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Existence and Uniqueness of the Solution for Hilfer Neural Networks with Delays*

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Abstract This paper mainly concerns with a class of nonlinear Hilfer fractional neutral recurrent neural networks with time varying delays. The existence and uniqueness of solutions in the space of weighted continuous functions are established by Banach's contraction principle. Finally, an example is provided to illustrate the application of the obtained results.

Keywords Hilfer fractional differential equation, existence and uniqueness, weighted space of continuous functions, fixed point theory

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1. Introduction

In the past three decades, fractional calculus has received an increasing attention due to its various applications in engineering, mechanics, signal processing, material sciences, etc [1–4]. The related theories about fractional differential equations have been extensively studied by many researchers [5–8]. Additionally, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative, was introduced in [1]. After that, a large number of fractional differential equations with Hilfer fractional derivatives were studied [9–13].

On the other hand, neutral neural networks with time delay have aroused naturally in a wide variety of fields like physics, chemistry, control, viscoelastic mechanics, porous media, electromagnetic and polymer rheology, etc. Therefore the issue concerning the existence, uniqueness of solutions of neutral neural networks has been widely discussed by many authors [14–19].

The successive approximation method [20, 21] and the fixed point theorems [20–30] have long been viewed as the main classical methods of studying existence and uniqueness problems in many areas of differential equations. Additionally, compared with fractional differential equations with delays, many dynamical systems not only depend on present and past states but also involve derivatives with delays, and neutral fractional differential equations with delays are often used to describe such systems [23, 31, 32]. Few authors studied the existence and uniqueness for a problem involving Hilfer fractional derivative. For example, C. Kou et

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al. studied the existence and uniqueness on fractional differential equation with Riemann-Liouville fractional derivative [33]. D.F. Luo et.al proved the uniqueness of the solution for the stochastic fractional delay system with Caputo fractional derivative, see [31]. K.M. Furtati et al. studied the existence and uniqueness of global solutions for a class of nonlinear fractional differential equations involving Hilfer fractional derivative, see [34]. However, to the best of our knowledge, the existence and uniqueness of nonlinear Hilfer fractional neutral recurrent neural networks have not been yet developed. In this paper, we will study the existence and uniqueness of a class of nonlinear Hilfer fractional neutral recurrent neural networks with time varying delays by using Banach's fixed point theory.

In this paper, we consider a general class of neural networks with discrete and distributed varying delays which is described by

$$\begin{aligned} D_{0^+}^{\alpha,\beta}\Big[x_i(t)-\sum_{j=1}^{n} q_{ij}x_j(t-\tau(t))\Big] &= \sum_{j=1}^{n} a_{ij}f_j(x_j(t))+\sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau(t))) \\ &\quad+\sum_{j=1}^{n} l_{ij}\int_{t-r(t)}^{t} h_j(x_j(s))ds, t\in[0,+\infty) \end{aligned} \tag{1.1}$$

or

$$D_{0^+}^{\alpha,\beta}\Big[x(t)-Qx(t-\tau(t))\Big]=Af(x(t))+Bg(x(t-\tau(t)))+W\int_{t-r(t)}^{t} h(x(s))ds,$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative with $0<\alpha<1$ and $0\leq\beta\leq 1$, for $i=1,2,3,...,n, x(t)=(x_1(t),x_2(t),...,x_n(t))^T\in\mathbb{R}^n, (x_i(t)\in C([\vartheta,\infty),R))$, are the status vector relating to the neurons; $A=(a_{ij})_{n\times n}, B=(b_{ij})_{n\times n}$ and $W=(l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j,g_j,h_j are activation functions, $f(x(t))=(f_1(x_1(t)),f_2(x_2(t)),...,f_n(x_n(t)))^T\in\mathbb{R}^n$, $g(x(t))=(g_1(x_1(t)),g_2(x_2(t)),...,g_n(x_n(t)))^T\in\mathbb{R}^n$, $h(x(t))=(h_1(x_1(t)),h_2(x_2(t)),...,$ $h_n(x_n(t)))^T\in\mathbb{R}^n$, and the mappings $f_j(\cdot), g_j(\cdot)$, and $h_j(\cdot)$ are globally Lipschitz continuous with constants α_j,β_j and $\gamma_j>0$, which satisfy $f(0)\equiv 0, g(0)\equiv 0, h(0)\equiv 0$, for $j=1,2,...,n$. Here $\tau(t)$ and $r(t)$ are nonnegative continuous functions that express discrete time varying delay and distributed time varying delay, respectively. Besides, the delays satisfy $\lim_{t\to\infty} t-\tau(t)\to\infty$ and $\lim_{t\to\infty} t-r(t)\to\infty$. The initial condition for the system (1) is given by

