P_2(c)=0$. Then, $P(\lambda^*)=0$ and $P'(\lambda^*)=0$. Since $R_0>1$, we can easily obtain that $b_0>0, b_1>0$ and $b_3<0$, then there exists a unique $c^*>0$ which satisfies $P_2(c^*)=0$ by the Descarte's rule of signs. Moreover, there is

$$P_2(c)\begin{cases}<0, & 0<c<c^*,\\ =0, & c=c^*,\\ >0, & c>c^*.\end{cases} \tag{2.7}$$

If $c=c^*$, we can get that $P(\lambda^*)=0$ and $P'(\lambda^*)=0$ by calculation. Since $P(\lambda)$ is a decreasing function of c, there exists the following lemma.

Lemma 2.1. *(i) If $0<c<c^*$, $P(\lambda)$ has two complex roots with positive real parts;*

(ii) If $c=c^*$, $P(\lambda)$ only has a positive root;

(iii) If $c>c^*$, $P(\lambda)$ has two positive roots.

Now, we discuss the non-existence of traveling waves.

Theorem 2.1. *Suppose that $R_0>1$ and $D>0$. Then system* (1.2) *doesn't have any traveling wave solution propagating at the speed $c\in(0,c^*)$.*

Proof. Since equation (2.4) exists two negative real roots and two complex roots with positive real parts, when $0<c<c^*$, there has a two-dimensional unstable manifold which is based on $\widehat{E}_0$. In addition, the critical point $\widehat{E}_0$ is a spiral point on the unstable manifold. As a result, for any heteroclinic orbit connecting $\widehat{E}_0$ and $\widehat{E}_1$, the trajectory approaching $\widehat{E}_0$ must satisfy that at least one of $v(\xi)$ or $w(\xi)$ is negative for some $\xi\in\mathbb{R}$, which is inconsistent with the non-negativity of traveling waves. □

Next, we prove the existence of traveling wave solutions, when $c\geq c^*$ and $0<D\ll 1$. In fact, when $R_0>1$, the following system

$$\begin{cases}\dfrac{\mathrm{d}s}{\mathrm{d}t}=\epsilon(1-n)s-is,\\ \dfrac{\mathrm{d}e}{\mathrm{d}t}=is-(\mu+\delta+\epsilon n)e,\\ \dfrac{\mathrm{d}i}{\mathrm{d}t}=\mu e-(d+\epsilon n)i\end{cases} \tag{2.8}$$

has two steady-state solutions $E_0(1,0,0)$ and $E_1(s^*,e^*,i^*)$. The details about the global properties of (2.8) can be referred to [16, Theorem 4.1], and one can get the following lemma.

Lemma 2.2. *The solution E_1 of (2.8) is globally asymptotically stable, when $R_0 > 1$, and the solution E_0 of (2.8) is globally asymptotically stable, when $R_0 \le 1$.*

Letting $\xi = D\eta$ in which $D > 0$ is sufficiently small, we can structure system (2.3) as the following

$$\begin{cases} c\dot{u} = D(\epsilon(1-n)u - wu), \\ c\dot{v} = D(wu - (\mu+\delta+\epsilon n)v), \\ \dot{w} = Dg, \\ \dot{g} = cg - \mu v + (d+\epsilon n)w. \end{cases} \tag{2.9}$$

Here, dots express differentiation with respect to η. It is obvious that systems (2.3) and (2.9) are equivalent, when $D > 0$, but the different time-scales lead to different limiting systems. In fact, if $D \to 0$, then system (2.3) becomes

$$\begin{cases} cu' = \epsilon(1-n)u - wu, \\ cv' = wu - (\mu+\delta+\epsilon n)v, \\ w' = g, \\ 0 = cg - \mu v + (d+\epsilon n)w. \end{cases} \tag{2.10}$$

Hence, the flow of (2.10) is limited on the submainfold

$$\mathfrak{M}_0 = \left\{ (u,v,w,g) \in \mathbb{R}^4 : g = \frac{\mu v - (d+\epsilon n)w}{c} \right\},$$

and only the first three equations determine dynamics of (2.10) on $\mathfrak{M}_0$. On the other hand, letting $D \to 0$ in (2.9), we have the system

$$\begin{cases} c\dot{u} = 0, \\ c\dot{v} = 0, \\ \dot{w} = 0, \\ \dot{g} = cg - \mu v + (d+\epsilon n)w. \end{cases} \tag{2.11}$$

Generally, system (2.3) is regarded as the slow system due to the slow time-scale ξ and $\mathfrak{M}_0$ is the slow manifold, while (2.9) is regarded as the fast one due to the fast variable η.

