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Existence and Uniqueness of Smooth Solution for a Four-waves Coupled System*

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Abstract In this paper, we consider a four-waves coupled system which describes the interaction between particles. Based on the uniform bound and the strong convergence property in the lower order norm, local existence and uniqueness of smooth solution are established by a limiting argument. Moreover, we show that the solution exists globally in the two-dimensional case under certain condition on the size for L^2 norm of the initial data.

Keywords Existence and uniqueness, global solution, four-waves coupled system

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1. Introduction

In this paper, we consider the Cauchy problem for a four-waves coupled system which reads

$$\left(\mathrm{i}(\partial_t + v_C\partial_y) + \alpha\Delta\right) A_C = \frac{b^2}{2} n A_C, \tag{1.1}$$

$$\left(\mathrm{i}(\partial_t + v_R\partial_y) + \beta\Delta\right) A_R = \frac{bc}{2} n A_R, \tag{1.2}$$

$$\left(\mathrm{i}\partial_t + \gamma\Delta\right) E = \frac{b}{2} n E, \tag{1.3}$$

$$\left(\partial_t^2 - v_s^2\Delta\right) n = a\Delta\left(|E|^2 + b\,|A_C|^2 + c\,|A_R|^2\right), \tag{1.4}$$

$$(A_C, A_R, E, n)(0) = (a_C, a_R, e, n_0),\ n_t(0) = n_1. \tag{1.5}$$

In this system, $A_C = A_0 + e^{-2iy}A_B$ where A_0 is the incident laser field and A_B is the Brillouin component, A_R is the Raman backscattered wave, E is the electronic-plasma wave and n is the variation of density of ions. Furthermore, $A_C,\ A_R,\ E$ and n are functions of $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ with $A_C,\ A_R$ and E the vector fields such that $A_C,\ A_R,\ E : \mathbb{R}^{d+1} \longrightarrow \mathbb{C}^d$, and with n the scalar field such that $n : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$. In this paper, we mainly consider the dimension $d = 2, 3$. The coefficients α, β, γ, v_C, v_R, a, b, c and v_s in the above system are real physical constants with $\alpha,\ \beta,\ \gamma > 0$.

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In equation (1.1) and (1.2), y represents the second component of the variable x, namely, $x = (x_1, y, x_3)$ when d=3 and $x = (x_1, y)$ when $d = 2$. With this notation, the Laplacian Δ is defined by

$$\Delta = \partial_{x_1}^2 + \partial_y^2 + \partial_{x_3}^2 \text{ if } d = 3, \quad \Delta = \partial_{x_1}^2 + \partial_y^2 \text{ if } d = 2.$$

In the following arguments of the paper, we often regard y to be x_2 for simplicity. System (1.1)-(1.5) was derived by M. Colin and T. Colin [1] which is a complete set of Zakharov's equations type describing laser-plasma interactions and we have omitted the quasilinear part in this context.

Ignoring the effect of the scattering fields A_C and A_R, system (1.1)-(1.5) is reduced to the classical Zakharov system [18]. Due to its physical importance, the Zakharov system has been studied intensively in mathematics since the works [6,15] and many important developments were obtained in the past decades ([5]). Further, omitting the term n_{tt}, the system is reduced to the cubic Schrödinger equation which has been studied by many researchers, see for example [2,8,12,13] and the references cited therein. For the three-waves (A_C, A_R and E) interacted system, the authors in [9–11] studied the local well-posedness theory.

The work is concerned with the existence and uniqueness of the smooth solution for the four-waves coupled system (1.1)-(1.5). We first introduce a vector-valued function V to rewrite the original system (1.1)-(1.5) as a Hamilton form.

$$\left(\mathrm{i}(\partial_t + v_C \partial_y) + \alpha\Delta\right) A_C = \frac{b^2}{2} n A_C, \tag{1.6}$$

$$\left(\mathrm{i}(\partial_t + v_R \partial_y) + \beta\Delta\right) A_R = \frac{bc}{2} n A_R, \tag{1.7}$$

$$\left(\mathrm{i}\partial_t + \gamma\Delta\right) E = \frac{b}{2} n E, \tag{1.8}$$

$$n_t + \nabla \cdot V = 0, \tag{1.9}$$

$$V_t + v_s^2 \nabla n + a\nabla\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right) = 0, \tag{1.10}$$

$$(A_C, A_R, E, n, V)(0) = (a_C, a_R, e, n_0, V_0). \tag{1.11}$$

Throughout the paper, we denote by $L^p(\mathbb{R}^d)$ the Lebesgue space equipped with the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^d} |u(x)|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \le p < +\infty$$

and

$$\|u\|_{L^\infty} = \text{esssup}\{|u(x)|; x \in \mathbb{R}^d\}.$$

For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\widehat{u}(\xi)|^2 d\xi < +\infty\},$$

where $\widehat{u}(\xi)$ is the Fourier transform of u.

The main results of the paper are stated in the following theorems.

Theorem 1.1. *Letting $d = 2, 3$, assume that a_C, a_R, $e \in H^m(\mathbb{R}^d)$, n_0, $V_0 \in H^{m-1}(\mathbb{R}^d)$, $m \ge 4$ is an integer and $ab > 0$. Then system (1.6)-(1.11) admits a unique solution (A_C, A_R, E, n, V) such that*

$$A_C,\ A_R,\ E \in C\left([0, T); H^m\left(\mathbb{R}^d\right)\right),\ n,\ V \in C\left([0, T); H^{m-1}\left(\mathbb{R}^d\right)\right),$$

where T depends on $\|a_C\|_{H^m}$, $\|a_R\|_{H^m}$, $\|e\|_{H^m}$, $\|n_0\|_{H^{m-1}}$ and $\|V_0\|_{H^{m-1}}$. Moreover, the solution satisfies the conserved quantities

$$\Phi_1(t) \equiv \Phi_1(0), \quad \Phi_2(t) \equiv \Phi_2(0), \quad \Phi_3(t) \equiv \Phi_3(0), \quad \Psi(t) \equiv \Psi(0),$$

where

$$\begin{aligned}
\Phi_1(t) :=& \|A_C(t)\|_{L^2}^2, \quad \Phi_2(t) := \|A_R(t)\|_{L^2}^2, \quad \Phi_3(t) := \|E(t)\|_{L^2}^2, \\
\Psi(t) :=& \alpha\|\nabla A_C\|_{L^2}^2 + \beta\,\|\nabla A_R\|_{L^2}^2 + \gamma\|\nabla E\|_{L^2}^2 + \frac{b}{4a}v_s^2\|n\|_{L^2}^2 + \frac{b}{4a}\|V\|_{L^2}^2 \\
&+ \int_{\mathbb{R}^d} n(t)\left(\frac{b^2}{2}|A_C(t)|^2 + \frac{bc}{2}|A_R(t)|^2 dx + \frac{b}{2}|E(t)|^2\right)dx \\
&+ \operatorname{Im}\int_{\mathbb{R}^d}(v_C\partial_y A_C\overline{A_C} + v_R\partial_y A_R\overline{A_R})dx.
\end{aligned}$$