$$I_{0^+}^{(1-\alpha)(1-\beta)}\Big[x_i(0)-\sum_{j=1}^{n} q_{ij}x_j(-\tau(0))\Big]=\phi_i(0)-\sum_{j=1}^{n} q_{ij}\phi_j(-\tau(0)), \tag{1.2}$$

$$x(t)=\phi(t), t\in[\vartheta,0], \tag{1.3}$$

where $I_{0^+}^{(1-\alpha)(1-\beta)}$ is the Riemann-Liouville fractional integral operator. Denote $\vartheta=\inf_{t\geq 0}\{t-\tau(t),t-r(t)\}, t\mapsto\phi(t)=(\phi_1(t),\phi_2(t),,...,\phi_n(t))^T\in C\big([\vartheta,0],L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\big)$.

The purpose of this paper is to investigate the existence and uniqueness of the solution to the neutral delayed neural networks (1.1) with initial conditions (1.2)-(1.3) through fixed point method. The paper is organized as follows. Some necessary concepts and related lemmas are reviewed in Section 2. In Section 3, we prove the existence and uniqueness of the solution. Examples to illustrate our main results are given in Section 4.

2. Preliminaries and related lemmas

In this section, some definitions and lemmas are given which will be used throughout this paper.

Definition 2.1. ([1]) The right-sided Riemann-Liouville fractional integral of order $\alpha>0, \alpha \in \mathbb{R}$ of a locally integrable function f is defined as

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy,$$

for $x>a$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. ([1]) The right-sided fractional derivative of order $0<\alpha<1$ and type $0 \leq \beta \leq 1$ with respect to x is defined by

$$D_{a^+}^{\alpha,\beta} f(x) = \left(I_{a^+}^{\beta(1-\alpha)} \frac{d}{dx}\left(I_{a^+}^{(1-\beta)(1-\alpha)} f\right)\right)(x),$$

for functions f for which the expression on the right hand side exists, the special cases are Riemann-Liouville fractional derivative for $\beta = 0$ and Caputo fractional difference for $\beta = 1$.

Different choices of norms can be considered. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction mapping with respect to the norm. For the system (1.1) with initial conditions (1.2)-(1.3), we consider the weighted spaces of continuous functions

$$C_\gamma[0,b] = \left\{f : (0,b] \to \mathbb{R} : x^\gamma f(x) \in C[a,b]\right\}, 0 \leq \gamma<1,$$

$$C_{1-\gamma}^{\alpha,\beta}[0,b] = \left\{f \in C_{1-\gamma}[0,b], D_{0^+}^{\alpha,\beta} f \in C_{1-\gamma}[0,b]\right\},$$

and

$$C_{1-\gamma}^{\gamma}[0,b] = \left\{f \in C_{1-\gamma}[0,b], D_{0^+}^{\gamma} f \in C_{1-\gamma}[0,b]\right\},$$

with the norms

$$||f||_{C_\gamma} = ||x^\gamma f(x)||_C.$$

Lemma 2.1 ([2])**.** *Let $0 \leq \gamma<1, 0<b<a, g \in C_\gamma[0,b], g \in C[b,a]$ and g be continuous at b. Then $g \in C_\gamma[0,a]$.*

The following lemmas provide some mapping properties of $I_{0^+}^{\alpha}$. Proofs can be found in [33].