When $\mathfrak{M}_0$ is normally hyperbolic, i.e., by linearizing the fast system (2.9), one can obtain dim$\mathfrak{M}_0$ eigenvalues with zero real part, there must exist a three-dimension perturbed invariant mainfold $\mathfrak{M}_D$ for any sufficiently small D asserted by the geometrical singular perturbation theory [7, 20]. Furthermore, all the dynamic behavior near the slow mainfold is dominated by the one on the slow mainfold [14].

Indeed, direct calculations show that the eigenvalues of the linearization of the fast system (2.9) on $\mathfrak{M}_0$ are $0,0,0,c$. Thus, $\mathfrak{M}_0$ is normally hyperbolic. Therefore, based on Fenichel's Invariant Manifold Theorem [7], it is clear to get the following three-dimensional locally invariant manifold

$$\mathfrak{M}_D = \left\{ (u,v,w,g) \in \mathbb{R}^4 : g = \frac{\mu v - (d+\epsilon n)w}{c} + Dh(u,v,w;D) \right\}.$$

Here, h is a smooth function and satisfies $h(1,0,0;D)=0$, when $0\le D<D_1$, and $D_1>0$ is small. Returning to the slow time scale, we have the projective system of slow one (2.3) on $\mathfrak{M}_D$ can be written as

$$\begin{cases} u' = \dfrac{\epsilon(1-n)u - wu}{c}, \\ v' = \dfrac{wu-(\mu+\delta+\epsilon n)v}{c}, \\ w' = \dfrac{\mu v-(d+\epsilon n)w}{c} + Dh(u,v,w;D). \end{cases} \tag{2.12}$$

Letting $D=0$, the flow on $\mathfrak{M}_0$ is

$$\begin{cases} u' = \dfrac{\epsilon(1-n)u - wu}{c}, \\ v' = \dfrac{wu-(\mu+\delta+\epsilon n)v}{c}, \\ w' = \dfrac{\mu v-(d+\epsilon n)w}{c}. \end{cases} \tag{2.13}$$

Since (2.13) and (2.8) are substantially equal, E_0 is unstable and E_1 is globally asymptotically stable for $R_0>1$ by Lemma 2.2. Furthermore, the eigenvalues of the linearization of (2.13) at E_0 are

$$\begin{aligned} \lambda_1 &= -\frac{\epsilon}{c}, \\ \lambda_2 &= -\frac{2\epsilon+\delta+d+\mu+\sqrt{(2\epsilon+\delta+d+\mu)^2+4(\mu-(d+\epsilon)(\mu+\epsilon+\delta))}}{2c}, \\ \lambda_3 &= -\frac{2\epsilon+\delta+d+\mu-\sqrt{(2\epsilon+\delta+d+\mu)^2+4(\mu-(d+\epsilon)(\mu+\epsilon+\delta))}}{2c}. \end{aligned}$$

When $c\ge c^*$, $\lambda_3>0$ is the unique eigenvalue with positive real part. According to [11, Theorem 6.1], there is a one-dimensional unstable manifold on E_0. Given global stability of E_1, the positive branch of the one-dimensional unstable manifold of E_0 for system (2.13), $\mathcal{N}^u(E_0)$, connects to E_1. Namely, there is a heteroclinic orbit which connects E_0 and E_1 for system (2.13). It is obvious that the manifolds $\mathcal{N}^u(E_0)$ and $\mathcal{N}^s(E_1)$ intersect transversally. Next, we prove that when $D>0$ is small, this intersection persists. Therefore, we consider that the equilibrium E_1 of system (2.12) is locally asymptotically stable for small $D>0$.