Theorem 1.2. *In addition to the hypothesis of Theorem 1.1 with $d = 2$, assume further*

$$\begin{aligned}
\|a_C\|_{L^2}^2 &< \frac{2\alpha v_s^2}{ab(b^2+c^2+1)}\|\psi\|_{L^2}^2, \\
\|a_R\|_{L^2}^2 &< \frac{2\beta v_s^2}{ab(b^2+c^2+1)}\|\psi\|_{L^2}^2, \\
\|e\|_{L^2}^2 &< \frac{2\gamma v_s^2}{ab(b^2+c^2+1)}\|\psi\|_{L^2}^2,
\end{aligned}$$

where ψ is the ground state solution of

$$\Delta\psi - \psi + \psi^3 = 0.$$

Then system (1.1)-(1.5) possesses a unique global solution (A_C, A_R, E, n) satisfying

$$A_C,\ A_R,\ E \in C\left(\mathbb{R}^+; H^m(\mathbb{R}^2)\right),\ n \in C\left(\mathbb{R}^+; H^{m-1}(\mathbb{R}^2)\right).$$

The paper is organized as follows. In Section 2, we construct a regularized system and give the existence and uniqueness result. In Section 3, local existence and uniqueness theorem of smooth solution will be established. Finally, we will consider global existence result of the solution in two-dimensional case.

2. Global solution for a regularized system

Theorem 1.1 will be proved via a limiting argument. To this end, we first introduce an approximated system for (1.6)-(1.11) with a regularized parameter $\varepsilon > 0$

$$\mathrm{i}A_{Ct}^\varepsilon + \mathrm{i}\varepsilon\Delta^2 A_{Ct}^\varepsilon = -\alpha\Delta A_C^\varepsilon - \mathrm{i}v_C\partial_y A_C^\varepsilon + \frac{b^2}{2}n^\varepsilon A_C^\varepsilon, \tag{2.1}$$

$$\mathrm{i}A_{Rt}^\varepsilon + \mathrm{i}\varepsilon\Delta^2 A_{Rt}^\varepsilon = -\beta\Delta A_R^\varepsilon - \mathrm{i}v_R\partial_y A_R^\varepsilon + \frac{bc}{2}n^\varepsilon A_R^\varepsilon, \tag{2.2}$$

$$\mathrm{i}E_t^\varepsilon + \mathrm{i}\varepsilon\Delta^2 E_t^\varepsilon = -\gamma\Delta E^\varepsilon + \frac{b}{2}n^\varepsilon E^\varepsilon, \tag{2.3}$$

$$A_C^\varepsilon(0) = a_C^\varepsilon,\ A_R^\varepsilon(0) = a_R^\varepsilon,\ E^\varepsilon(0) = e^\varepsilon, \tag{2.4}$$

where n^ε in (2.1)-(2.3) is determined through the equations

$$\begin{aligned} n_t^\varepsilon + \nabla\cdot V^\varepsilon &= 0,\\ V_t^\varepsilon + v_s^2\nabla n^\varepsilon &= -a\nabla\left(|E^\varepsilon|^2 + b|A_C^\varepsilon|^2 + c|A_R^\varepsilon|^2\right),\\ n^\varepsilon(0) = n_0^\varepsilon,\ V^\varepsilon(0) &= V_0^\varepsilon. \end{aligned} \tag{2.5}$$

For $\varepsilon > 0$, we define the Fourier operator $\Lambda: \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\Lambda := (I + \varepsilon\Delta^2)^{-1}.$$

Note that for any $s \in \mathbb{R}$, $\Lambda: H^s(\mathbb{R}^d) \to H^{s+4}(\mathbb{R}^d)$ is a bounded operator. Also, it is easy to see that Λ satisfies the following properties:

(i) $\|\Lambda f\|_{H^k} \le \|f\|_{H^k}, \forall k \in \mathbb{R}$;

(ii) $(\Lambda f, f) = \int_{\mathbb{R}^d}(\Lambda f)\cdot \overline{f}dx \ge 0$;

(iii) $(\Lambda f, g) = (f, \Lambda g)$;

(iv) Λ commutes with Fourier multipliers such as $\Lambda^s, \nabla, \Delta$.

These properties will be frequently used in the succeeding energy estimates.

Now the solution of (2.1)-(2.3) can be rewritten in the integral form

$$A_C^\varepsilon(t) = U_1^\varepsilon(t)a_C^\varepsilon + \int_0^t U_1^\varepsilon(t-\tau)f_1^\varepsilon(\tau)d\tau, \tag{2.6}$$

$$A_R^\varepsilon(t) = U_2^\varepsilon(t)a_R^\varepsilon + \int_0^t U_2^\varepsilon(t-\tau)f_2^\varepsilon(\tau)d\tau, \tag{2.7}$$

$$E^\varepsilon(t) = U_3^\varepsilon(t)e^\varepsilon + \int_0^t U_3^\varepsilon(t-\tau)f_3^\varepsilon(\tau)d\tau, \tag{2.8}$$

where the linear semigroups $U_1^\varepsilon(t),\ U_2^\varepsilon(t),\ U_3^\varepsilon(t)$ are defined by

$$\begin{aligned} U_1^\varepsilon(t) &= \exp\left[(\mathrm{i}\alpha\Lambda\Delta - v_C\Lambda\partial_y)t\right],\\ U_2^\varepsilon(t) &= \exp\left[(\mathrm{i}\beta\Lambda\Delta - v_R\Lambda\partial_y)t\right],\\ U_3^\varepsilon(t) &= \exp\left[\mathrm{i}\gamma\Lambda\Delta t\right], \end{aligned}$$

and the nonlinear terms $f_1^\varepsilon(t),\ f_2^\varepsilon(t),\ f_3^\varepsilon(t)$ are defined by

$$f_1^\varepsilon = -\mathrm{i}\Lambda(\frac{b^2}{2}n^\varepsilon A_C^\varepsilon),\ f_2^\varepsilon = -\mathrm{i}\Lambda(\frac{bc}{2}n^\varepsilon A_R^\varepsilon),\ f_3^\varepsilon = -\mathrm{i}\Lambda(\frac{b}{2}n^\varepsilon E^\varepsilon).$$

Since n^ε satisfies the nonlinear wave equation, then it can be expressed by([16, Chapter2])

$$\begin{aligned} n^\varepsilon =& \cos(v_s t\sqrt{-\Delta})n_0^\varepsilon + \frac{\sin(v_s t\sqrt{-\Delta})}{v_s\sqrt{-\Delta}}n_1^\varepsilon\\ &+ \int_0^t \frac{\sin(v_s(t-\tau)\sqrt{-\Delta})}{v_s\sqrt{-\Delta}}\Delta\left(a(|E^\varepsilon|^2 + b\,|A_C^\varepsilon|^2 + c\,|A_R^\varepsilon|^2)\right)d\tau. \end{aligned} \tag{2.9}$$

The main result for (2.1)-(2.5) is stated as follows.