Lemma 2.2 ([2])**.** *For $x>0$ we have*

$$\left[I_{0^+}^{\alpha} t^{\beta-1}\right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} x^{\beta+\alpha-1}, \alpha \geq 0, \beta>0.$$

Lemma 2.3 ([2])**.** *For $\alpha>0$, the fractional integration operator $I_{0^+}^{\alpha}$ maps $C[0,b]$ into $C[0,b]$.*

Lemma 2.4 ([2]). *For $\alpha>0$ and $0 \le \gamma<1$, then the fractional integration operator $I^{\alpha}_{0^+}$ is bounded from $C_{\gamma}[0,b]$ into $C_{\gamma}[0,b]$.*

Lemma 2.5 ([34]). *Let $f \in L^1(a,c)$. Then*

$$\lim_{x\to c^+}\int_a^c (x-t)^{\alpha-1}f(t)dt = \int_a^c (c-t)^{\alpha-1}f(t)dt = \Gamma(\alpha)I^{\alpha}_{a^+}f(c), \alpha>0.$$

3. Main result

In this section, we establish the existence of a unique solution to the problem (1.1)-(1.3) in the space $C^{\alpha,\beta}_{1-\gamma}[0,b]$ by reducing the problem to a Volterra integral equation and then applying the Banach fixed point theorem. We start with some preparations.

Lemma 3.1 (Theorem 23, [34]). *Let $\gamma=\alpha+\beta-\alpha\beta$ where $0<\alpha<1$ and $0 \le \beta \le 1$. Let $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(\cdot,y(\cdot)) \in C_{1-\gamma}[a,b]$ for any $y \in C_{1-\gamma}[a,b]$. If $y \in C^{\gamma}_{1-\gamma}[a,b]$, then y satisfies*

$$\begin{cases} D^{\alpha,\beta}_{a^+}y(x)=f(x,y), x>a, 0<\alpha<1, 0\le\beta\le 1, \\ I^{1-\gamma}_{a^+}y(a^+)=y_a, \end{cases}$$

if and only if y satisfies

$$y(x)=\frac{y_a}{\Gamma(\alpha+\beta-\alpha\beta)}(x-a)^{(\alpha-1)(1-\beta)}+\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t,y(t))dt.$$

According to Lemma 3.1, one can easily express the equivalent Volterra integral equation of the system (1.1)-(1.3) in the following form

$$\begin{aligned} x(t)=&\frac{\left[\phi_i(0)-\sum_{j=1}^n q_{ij}\phi_j(-\tau(0))\right]}{\Gamma(\alpha+\beta-\alpha\beta)}t^{(\alpha-1)(1-\beta)}+\sum_{j=1}^n q_{ij}x_j(t-\tau(t)) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sum_{j=1}^n a_{ij}f_j(x_j(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sum_{j=1}^n b_{ij}g_j\big(x_j(s-\tau(s))\big)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(x_j(u))duds. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $0\le\alpha\le 1$, $0<\beta<1$, and $\gamma=\alpha+\beta-\alpha\beta$. Let $f,g,h:(0,a]\times\mathbb{R}\to\mathbb{R}$ be functions such that $f(x(\cdot)),g(x(\cdot)),h(x(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma}[0,a]$ for any $x \in C_{1-\gamma}[0,a]$. Besides, $f(x(\cdot)),g(x(\cdot)),h(x(\cdot))$ are globally Lipschitz continuous with constants α_j,β_j and $\gamma_j>0$, satisfying that $f(0)\equiv 0, g(0)\equiv 0, h(0)\equiv 0$, for $j=1,2,...,n$, respectively. Then there exists a unique solution $x(t)$ to the problem (1.1)-(1.3) such that $x(t)-\sum_{j=1}^n q_{ij}x_j(t-\tau(t))$ in the space $C^{\gamma}_{1-\gamma}[0,a]$.*