Lemma 2.3. *For system* (2.12), *suppose that* $R_0>1$. *Then for* $c\ge c^*$, *there is* $D_0>0$ *such that* E_1 *is locally asymptotically stable, when* $0<D<D_0$.

Proof. Let $h_1(D)=\frac{\partial h}{\partial u}(E_1), h_2(D)=\frac{\partial h}{\partial v}(E_1), h_3(D)=\frac{\partial h}{\partial w}(E_1)$. Linearizing (2.12) at E_1, there is the following characteristic equation

$$\lambda^3+A_1(D)\lambda^2+A_2(D)\lambda+A_3(D)=0, \tag{2.14}$$

where

$$\begin{aligned}A_1(D) =&\frac{1}{c}[\mu-\epsilon+d+\delta+w^*+4\epsilon(u^*+v^*+w^*)-cDh_3(D)],\\ A_2(D) =&\frac{1}{c^2}[(\mu-\epsilon+\delta+w^*+4\epsilon(u^*+v^*+w^*))(d+\epsilon-cDh_3(D))\\ &-\delta u^*(\mu+cDh_2(D))+\delta u^* cDh_1(D)-\epsilon^2],\\ A_3(D) =&\frac{1}{c^3}[\epsilon(\mu-\epsilon+\delta+w^*+4\epsilon(u^*+v^*+w^*))(d-cDh_3(D))\\ &-\epsilon\delta u^*(\mu+cDh_2(D))+\epsilon\delta u^* cDh_1(D)].\end{aligned}$$

For $c \geq c^*$, there has $\tilde{c} \geq 0$ which satisfies $c = c^* + \tilde{c}$, and let

$$\begin{aligned}&H_1(D,\tilde{c}) = A_1(D, c = c^*+\tilde{c}),\\ &H_2(D,\tilde{c}) = A_3(D, c = c^*+\tilde{c}),\\ &H_3(D,\tilde{c}) = A_1(D, c = c^*+\tilde{c})A_2(D, c = c^*+\tilde{c}) - A_3(D, c = c^*+\tilde{c}).\end{aligned} \tag{2.15}$$

Then, we can obtain

$$\begin{aligned}H_1(0,\tilde{c}) =&\frac{(\epsilon+\delta)(R_0-1)+d+\mu+\frac{\epsilon\mu}{(\epsilon+\mu+\delta)(d+\epsilon)}}{c^*+\tilde{c}} > 0,\\ H_2(0,\tilde{c}) =&\frac{\mu\epsilon(\epsilon+\delta)(R_0-1)}{(c^*+\tilde{c})^3} > 0,\\ H_3(0,\tilde{c}) =&\frac{1}{(c^*+\tilde{c})^3}\left[(\epsilon+\delta)(R_0-1)+d+\mu+\frac{\epsilon\mu}{(\epsilon+\mu+\delta)(d+\epsilon)}\right]\\ &\left[(d+\mu)(\epsilon+\delta)(R_0-1)+\frac{\delta\epsilon\mu}{(\epsilon+\mu+\delta)(d+\epsilon)}\right]+\frac{1}{(c^*+\tilde{c})^3}\\ &d\mu\left[d+\mu+\frac{\epsilon\mu}{(\epsilon+\mu+\delta)(d+\epsilon)}\right] > 0.\end{aligned}$$

Since $H_i(0,\tilde{c}) > 0, i = 1,2,3$, for any $0 \leq \tilde{c}$, there is $\widehat{D}(\tilde{c}) > 0$ such that $H_i(D,\tilde{c}) > 0$ for every $0 < D < \widehat{D}(\tilde{c})$. Let $D_2 = inf\left\{\widehat{D}(\tilde{c}) | 0 \leq \tilde{c}\right\}$ and $D_0 = min\{D_1, D_2\}$. Then, $A_1(D) > 0, A_3(D) > 0, A_1(D)A_2(D) - A_3(D) > 0$ for $0 < D < D_0$ and $c^* \leq c$. According to Hurwitz criterion, all eigenvalues of (2.14) have negative real parts. Hence, E_1 is locally asymptotically stable. □

Therefore, by [20, Theorem 3.1], we have the following theorem.