Theorem 2.1. *Assume that* $a_C^\varepsilon,\ a_R^\varepsilon,\ e^\varepsilon \in H^k(\mathbb{R}^d),\ n_0^\varepsilon,\ V_0^\varepsilon \in H^{k-1}(\mathbb{R}^d)$ *with* $k \geq 5$. *Then for any given* $\varepsilon > 0$, *system* (2.1)-(2.5) *has a unique solution* $A_C^\varepsilon,\ A_R^\varepsilon,\ E^\varepsilon \in C(\mathbb{R}^+; H^k(\mathbb{R}^d))$ *and* $n^\varepsilon,\ V^\varepsilon \in C(\mathbb{R}^+; H^{k-1}(\mathbb{R}^d))$.

According to the integral system (2.6)-(2.9), the above theorem can be proved by using the contractive mapping principle. The regularized operator $\varepsilon\Delta^2$ in (2.1)-(2.3) plays two important roles during the proof. On one hand, it counteracts the difficulty in the nonlinear estimates caused by different regular conditions on the solution. On the other hand, Proposition 2.1 below yields the L^∞ estimate which enables the solution to be extended globally. Since the proof is standard, we omit further details and refer to similar arguments in [7, 14].

Proposition 2.1. *The solution obtained in Theorem 2.1 admits the mass conservation and the energy conservation*

$$\Phi_1^\varepsilon(t) \equiv \Phi_1^\varepsilon(0),\ \Phi_2^\varepsilon(t) \equiv \Phi_2^\varepsilon(0),\ \Phi_3^\varepsilon(t) \equiv \Phi_3^\varepsilon(0),\ \Psi^\varepsilon(t) \equiv \Psi^\varepsilon(0),$$

where

$$\begin{aligned}
\Phi_1^\varepsilon(t) &:= \|A_C^\varepsilon\|_{L^2}^2 + \varepsilon\, \|\Delta A_C^\varepsilon\|_{L^2}^2\,, \\
\Phi_2^\varepsilon(t) &:= \|A_R^\varepsilon\|_{L^2}^2 + \varepsilon\, \|\Delta A_R^\varepsilon\|_{L^2}^2\,, \\
\Phi_3^\varepsilon(t) &:= \|E^\varepsilon\|_{L^2}^2 + \varepsilon\, \|\Delta E^\varepsilon\|_{L^2}^2\,, \\
\Psi^\varepsilon(t) &:= \alpha\|\nabla A_C^\varepsilon\|_{L^2}^2 + \beta\, \|\nabla A_R^\varepsilon\|_{L^2}^2 + \gamma\|\nabla E^\varepsilon\|_{L^2}^2 + \frac{b}{4a} v_s^2 \|n^\varepsilon\|_{L^2}^2 + \frac{b}{4a}\|V^\varepsilon\|_{L^2}^2 \\
&\quad + \int_{\mathbb{R}^d} n^\varepsilon(t) \left(\frac{b^2}{2}|A_C^\varepsilon(t)|^2 + \frac{bc}{2}|A_R^\varepsilon(t)|^2 dx + \frac{b}{2}|E^\varepsilon(t)|^2 \right) dx \\
&\quad + \operatorname{Im} \int_{\mathbb{R}^d} (v_C \partial_y A_C^\varepsilon \overline{A_C^\varepsilon} + v_R \partial_y A_R^\varepsilon \overline{A_R^\varepsilon}) dx.
\end{aligned}$$

Proof. Multiplying both sides of (2.1)-(2.3) with $\overline{A_C^\varepsilon}$, $\overline{A_R^\varepsilon}$ and $\overline{E^\varepsilon}$, respectively, integrating the resulted equations over $\mathbb{R}^d$ and comparing the imaginary parts of the integral equations, we can obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} (|A_C^\varepsilon|^2 + \varepsilon^2\, |\Delta A_C^\varepsilon|^2) dx = 0, \\
\frac{d}{dt}\int_{\mathbb{R}^d} (|A_R^\varepsilon|^2 + \varepsilon^2\, |\Delta A_R^\varepsilon|^2) dx = 0, \\
\frac{d}{dt}\int_{\mathbb{R}^d} (|E^\varepsilon|^2 + \varepsilon^2\, |\Delta E^\varepsilon|^2) dx = 0,
\end{aligned}$$

which gives the mass conservation. To obtain the conservation of energy, we multiply both sides of (2.1)-(2.3) with $\overline{A_{Ct}^\varepsilon}$, $\overline{A_{Rt}^\varepsilon}$ and $\overline{E_t^\varepsilon}$, respectively, then integrate the resulted equations over $\mathbb{R}^d$ and compare the real parts of the equations. There hold

$$\alpha\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla A_C^\varepsilon|_t^2\, dx + \frac{b^2}{2}\int_{\mathbb{R}^d} n^\varepsilon\, |A_C^\varepsilon|_t^2\, dx + 2\operatorname{Im}\int_{\mathbb{R}^d} v_C \partial_y A_C^\varepsilon \overline{A_{Ct}^\varepsilon} dx = 0, \tag{2.10}$$

$$\beta\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla A_R^\varepsilon|_t^2\, dx + \frac{bc}{2}\int_{\mathbb{R}^d} n^\varepsilon\, |A_R^\varepsilon|_t^2\, dx + 2\operatorname{Im}\int_{\mathbb{R}^d} v_R \partial_y A_R^\varepsilon \overline{A_{Rt}} dx = 0, \tag{2.11}$$

$$\gamma\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla E^\varepsilon|^2 dx + \frac{b}{2}\int_{\mathbb{R}^d} n^\varepsilon |E^\varepsilon|_t^2 dx = 0. \tag{2.12}$$

Taking the inner products of either side of the first and second equations of (2.5) with n^ϵ and V^ϵ, respectively, we can see that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|n^\varepsilon|^2dx-\int_{\mathbb{R}^d}V^\varepsilon\cdot\nabla n^\varepsilon dx=0, \tag{2.13}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|V^\varepsilon|^2dx+v_s^2\int_{\mathbb{R}^d}V^\varepsilon\cdot\nabla n^\varepsilon dx+a\int_{\mathbb{R}^d}n_t^\varepsilon\left(|E^\varepsilon|^2+b\,|A_C^\varepsilon|^2+c\,|A_R^\varepsilon|^2\right)dx=0. \tag{2.14}$$

Combining with the identities (2.10)-(2.14), we thus obtain the second conserved law. □

3. Local existence and uniqueness

This section is devoted to proving Theorem 1.1. The main steps of the proof are the conclusions established by Proposition 3.1 and Proposition 3.2 below.