Proof. Define an operator Q by

$$
\begin{aligned}
Qx(t) =& \frac{\left[\phi_i(0) - \sum_{j=1}^{n} q_{ij}\phi_j(-\tau(0))\right]}{\Gamma(\alpha+\beta-\alpha\beta)} t^{(\alpha-1)(1-\beta)} + \sum_{j=1}^{n} q_{ij} x_j(t-\tau(t)) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} a_{ij} f_j(x_j(s)) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} b_{ij} g_j\big(x_j(s-\tau(s))\big) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u)) du ds, t \geq 0.
\end{aligned}
\tag{3.2}
$$

First we prove the existence of a unique solution $x(t)$ in the space $C_{1-\gamma}[0,a]$. Our proof is based on partitioning the interval $(0,a]$ into subintervals on which the operator Q is a contraction, and based on the Banach's fixed point theorem. Note that $C_{1-\gamma}[b_1,b_2], 0 \leq b_1 < b_2 \leq a$ is a complete metric space with the metric d defined by

$$
d(x_1,x_2) = ||x_1 - x_2||_{C_{1-\gamma}[b_1,b_2]} := \max_{t\in[b_1,b_2]} \left|t^{1-\gamma}[x_1(t) - x_2(t)]\right|.
$$

Choose $t_1 \in (0,a]$ such that

$$
\begin{aligned}
\omega_1 =& \Bigg(\sum_{j=1}^{n} |q_{ij}| \max_{t\in[0,t_1]} \left|1+\frac{\tau(t)}{t-\tau(t)}\right|^{1-\gamma} + \Bigg[\sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| \max_{t\in[0,t_1]} \left|1+\frac{\tau(t)}{t-\tau(t)}\right|^{1-\gamma} \\
&+ \sum_{j=1}^{n} \frac{|l_{ij}|}{\gamma} \left(1 - \max_{t\in[0,t_1]} \left|1-\frac{r(t)}{t}\right|^{\gamma}\right)\Bigg] \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} t_1^{\alpha} \Bigg) < 1.
\end{aligned}
\tag{3.3}
$$

Clearly, by Lemma 2.4, we obtain that $Qx(t) \in C_{1-\gamma}[0,t_1]$. Therefore $Q : C_{1-\gamma}[0,t_1] \to C_{1-\gamma}[0,t_1]$. Next we prove that Q is a contraction mapping. For any $\varphi, \psi \in C_{1-\gamma}[0,t_1]$, from Lemma 2.2, we have