Theorem 2.2. *Suppose that $R_0 > 1$. Then for any D such that $0 < D < D_0$, system* (1.2) *has the traveling wave solutions $(u(x+ct), v(x+ct), w(x+ct))$, given that $c \geq c^*$.*

3. Numerical simulation of the traveling waves

In this section, the numerical simulation is exhibited to illustrate the existence of traveling waves for system (1.2). First, we take the parameter values suggested by Anderson et al. [1] as follows:
Therefore, $\epsilon = 0.5/82.4$, $\delta = 0.5/82.4$, $\mu = 365/2307.2$, $d = 73.5/82.4$ and $D = 0.04$ in system (1.2). In addition, the below functions are taken as initial data, which

Table 1. Parameter values selected for simulation

parameter	symbol	value
average birth rate	a	1 per year
average intrinsic death rate	b	0.5 per year
average duration of clinical disease	$1/\alpha$	5 days
average incubation time	$1/\sigma$	28 days
disease transmission coefficient	β	80 km^{-2} per year
carrying capacity	K	1.03 foxese km^{-2}

are shown as

$$s(0,x) = \begin{cases} 1, & x < 0, \\ s^*, & x \geq 0, \end{cases} \quad e(0,x) = \begin{cases} 0, & x < 0, \\ e^*, & x \geq 0, \end{cases} \quad i(0,x) = \begin{cases} 0, & x < 0, \\ i^*, & x \geq 0. \end{cases} \tag{3.1}$$

Here, $(s^*, e^*, i^*) = (0.9656, 1.139 \times 10^{-3}, 2.006 \times 10^{-4})$, which is an endemic equilibrium point. Using (2.7), we can get the minimal speed $c^* = 0.0054$. When $c = 0.0093$, i.e., $c > c^*$, Figure 1 and Figure 2 show that system (1.2) has a traveling wave connecting E_0 and E_1, and fox rabies thus spreads to all the space.

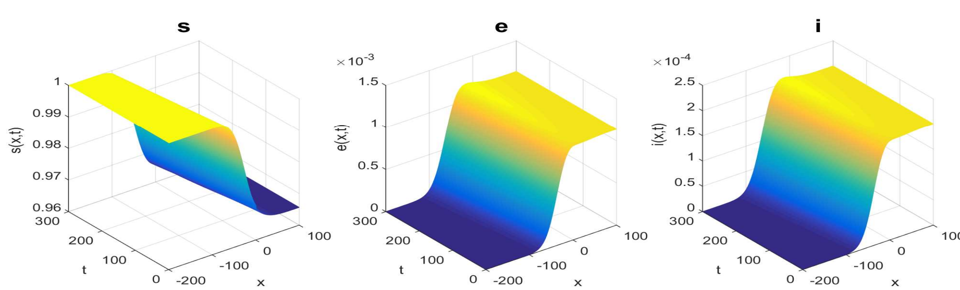

Figure 1. The three-dimensional profiles of traveling wave solutions of system (1.2)

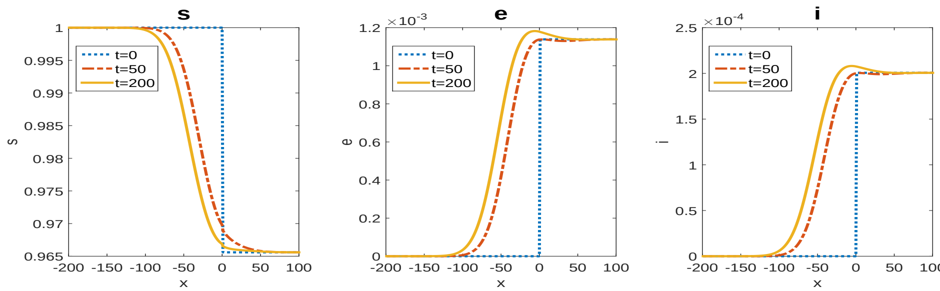

Figure 2. The two-dimensional asymptotic profiles of s, e, i as $t = 0$, $t = 50$, $t = 200$

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