For given initial data $a_C,\ a_R,\ e\in H^m(\mathbb{R}^d)$ and $n_0,\ V_0\in H^{m-1}(\mathbb{R}^d)$, we choose $a_C^\varepsilon,\ a_R^\varepsilon,\ e^\varepsilon\in H^k(\mathbb{R}^d)$ and $n_0^\varepsilon,\ V_0^\varepsilon\in H^{k-1}(\mathbb{R}^d)$ satisfying

$$\|a_C^\varepsilon-a_C\|_{H^m}\to 0,\ \|a_R^\varepsilon-a_R\|_{H^m}\to 0,\ \|e^\varepsilon-e\|_{H^m}\to 0 \tag{3.1}$$

and

$$\|n_0^\varepsilon-n_0\|_{H^{m-1}}\to 0,\ \|V_0^\varepsilon-V_0\|_{H^{m-1}}\to 0 \tag{3.2}$$

as $\varepsilon\to 0$. Here k can be selected large enough, for example $k>m+10$. According to Theorem 2.1, the regularized system (2.1)-(2.5) has a unique smooth solution. Hence, all the differential operations or integration by part appearing in Proposition 3.1 and Proposition 3.2 below are meaningful as the approximated solution is regular enough and decays at infinity.

Proposition 3.1. *Assume that* (3.1) *and* (3.2) *hold with* $m\geq 4$ *be an integer, then there exist* $T>0$ *and* $C>0$ *such that*

$$\|A_C^\varepsilon\|_{H^m}+\|A_R^\varepsilon\|_{H^m}+\|E^\varepsilon\|_{H^m}+\|n^\varepsilon\|_{H^{m-1}}+\|V^\varepsilon\|_{H^{m-1}}\leq C,\quad \forall t\in[0,T],$$

where both C *and* T *are independent of* ε.

Proof. To make the proof clear, we only prove the case $m=4$. Moreover, for sake of simplicity, the superscript ε is omitted in this proof. By the mass conservation of Proposition 2.1, we see

$$\|A_C\|_{L^2}+\|A_R\|_{L^2}+\|E\|_{L^2}\leq C,\quad t\geq 0. \tag{3.3}$$

Also, it follows from (2.13) and (2.14) that

$$\frac{d}{dt}(v_s^2\|n\|_{L^2}^2+\|V\|_{L^2}^2)\leq C\|\nabla V\|_{L^2}(\|A_C\|_{H^2}+\|A_R\|_{H^2}+\|E\|_{H^2}). \tag{3.4}$$

(3.3) and (3.4) give the lowest order estimates of the solution. Now we deal with the highest order estimates which should be treated precisely.

Taking the inner product of $(\partial_t^2 - v_s^2\Delta)n = a\Delta\left(|E|^2 + b\,|A_C|^2 + c\,|A_R|^2\right)$ with $\Delta^2 n_t$ yields

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta n_t\|_{L^2}^2 + v_s^2\|\nabla\Delta n\|_{L^2}^2 + 2ab\int_{\mathbb{R}^d}\nabla\Delta|A_C|^2\nabla\Delta n dx\\
&\quad + 2ac\int_{\mathbb{R}^d}\nabla\Delta|A_R|^2\nabla\Delta n dx + 2a\int_{\mathbb{R}^d}\nabla\Delta|E|^2\nabla\Delta n dx)\\
=&2ab\int_{\mathbb{R}^d}\nabla\Delta|A_C|_t^2\nabla\Delta n dx + 2ac\int_{\mathbb{R}^d}\nabla\Delta|A_R|_t^2\nabla\Delta n dx + 2a\int_{\mathbb{R}^d}\nabla\Delta|E|_t^2\nabla\Delta n dx.
\end{aligned}$$

Since $A_C,\ A_R$ and E satisfy

$$|A_C|_t^2 = 2\,\mathrm{Im}((-\alpha(\Lambda\Delta A_C) + \frac{b^2}{2}\Lambda(nA_C) - \mathrm{i}v_C\Lambda\partial_y A_C)\cdot\overline{A_C}), \tag{3.5}$$

$$|A_R|_t^2 = 2\,\mathrm{Im}((-\beta(\Lambda\Delta A_R) + \frac{bc}{2}\Lambda(nA_R) - \mathrm{i}v_R\Lambda\partial_y A_R)\cdot\overline{A_R}), \tag{3.6}$$

$$|E|_t^2 = 2\,\mathrm{Im}((-\gamma(\Lambda\Delta E) + \frac{b}{2}\Lambda(nE))\cdot\overline{E}), \tag{3.7}$$

respectively, then using identities (3.5)-(3.7), we can obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}\nabla\Delta|A_C|_t^2\nabla\Delta n dx =& -2\alpha\,\mathrm{Im}\int_{\mathbb{R}^d}\nabla\Delta\left((\Lambda\Delta A_C)\cdot\overline{A_C}\right)\nabla\Delta n dx\\
&+ b^2\,\mathrm{Im}\int_{\mathbb{R}^d}\nabla\Delta\left(\Lambda(nA_C)\cdot\overline{A_C}\right)\nabla\Delta n dx\\
&- 2v_C\,\mathrm{Im}\,\mathrm{i}\int_{\mathbb{R}^d}\nabla\Delta\left(\Lambda\partial_y A_C\cdot\overline{A_C}\right)\nabla\Delta n dx,
\end{aligned}\tag{3.8}$$

$$\begin{aligned}
\int_{\mathbb{R}^d}\nabla\Delta|A_R|_t^2\nabla\Delta n dx =& -2\beta\,\mathrm{Im}\int_{\mathbb{R}^d}\nabla\Delta((\Lambda\Delta A_R)\cdot\overline{A_R})\nabla\Delta n dx\\
&+ bc\,\mathrm{Im}\int_{\mathbb{R}^d}\nabla\Delta(\Lambda(nA_R)\cdot\overline{A_R})\nabla\Delta n dx\\
&- 2v_R\,\mathrm{Im}\,\mathrm{i}\int_{\mathbb{R}^d}\nabla\Delta\left(\Lambda\partial_y A_R\cdot\overline{A_R}\right)\nabla\Delta n dx,
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^d}\nabla\Delta|E|_t^2\nabla\Delta n dx =& -2\gamma\,\mathrm{Im}\int_{\mathbb{R}^d}\nabla\Delta((\Lambda\Delta E)\cdot\overline{E})\nabla\Delta n dx\\
&+ b\,\mathrm{Im}\int_{\mathbb{R}^d}\nabla\Delta(\Lambda(nE)\cdot\overline{E})\nabla\Delta n dx.
\end{aligned}$$