$$
\begin{aligned}
&\left|\left|(Q\varphi)_i(t) - (Q\psi)_i(t)\right|\right|_{C_{1-\gamma}[0,t_1]} \\
&\leq \sum_{j=1}^{n} |q_{ij}| \max_{t\in[0,t_1]} \left| \frac{t^{1-\gamma}}{(t-\tau(t))^{1-\gamma}} (t-\tau(t))^{1-\gamma} \Big(\varphi_j(t-\tau(t)) - \psi_j(t-\tau(t))\Big) \right| \\
&\quad + \max_{t\in[0,t_1]} \left| t^{1-\gamma} \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} a_{ij} \int_0^t \frac{(t-s)^{\alpha-1}}{s^{1-\gamma}} s^{1-\gamma} \big(f_j(\varphi_j(s)) - f_j(\psi_j(s))\big) ds \right| \\
&\quad + \max_{t\in[0,t_1]} \Bigg| t^{1-\gamma} \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} b_{ij} \int_0^t \frac{(t-s)^{\alpha-1}}{(s-\tau(s))^{1-\gamma}} (s-\tau(s))^{1-\gamma} \\
&\qquad \big[g_j\big(\varphi_j(s-\tau(s))\big) - g_j\big(\psi_j(s-\tau(s))\big)\big] ds \Bigg| \\
&\quad + \max_{t\in[0,t_1]} \left| t^{1-\gamma} \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} l_{ij} \int_0^t (t-s)^{\alpha-1} \int_{s-r(s)}^{s} \big[h_j\big(\varphi_j(u)\big) - h_j\big(\psi_j(u)\big)\big] du ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n}|q_{ij}|\max_{t\in[0,t_1]}\left|\frac{t^{1-\gamma}}{(t-\tau(t))^{1-\gamma}}\right|\big|\big|\varphi_j(t-\tau(t))-\psi_j(t-\tau(t))\big|\big|_{C_{1-\gamma}[0,t_1]}\\
&\quad+\sum_{j=1}^{n}|a_{ij}|\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}t_1^{\alpha}\big|\big|\varphi_j(t)-\psi_j(t)\big|\big|_{C_{1-\gamma}[0,t_1]}\\
&\quad+\sum_{j=1}^{n}|b_{ij}|\max_{t\in[0,t_1]}\left|1+\frac{\tau(t)}{t-\tau(t)}\right|^{1-\gamma}\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}t_1^{\alpha}\big|\big|\varphi_j(t)-\psi_j(t)\big|\big|_{C_{1-\gamma}[0,t_1]}\\
&\quad+\sum_{j=1}^{n}\frac{|l_{ij}|}{\gamma}\left(1-\max_{t\in[0,t_1]}\left|1-\frac{r(t)}{t}\right|^{\gamma}\right)\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}t_1^{\alpha}\big|\big|\varphi_j(t)-\psi_j(t)\big|\big|_{C_{1-\gamma}[0,t_1]}\\
&\le\left(\sum_{j=1}^{n}|q_{ij}|\max_{t\in[0,t_1]}\left|1+\frac{\tau(t)}{t-\tau(t)}\right|^{1-\gamma}+\left[\sum_{j=1}^{n}|a_{ij}|+\sum_{j=1}^{n}|b_{ij}|\max_{t\in[0,t_1]}\left|1+\frac{\tau(t)}{t-\tau(t)}\right|^{1-\gamma}\right.\right.\\
&\qquad\left.\left.+\sum_{j=1}^{n}\frac{|l_{ij}|}{\gamma}\left(1-\max_{t\in[0,t_1]}\left|1-\frac{r(t)}{t}\right|^{\gamma}\right)\right]\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}t_1^{\alpha}\right)\big|\big|\varphi_j(t)-\psi_j(t)\big|\big|_{C_{1-\gamma}[0,t_1]}.
\end{aligned}
$$

Therefore,

$$
\big|\big|(Q\varphi)_i(t)-(Q\psi)_i(t)\big|\big|_{C_{1-\gamma}[0,t_1]}\le\omega_1\big|\big|\varphi_j(t)-\psi_j(t)\big|\big|_{C_{1-\gamma}[0,t_1]}.
$$

By the contraction mapping principle, we obtain that Q has a unique fixed point $x_0^*(t)\in C_{1-\gamma}[0,t_1]$.

If $t_1\neq a$, then we consider the interval $[t_1,a]$. On this interval we consider solution $x(t)\in C[t_1,a]$ to the equation

$$
\begin{aligned}
x(t)=Qx(t):=&\frac{\left[\phi_i(0)-\sum_{j=1}^{n}q_{ij}\phi_j(-\tau(0))\right]}{\Gamma(\alpha+\beta-\alpha\beta)}t^{(\alpha-1)(1-\beta)}+\sum_{j=1}^{n}q_{ij}x_j(t-\tau(t))\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t-s)^{\alpha-1}\sum_{j=1}^{n}a_{ij}f_j(x_0^*(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\sum_{j=1}^{n}a_{ij}f_j(x_j(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t-s)^{\alpha-1}\sum_{j=1}^{n}b_{ij}g_j\big(x_0^*(s-\tau(s))\big)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\sum_{j=1}^{n}b_{ij}g_j\big(x_j(s-\tau(s))\big)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t-s)^{\alpha-1}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}h_j(x_0^*(u))duds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}h_j(x_j(u))duds.
\end{aligned}\tag{3.4}
$$