Using Hölder's inequality and Sobolev's inequality, the terms in (3.8) are estimated by

$$\begin{aligned}
&-2\alpha\int_{\mathbb{R}^d}\nabla\Delta((\Lambda\Delta A_C)\cdot\overline{A_C})\nabla\Delta n dx\\
&\quad= -2\alpha\int_{\mathbb{R}^d}(\Lambda\nabla\Delta^2 A_C)\cdot\overline{A_C}\nabla\Delta n dx - 6\alpha\int_{\mathbb{R}^d}(\Lambda\Delta^2 A_C)\cdot\nabla\overline{A_C}\nabla\Delta n dx\\
&\quad\quad - 6\alpha\int_{\mathbb{R}^d}\Lambda\nabla\Delta A_C\cdot\Delta\overline{A_C}\nabla\Delta n dx - 2\alpha\int_{\mathbb{R}^d}\Lambda\Delta A_C\cdot\nabla\Delta\overline{A_C}\nabla\Delta n dx
\end{aligned}$$

$$
\begin{aligned}
&\le -2\alpha \int_{\mathbb{R}^d} (\Lambda\nabla\Delta^2 A_C)\cdot\overline{A_C}\nabla\Delta n dx + C\left\|\Delta^2 A_C\right\|_{L^2}\|\nabla A_C\|_{L^\infty}\|\nabla\Delta n\|_{L^2}\\
&\quad + C\|\nabla\Delta A_C\|_{L^2}\|\Delta A_C\|_{L^\infty}\|\nabla\Delta n\|_{L^2}\\
&\le -2\alpha \int_{\mathbb{R}^d} (\Lambda\nabla\Delta^2 A_C)\cdot\overline{A_C}\nabla\Delta n dx + C\|n\|_{H^3}\|A_C\|_{H^4}^2, \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
&b^2\int_{\mathbb{R}^d}\nabla\Delta(\Lambda(nA_C)\cdot\overline{A_C})\nabla\Delta n dx\\
&= b^2\int_{\mathbb{R}^d}\nabla\Delta\Lambda(nA_C)\cdot\overline{A_C}\nabla\Delta n dx + 3b^2\int_{\mathbb{R}^d}\Delta\Lambda(nA_C)\cdot\nabla\overline{A_C}\nabla\Delta n dx\\
&\quad + 3b^2\int_{\mathbb{R}^d}\nabla\Lambda(nA_C)\cdot\Delta\overline{A_C}\nabla\Delta n dx + b^2\int_{\mathbb{R}^d}\Lambda(nA_C)\nabla\Delta\overline{A_C}\nabla\Delta n dx\\
&\le C\|\nabla\Delta(nA_C)\|_{L^2}\|A_C\|_{L^\infty}\|\nabla\Delta n\|_{L^2} + C\|\Delta(nA_C)\|_{L^2}\|\nabla\overline{A_C}\|_{L^\infty}\|\nabla\Delta n\|_{L^2}\\
&\quad + C\|\nabla(nA_C)\|_{L^2}\|\Delta\overline{A_C}\|_{L^\infty}\|\nabla\Delta n\|_{L^2} + C\|nA_C\|_{L^4}\|\nabla\Delta\overline{A_C}\|_{L^4}\|\nabla\Delta n\|_{L^2}\\
&\le C\|n\|_{H^3}^2\|A_C\|_{H^4}^2, \qquad (3.11)
\end{aligned}
$$

and

$$
\left|2v_C\,\mathrm{Im}\,\mathrm{i}\int_{\mathbb{R}^d}\nabla\Delta\left(\Lambda\partial_y A_C\cdot\overline{A_C}\right)\nabla\Delta n dx\right| \le C\|n\|_{H^3}\|A_C\|_{H^4}^2. \qquad (3.12)
$$

From the bounds (3.10)-(3.12), we get

$$
\begin{aligned}
2ab\int_{\mathbb{R}^d}\nabla\Delta|A_C|_t^2\nabla\Delta n dx \le& C\left(\|n\|_{H^3}\|A_C\|_{H^4}^2 + \|n\|_{H^3}^2\|A_C\|_{H^4}^2\right)\\
&- 4ab\alpha\,\mathrm{Im}\int_{\mathbb{R}^d}(\Lambda\nabla\Delta^2 A_C)\cdot\overline{A_C}\nabla\Delta n dx. \qquad (3.13)
\end{aligned}
$$

Note that the last term in (3.13) contains the fifth order derivative for A_C. For the terms in (3.9) and (3.10), we apply similar arguments to obtain

$$
\begin{aligned}
2ac\int_{\mathbb{R}^d}\nabla\Delta|A_R|_t^2\nabla\Delta n dx \le& C\left(\|n\|_{H^3}\|A_R\|_{H^4}^2 + \|n\|_{H^3}^2\|A_R\|_{H^4}^2\right)\\
&- 4ac\beta\,\mathrm{Im}\int_{\mathbb{R}^d}(\Lambda\nabla\Delta^2 A_R)\cdot\overline{A_R}\nabla\Delta n dx, \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
2a\int_{\mathbb{R}^d}\nabla\Delta|E|_t^2\nabla\Delta n dx \le& C\left(\|n\|_{H^3}\|E\|_{H^4}^2 + \|n\|_{H^3}^2\|E\|_{H^4}^2\right)\\
&- 4a\gamma\,\mathrm{Im}\int_{\mathbb{R}^d}(\Lambda\nabla\Delta^2 E)\cdot\overline{E}\nabla\Delta n dx. \qquad (3.15)
\end{aligned}
$$

To cancel the integral terms in (3.13)-(3.15), we first take the fourth order energy estimate for the equation

$$
\mathrm{i}A_{C_t} = -\alpha\Lambda\Delta A_C + \frac{b^2}{2}\Lambda(nA_C) - \mathrm{i}v_C\Lambda\partial_y A_C
$$

to get

$$
\frac{d}{dt}\|\Delta^2 A_C\|_{L^2}^2 = -\,b^2\,\mathrm{Im}\int_{\mathbb{R}^d}\Lambda\nabla\Delta\left(nA_C\right)\nabla\Delta^2\overline{A_C}dx
$$

$$
\begin{aligned}
=&b^2\mathrm{Im}\int_{\mathbb{R}^d}\Lambda\nabla\Delta\left(n\overline{A_C}\right)\nabla\Delta^2A_Cdx\\
=&b^2\mathrm{Im}\int_{\mathbb{R}^d}\Lambda\left(\overline{A_C}\nabla\Delta n\right)\nabla\Delta^2A_Cdx+3b^2\mathrm{Im}\int_{\mathbb{R}^d}\Lambda(\nabla\overline{A_C}\Delta n)\nabla\Delta^2A_Cdx\\
&+3b^2\,\mathrm{Im}\int_{\mathbb{R}^d}\Lambda(\Delta\overline{A_C}\nabla n)\nabla\Delta^2A_Cdx+b^2\,\mathrm{Im}\int_{\mathbb{R}^d}\Lambda(n\Delta\overline{A_C})\nabla\Delta^2A_Cdx\\
\le&C\|n\|_{H^3}\|A_C\|_{H^4}^2+b^2\,\mathrm{Im}\int_{\mathbb{R}^d}\Lambda(\overline{A_C}\nabla\Delta n)\nabla\Delta^2A_Cdx. \qquad (3.16)
\end{aligned}
$$

In the same way, we can obtain

$$
\frac{d}{dt}\|\Delta^2A_R\|_{L^2}^2\le C\|n\|_{H^3}\|A_R\|_{H^4}^2+bc\,\mathrm{Im}\int_{\mathbb{R}^d}\Lambda(\overline{A_R}\nabla\Delta n)\nabla\Delta^2A_Rdxdx, \qquad (3.17)
$$

and

$$
\frac{d}{dt}\|\Delta^2E\|_{L^2}^2\le C\|n\|_{H^3}\|E\|_{H^4}^2+b\,\mathrm{Im}\int_{\mathbb{R}^d}\Lambda(\overline{E}\nabla\Delta n)\nabla\Delta^2Edx. \qquad (3.18)
$$

Combining (3.13)-(3.18) gives

$$
\begin{aligned}
&\frac{d}{dt}(\|\Delta n_t\|_{L^2}^2+v_s^2\|\nabla\Delta n\|_{L^2}^2+\frac{4a\alpha}{b}\|\Delta^2A_C\|_{L^2}^2+\frac{4a\beta}{b}\left\|\Delta^2A_R\right\|_{L^2}^2+\frac{4a\gamma}{b}\|\Delta^2E\|_{L^2}^2\\
&\quad+2ab\int_{\mathbb{R}^d}\nabla\Delta\left|A_C\right|^2\nabla\Delta ndx+2ac\int_{\mathbb{R}^d}\nabla\Delta\left|A_R\right|^2\nabla\Delta ndx\\
&\quad+2a\int_{\mathbb{R}^d}\nabla\Delta|E|^2\nabla\Delta ndx)\\
&\le C\left(1+\|n\|_{H^3}^2+\|A_C\|_{H^4}^2+\|A_R\|_{H^4}^2+\|E\|_{H^4}^2\right)^2,
\end{aligned}
$$

namely,

$$
\begin{aligned}
&\frac{b}{4a}\|\Delta n_t\|_{L^2}^2+\frac{bv_s^2}{4a}\|\nabla\Delta n\|_{L^2}^2+\alpha\|\Delta^2A_C\|_{L^2}^2+\beta\left\|\Delta^2A_R\right\|_{L^2}^2+\gamma\|\Delta^2E\|_{L^2}^2\\
\le&C+C\int_0^t\left(1+\|n\|_{H^3}^2+\|A_C\|_{H^4}^2+\|A_R\|_{H^4}^2+\|E\|_{H^4}^2\right)^2d\tau\\
&+\frac{b^2}{2}\left|\int_{\mathbb{R}^d}\nabla\Delta\left|A_C\right|^2\nabla\Delta ndx\right|+\frac{bc}{2}\left|\int_{\mathbb{R}^d}\nabla\Delta\left|A_R\right|^2\nabla\Delta ndx\right|\\
&+\frac{b}{2}\left|\int_{\mathbb{R}^d}\nabla\Delta|E|^2\nabla\Delta ndx\right|\\
\le&C+C\int_0^t\left(1+\|n\|_{H^3}^2+\|A_C\|_{H^4}^2+\|A_R\|_{H^4}^2+\|E\|_{H^4}^2\right)^2d\tau+I. \qquad (3.19)
\end{aligned}
$$

Note that

$$
\begin{aligned}
I:=&\frac{b^2}{2}\left|\int_{\mathbb{R}^d}\nabla\Delta|A_C|^2\nabla\Delta n\right|+\frac{bc}{2}\left|\int_{\mathbb{R}^d}\nabla\Delta|A_R|^2\nabla\Delta n\right|+\frac{b}{2}\left|\int_{\mathbb{R}^d}\nabla\Delta|E|^2\nabla\Delta n\right|\\
\le&3\varepsilon\|n\|_{H^3}^2+\frac{4}{\varepsilon}\left(\frac{b^4}{4}\|\nabla\Delta|A_C|^2\|_{L^2}^2+\frac{b^2c^2}{4}\|\nabla\Delta|A_R|^2\|_{L^2}^2+\frac{b^2}{4}\|\nabla\Delta|E|^2\|_{L^2}^2\right).
\end{aligned}
$$

Using Hölder's inequality and Sobolev's inequality, it is easy to see

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\Delta|A_C|^2\|_{L^2}^2 \\
=&2\int_{\mathbb{R}^d} \nabla\Delta|A_C|^2\nabla\Delta|A_C|_t^2 dx \\
=&4\int_{\mathbb{R}^d} \nabla\Delta \operatorname{Im}((-\alpha(\Lambda\Delta A_C) + \frac{b}{2}\Lambda(nA_C) - \mathrm{i}\Lambda v_C\partial_y A_C)\cdot\overline{A_C})\cdot\nabla\Delta|A_C|^2 dx \\
\le& C\left(\|A_C\|_{H^4}^4 + \|n\|_{H^3}\|A_C\|_{H^4}^4\right).
\end{aligned}$$

The estimates for $\|\nabla\Delta|A_R|^2\|_{L^2}^2$ and $\|\nabla\Delta|E|^2\|_{L^2}^2$ can be treated similarly. Inserting these estimates into (3.19) and using also (3.3)-(3.4), we actually obtain

$$\begin{aligned}
&\|A_C\|_{H^4}^2 + \|A_R\|_{H^4}^2 + \|E\|_{H^4}^2 + \|n\|_{H^3}^2 + \|V\|_{H^3}^2 \\
&\quad \le C + C\int_0^t \left(1 + \|A_C\|_{H^4}^2 + \|A_R\|_{H^4}^2 + \|E\|_{H^4}^2 + \|n\|_{H^3}^2\right)^3 d\tau. \qquad (3.20)
\end{aligned}$$

Hence, it follows from (3.20) that there exist $C,\ T>0$ (independent of ε) such that

$$\|A_C\|_{H^4}^2 + \|A_R\|_{H^4}^2 + \|E\|_{H^4}^2 + \|n\|_{H^3}^2 + \|V\|_{H^3}^2 \le C, \quad \forall t\in[0,T].$$