Now we select $t_2 \in [t_1, a]$ such that

$$\omega_2 = \sum_{j=1}^{n} |q_{ij}| + \frac{(t_2 - t_1)^\alpha}{\alpha\Gamma(\alpha)} \Big(\sum_{j=1}^{n} |a_{ij}\alpha_j| + \sum_{j=1}^{n} |b_{ij}\beta_j| + r \sum_{j=1}^{n} |l_{ij}\gamma_j| \Big) < 1. \tag{3.5}$$

Since $x_i(t) \in C[t_1, t_2]$, and $f(x(t)), g(x(t)), h(x(t)) \in C[t_1, t_2]$ for any $x(t) \in C[t_1, t_2]$, Lemma 2.3 implies that the right-hand side of (3.4) is in $C[t_1, t_2]$. Therefore $Q : C[t_1, t_2] \to C[t_1, t_2]$. For any $\varphi, \psi \in C[t_1, t_2]$, we have

$$\begin{aligned}
&\big|\big|(Q\varphi)_i(t) - (Q\psi)_i(t)\big|\big|_{C[t_1,t_2]} \\
&\le \Big|\Big| \sum_{j=1}^{n} q_{ij}\big(\varphi_j(t-\tau(t)) - \psi_j(t-\tau(t))\big)\Big|\Big|_{C[t_1,t_2]} \\
&\quad + \Big|\Big| \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} a_{ij} \int_{t_1}^{t} (t-s)^{\alpha-1}\big(f_j(\varphi_j(s)) - f_j(\psi_j(s))\big) ds \Big|\Big|_{C[t_1,t_2]} \\
&\quad + \Big|\Big| \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} b_{ij} \int_{t_1}^{t} (t-s)^{\alpha-1}\big[g_j\big(\varphi_j(s-\tau(s))\big) - g_j\big(\psi_j(s-\tau(s))\big)\big] ds \Big|\Big|_{C[t_1,t_2]} \\
&\quad + \Big|\Big| \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} l_{ij} \int_{t_1}^{t} (t-s)^{\alpha-1} \int_{s-r(s)}^{s} \big[h_j\big(\varphi_j(u)\big) - h_j\big(\psi_j(u)\big)\big] du ds \Big|\Big|_{C[t_1,t_2]} \\
&\le \sum_{j=1}^{n} |q_{ij}| \max\Big\{ \max_{t\in[t_1-\tau, t_1]} \big|\varphi_j(t) - \psi_j(t)\big|, \max_{t\in[t_1,t_2]} \big|\varphi_j(t) - \psi_j(t)\big| \Big\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} |a_{ij}\alpha_j| \Big|\Big| \max_{s\in[t_1,t]} \big(\varphi_j(s) - \psi_j(s)\big) \int_{t_1}^{t} (t-s)^{\alpha-1} ds \Big|\Big|_{C[t_1,t_2]} \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} |b_{ij}\beta_j| \Big|\Big| \max_{s\in[t_1,t]} \big(\varphi_j(s-\tau(s)) - \psi_j(s-\tau(s))\big) \int_{t_1}^{t} (t-s)^{\alpha-1} ds \Big|\Big|_{C[t_1,t_2]} \\
&\quad + \frac{r}{\Gamma(\alpha)} \sum_{j=1}^{n} |l_{ij}\gamma_j| \Big|\Big| \max_{u\in[s-r(s),s]} \big(\varphi_j(u) - \psi_j(u)\big) \int_{t_1}^{t} (t-s)^{\alpha-1} ds \Big|\Big|_{C[t_1,t_2]} \\
&\le \sum_{j=1}^{n} |q_{ij}| \big|\big|\varphi_j(t) - \psi_j(t)\big|\big|_{C[t_1,t_2]} + \frac{(t_2-t_1)^\alpha}{\alpha\Gamma(\alpha)} \sum_{j=1}^{n} |a_{ij}\alpha_j| \big|\big|\varphi_j(t) - \psi_j(t)\big|\big|_{C[t_1,t_2]} \\
&\quad + \frac{(t_2-t_1)^\alpha}{\alpha\Gamma(\alpha)} \sum_{j=1}^{n} |b_{ij}\beta_j| \big|\big|\varphi_j(t) - \psi_j(t)\big|\big|_{C[t_1,t_2]} \\
&\quad + \frac{r(t_2-t_1)^\alpha}{\alpha\Gamma(\alpha)} \sum_{j=1}^{n} |l_{ij}\gamma_j| \big|\big|\varphi_j(t) - \psi_j(t)\big|\big|_{C[t_1,t_2]} \\
&= \left(\sum_{j=1}^{n} |q_{ij}| + \frac{(t_2-t_1)^\alpha}{\alpha\Gamma(\alpha)} \Big(\sum_{j=1}^{n} |a_{ij}\alpha_j| + \sum_{j=1}^{n} |b_{ij}\beta_j| + r \sum_{j=1}^{n} |l_{ij}\gamma_j| \Big) \right) \\
&\qquad \big|\big|\varphi_j(t) - \psi_j(t)\big|\big|_{C[t_1,t_2]}.
\end{aligned}$$

Since $0 < \omega_2 < 1$, Q is a contraction. By the contraction mapping principle, we obtain that Q has a unique fixed point $x_1^*(t) \in C[t_1, t_2]$ to (3.2). Moreover, it follows from Lemma 2.5 that $x_0^*(t_1) = x_1^*(t_1)$. Therefore, if

$$x^*(t) = \begin{cases} x_0^*(t),\ 0<t \le t_1, \\ x_1^*(t),\ t_1<t \le t_2, \end{cases}$$

then by Lemma 2.1, $x^*(t) \in C_{1-\gamma}[0, t_2]$. So $x^*(t)$ is the unique solution of (3.2) in $C_{1-\gamma}[0, t_2]$ on the interval $[0, t_2]$.

If $t_2 \neq a$, we repeat the process as necessary, such as $N-2$ times, to obtain the unique solution $x_k^*(t) \in C[t_k, t_{k+1}], k = 2, 3, \cdots, N$, where $0 = t_0<t_1<\cdots<t_N = a$, such that

$$\omega_{k+1} = \sum_{j=1}^{n} |q_{ij}| + \frac{(t_{k+1}-t_k)^\alpha}{\alpha\Gamma(\alpha)}\Big(\sum_{j=1}^{n}|a_{ij}\alpha_j| + \sum_{j=1}^{n}|b_{ij}\beta_j| + r\sum_{j=1}^{n}|l_{ij}\gamma_j|\Big)<1.$$

As a result we have the unique solution $x^*(t) \in C_{1-\gamma}[0, a]$ of (3.2) given by

$$x^*(t) = x_k^*(t), t \in (t_k, t_{k+1}], k = 0, 1, \cdots, N-1.$$

It remains to show that such a unique solution $x^*(t) \in C_{1-\gamma}[0, a]$ is actually in $C_{1-\gamma}^{\gamma}[0, a]$. Applying $D_{0^+}^{\gamma}$ to both sides yields of (3.1)

$$\begin{aligned}
&D_{0^+}^{\gamma}\left[x^*(t) - \sum_{j=1}^{n} q_{ij}x_j^*(t-\tau(t))\right] \\
&\quad = D_{0^+}^{\gamma}\left[I_{0^+}^{\alpha}\left(\sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j\big(x_j(t-\tau(t))\big) + \sum_{j=1}^{n} l_{ij}\int_{t-r(t)}^{t} h_j(x_j(s))ds\right)\right] \\
&\quad = D_{0^+}^{\beta(1-\alpha)}\left[\sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j\big(x_j(t-\tau(t))\big) + \sum_{j=1}^{n} l_{ij}\int_{t-r(t)}^{t} h_j(x_j(s))ds\right].
\end{aligned}$$

By hypothesis, the right hand side is in $C_{1-\gamma}[0, a]$ and thus $x^*(t) - \sum_{j=1}^{n} q_{ij}x_j^*(t-\tau(t)) \in C_{1-\gamma}^{\gamma}[0, a]$.