This ends the proof of Proposition 3.1. □

Proposition 3.2. *Under the same assumptions as Proposition 3.1, for $0<\varepsilon'<\varepsilon$, there exists a constant $C>0$ independent of ε and ε' such that*

$$\max_{t\in[0,T)}(\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{H^2} + \|A_R^\varepsilon - A_R^{\varepsilon'}\|_{H^2} + \|E^\varepsilon - E^{\varepsilon'}\|_{H^2}) \le C\varepsilon^{\frac14} + CI_0^{\varepsilon,\varepsilon'}, \quad (3.21)$$

$$\max_{t\in[0,T)}(\|n^\varepsilon - n^{\varepsilon'}\|_{H^1} + \|V^\varepsilon - V^{\varepsilon'}\|_{H^1}) \le C\varepsilon^{\frac14} + CJ_0^{\varepsilon,\varepsilon'}, \quad (3.22)$$

where T is obtained by Proposition 3.1, and $I_0^{\varepsilon,\varepsilon'}$, $J_0^{\varepsilon,\varepsilon'}$ are defined by

$$\begin{aligned}
I_0^{\varepsilon,\varepsilon'} &:= \|a_C^\varepsilon - a_C^{\varepsilon'}\|_{H^2} + \|a_R^\varepsilon - a_R^{\varepsilon'}\|_{H^2} + \|e^\varepsilon - e^{\varepsilon'}\|_{H^2}, \\
J_0^{\varepsilon,\varepsilon'} &:= \|n_0^\varepsilon - n_0^{\varepsilon'}\|_{H^1} + \|V_0^\varepsilon - V_0^{\varepsilon'}\|_{H^1}.
\end{aligned}$$

Proof. In this proof, we recover the superscript ε and also denote $\Lambda^\varepsilon = \Lambda = (1+\varepsilon\Delta^2)^{-1}$ to emphasize the reliance on ε. To prove the property of Cauchy sequence for the approximate solution, we first write out the equation for $n^\varepsilon - n^{\varepsilon'}$ and $V^\varepsilon - V^{\varepsilon'}$. Indeed, from (2.5), one has

$$\begin{aligned}
&(n^\varepsilon - n^{\varepsilon'})_t + \nabla\cdot(V^\varepsilon - V^{\varepsilon'}) = 0, \\
&(V^\varepsilon - V^{\varepsilon'})_t + v_s^2\nabla(n^\varepsilon - n^{\varepsilon'}) = -a\nabla(|E^\varepsilon|^2 - |E^{\varepsilon'}|^2 \qquad (3.23)\\
&\qquad + b|A_C^\varepsilon|^2 - b|A_C^{\varepsilon'}|^2 + c|A_R^\varepsilon|^2 - c|A_R^{\varepsilon'}|^2).
\end{aligned}$$

Performing the energy estimate for (3.23) at H^1 level and using the uniform bound of Proposition 3.1, we have

$$\frac12\frac{d}{dt}(v_s^2\|n^\varepsilon - n^{\varepsilon'}\|_{H^1}^2 + \|V^\varepsilon - V^{\varepsilon'}\|_{H^1}^2)$$

$$\leq C(\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{H^2} + \|A_R^\varepsilon - A_R^{\varepsilon'}\|_{H^2} + \|E^\varepsilon - E^{\varepsilon'}\|_{H^2})\|V^\varepsilon - V^{\varepsilon'}\|_{H^1}. \tag{3.24}$$

Next we turn to deal with the estimate for $A_C^\varepsilon - A_C^{\varepsilon'}$ whose equation is given by

$$\begin{aligned}
&\mathrm{i}(A_C^\varepsilon - A_C^{\varepsilon'})_t + \alpha(\Lambda^\varepsilon - \Lambda^{\varepsilon'})\Delta A_C^\varepsilon + \alpha\Lambda^{\varepsilon'}\Delta(A_C^\varepsilon - A_C^{\varepsilon'}) \\
&\quad + \mathrm{i}v_C(\Lambda^\varepsilon - \Lambda^{\varepsilon'})\partial_y A_C^\varepsilon + \mathrm{i}v_C\Lambda^{\varepsilon'}\partial_y(A_C^\varepsilon - A_C^{\varepsilon'}) \\
=&\frac{b^2}{2}(\Lambda^\varepsilon - \Lambda^{\varepsilon'})(n^\varepsilon A_C^\varepsilon) + \frac{b^2}{2}\Lambda^{\varepsilon'}[(n^\varepsilon - n^{\varepsilon'})A_C^\varepsilon + n^{\varepsilon'}(A_C^\varepsilon - A_C^{\varepsilon'})].
\end{aligned} \tag{3.25}$$

Note that

$$\frac{1}{1+\varepsilon'|\xi|^4} - \frac{1}{1+\varepsilon|\xi|^4} \leq \frac{\varepsilon|\xi|^4}{(1+\varepsilon'|\xi|^4)(1+\varepsilon|\xi|^4)} \leq \varepsilon^{\frac14}|\xi|,$$

where we have used Young's inequality to get

$$\varepsilon|\xi|^4 = (\varepsilon^{\frac14}|\xi|)^3\varepsilon^{\frac14}|\xi| \leq (\frac{3\varepsilon|\xi|^4}{4} + \frac14)\varepsilon^{\frac14}|\xi| \leq (1+\varepsilon|\xi|^4)\varepsilon^{\frac14}|\xi|.$$

Hence, we have

$$\|(\Lambda^\varepsilon - \Lambda^{\varepsilon'})f\|_{H^s} \leq \varepsilon^{\frac14}\|f\|_{H^{s+1}}.$$

Now we take L^2 estimate for the equation (3.25) to get

$$\begin{aligned}
&\frac12\frac{d}{dt}\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{L^2}^2 \\
\leq& C(\varepsilon^{\frac14}\|A_C^\varepsilon\|_{H^3} + \varepsilon^{\frac14}\|n^\varepsilon A_C^\varepsilon\|_{H^1})\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{L^2} \\
&+ C(\|n^\varepsilon - n^{\varepsilon'}\|_{L^2}\|A_C^\varepsilon\|_{L^\infty} + \|n^{\varepsilon'}\|_{L^\infty}\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{L^2})\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{L^2} \\
\leq& C\varepsilon^{\frac14} + C(\|n^\varepsilon - n^{\varepsilon'}\|_{L^2} + \|A_C^\varepsilon - A_C^{\varepsilon'}\|_{L^2})\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{L^2}.
\end{aligned} \tag{3.26}$$

Taking the derivative of equation (3.25) with respect to t and doing L^2 energy estimate for $A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}$, then we can obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2}^2 \\
\leq& C(\varepsilon^{\frac14}\|\Delta A_{Ct}^\varepsilon\|_{L^2} + \varepsilon^{\frac14}\|\nabla A_{Ct}^\varepsilon\|_{L^2} + \varepsilon^{\frac14}\|(n^\varepsilon A_C^\varepsilon)_t\|_{H^1})\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2} \\
&+ C(\|n_t^\varepsilon - n_t^{\varepsilon'}\|_{L^2}\|A_C^\varepsilon\|_{L^\infty} + \|n^\varepsilon - n^{\varepsilon'}\|_{H^1}\|A_{Ct}^\varepsilon\|_{H^1})\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2} \\
&+ C(\|n_t^{\varepsilon'}\|_{H^1}\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{H^1} + \|n^{\varepsilon'}\|_{L^\infty}\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2})\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2} \\
\leq& C\varepsilon^{\frac14} + C(\|V^\varepsilon - V^{\varepsilon'}\|_{L^2} + \|n^\varepsilon - n^{\varepsilon'}\|_{H^1})\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2} \\
&+ C(\|A_C^\varepsilon - A_C^{\varepsilon'}\|_{H^1} + \|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2})\|A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'}\|_{L^2}.
\end{aligned} \tag{3.27}$$