Therefore, by Lemma 3.1, $x^*(t)$ is the unique solution of (1.1)-(1.3) such that $x^*(t) - \sum_{j=1}^{n} q_{ij}x_j^*(t-\tau(t))$ in the space $C_{1-\gamma}^{\gamma}[0, a]$. □

4. Example

Example 4.1. Consider the following two-dimensional Hilfer fractional neutral neural network

$$D_{0^+}^{\alpha,\beta}\Big[x(t) - Qx(t-\tau(t))\Big] = Af(x(t)) + Bg(x(t-\tau(t))) + W\int_{t-r(t)}^{t} h(x(s))ds, \tag{4.1}$$

where $\alpha = 0.8, \beta = 0.25$,

$$Q = \begin{pmatrix} -0.1 & 0 \\ 0.1 & 0.1 \end{pmatrix}, A = \begin{pmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{pmatrix}, B = \begin{pmatrix} -0.1 & 0.2 \\ 0.1 & 0.2 \end{pmatrix}, W = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

$f(x) = g(x) = h(x) = 0.2\tanh(x)$. $\tau(t), r(t)$ are continuous functions such that $|\tau(t)| \leq 1$ and $|r(t)| \leq 1$.

In this example, Letting $\alpha_j = 0.2, \beta_j = 0.25, \gamma_j = 0.1, j = 1, 2,\ k = 2, a = 1, t_1 = 0.1, t_2 = 0.2, t_3 = a = 1$, we get

$$\omega_1 = \Bigg(\sum_{j=1}^{2} |q_{ij}| \max_{t\in[0,t_1]} \left| 1 + \frac{\tau(t)}{t-\tau(t)} \right|^{1-\gamma} + \Bigg[\sum_{j=1}^{2} |a_{ij}| + \sum_{j=1}^{2} |b_{ij}| \max_{t\in[0,t_1]} \left| 1 + \frac{\tau(t)}{t-\tau(t)} \right|^{1-\gamma} + \sum_{j=1}^{2} \frac{|l_{ij}|}{\gamma} \left(1 - \max_{t\in[0,t_1]} \left| 1 - \frac{r(t)}{t} \right|^{\gamma} \right) \Bigg] \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} t_1^{\alpha} \Bigg) \leq 0.671 < 1,$$

$$\omega_2 = \sum_{j=1}^{n} |q_{ij}| + \frac{(t_2 - t_1)^{\alpha}}{\alpha\Gamma(\alpha)} \Big(\sum_{j=1}^{n} |a_{ij}\alpha_j| + \sum_{j=1}^{n} |b_{ij}\beta_j| + r \sum_{j=1}^{n} |l_{ij}\gamma_j| \Big) \leq 0.604 < 1.$$

Letting $k = 2$, we have

$$\omega_3 = \sum_{j=1}^{2} |q_{ij}| + \frac{(t_{k+1} - t_k)^{\alpha}}{\alpha\Gamma(\alpha)} \Big(\sum_{j=1}^{2} |a_{ij}\alpha_j| + \sum_{j=1}^{2} |b_{ij}\beta_j| + r \sum_{j=1}^{2} |l_{ij}\gamma_j| \Big) \leq 0.897 < 1.$$

Applying Theorem 3.1, the initial value problem (4.1) has a unique solution $x(t)$ such that $x(t) - \sum_{j=1}^{n} q_{ij} x_j(t - \tau(t))$ in the space $C_{1-\gamma}^{\gamma}[0, a]$.

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