Remarked in (3.27), the problematic term is

$$I := \alpha\mathrm{Im}\int_{\mathbb{R}^d}(\Lambda^\varepsilon - \Lambda^{\varepsilon'})\Delta A_{Ct}^\varepsilon \cdot (A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'})dx,$$

in which $\Delta A_{Ct}^\varepsilon$ contains a fourth order term. This term can be estimated as

$$|I| \leq C\|(\Lambda^\varepsilon - \Lambda^{\varepsilon'})\nabla A_{Ct}^\varepsilon\|_{L^2}\|\nabla(A_{Ct}^\varepsilon - A_{Ct}^{\varepsilon'})\|_{L^2}$$

$$
\begin{aligned}
&\le C\varepsilon^{\frac14}\|\Delta A^{\varepsilon}_{Ct}\|_{L^2}\|\nabla(A^{\varepsilon}_{Ct}-A^{\varepsilon'}_{Ct})\|_{L^2}\\
&\le C\varepsilon^{\frac14}.
\end{aligned}
$$

In addition, it is obvious that (see (3.25))

$$
\begin{aligned}
\alpha\|\Delta(A^{\varepsilon}_{C}-A^{\varepsilon'}_{C})\|_{L^2} \le & C\varepsilon^{\frac14}+\|A^{\varepsilon}_{Ct}-A^{\varepsilon'}_{Ct}\|_{L^2}+C\|n^{\varepsilon}-n^{\varepsilon'}\|_{L^2}\\
&+C\|A^{\varepsilon}_{C}-A^{\varepsilon'}_{C}\|_{L^2}+C\|\partial_y(A^{\varepsilon}_{C}-A^{\varepsilon'}_{C})\|_{L^2},
\end{aligned}
$$

and by interpolating for the last term, we get

$$
\|\Delta(A^{\varepsilon}_{C}-A^{\varepsilon'}_{C})\|_{L^2} \le C\varepsilon^{\frac14}+C\|A^{\varepsilon}_{Ct}-A^{\varepsilon'}_{Ct}\|_{L^2}+C\|n^{\varepsilon}-n^{\varepsilon'}\|_{L^2}+C\|A^{\varepsilon}_{C}-A^{\varepsilon'}_{C}\|_{L^2}. \tag{3.28}
$$

Similar estimates as (3.26)-(3.28) hold for $A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}$ and $E^{\varepsilon}-E^{\varepsilon'}$, namely, we have

$$
\frac12\frac{d}{dt}\|A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}\|^2_{L^2}\le C\varepsilon^{\frac14}+C(\|n^{\varepsilon}-n^{\varepsilon'}\|_{L^2}+\|A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}\|_{L^2})\|A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}\|_{L^2}, \tag{3.29}
$$

$$
\begin{aligned}
\frac12\frac{d}{dt}\|A^{\varepsilon}_{Rt}-A^{\varepsilon'}_{Rt}\|^2_{L^2}\le & C\varepsilon^{\frac14}+C(\|V^{\varepsilon}-V^{\varepsilon'}\|_{L^2}+\|n^{\varepsilon}-n^{\varepsilon'}\|_{H^1})\|A^{\varepsilon}_{Rt}-A^{\varepsilon'}_{Rt}\|_{L^2}\\
&+C(\|A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}\|_{H^1}+\|A^{\varepsilon}_{Rt}-A^{\varepsilon'}_{Rt}\|_{L^2})\|A^{\varepsilon}_{Rt}-A^{\varepsilon'}_{Rt}\|_{L^2},
\end{aligned} \tag{3.30}
$$

$$
\|\Delta(A^{\varepsilon}_{R}-A^{\varepsilon'}_{R})\|_{L^2} \le C\varepsilon^{\frac14}+C\|A^{\varepsilon}_{Rt}-A^{\varepsilon'}_{Rt}\|_{L^2}+C\|n^{\varepsilon}-n^{\varepsilon'}\|_{L^2}+C\|A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}\|_{L^2}, \tag{3.31}
$$

and

$$
\frac12\frac{d}{dt}\|E^{\varepsilon}-E^{\varepsilon'}\|^2_{L^2}\le C\varepsilon^{\frac14}+C(\|n^{\varepsilon}-n^{\varepsilon'}\|_{L^2}+\|E^{\varepsilon}-E^{\varepsilon'}\|_{L^2})\|E^{\varepsilon}-E^{\varepsilon'}\|_{L^2}, \tag{3.32}
$$

$$
\begin{aligned}
\frac12\frac{d}{dt}\|E^{\varepsilon}_{t}-E^{\varepsilon'}_{t}\|^2_{L^2}\le & C\varepsilon^{\frac14}+C(\|V^{\varepsilon}-V^{\varepsilon'}\|_{L^2}+\|n^{\varepsilon}-n^{\varepsilon'}\|_{H^1})\|E^{\varepsilon}_{t}-E^{\varepsilon'}_{t}\|_{L^2}\\
&+C(\|E^{\varepsilon}-E^{\varepsilon'}\|_{H^1}+\|E^{\varepsilon}_{t}-E^{\varepsilon'}_{t}\|_{L^2})\|E^{\varepsilon}_{t}-E^{\varepsilon'}_{t}\|_{L^2},
\end{aligned} \tag{3.33}
$$

$$
\|\Delta(E^{\varepsilon}-E^{\varepsilon'})\|_{L^2} \le C\varepsilon^{\frac14}+C\|E^{\varepsilon}_{t}-E^{\varepsilon'}_{t}\|_{L^2}+C\|n^{\varepsilon}-n^{\varepsilon'}\|_{L^2}+C\|E^{\varepsilon}-E^{\varepsilon'}\|_{L^2}. \tag{3.34}
$$

Finally, collecting the estimates (3.24) and (3.26)-(3.31) together, we thus arrive at

$$
\frac{d}{dt}F(t)\le C(\varepsilon^{\frac14}+F(t))
$$

with

$$
\begin{aligned}
F(t):=&\|A^{\varepsilon}_{C}-A^{\varepsilon'}_{C}\|^2_{L^2}+\|A^{\varepsilon}_{R}-A^{\varepsilon'}_{R}\|^2_{L^2}+\|E-E^{\varepsilon'}\|^2_{L^2}+\|A^{\varepsilon}_{Ct}-A^{\varepsilon'}_{Ct}\|^2_{L^2}\\
&+\|A^{\varepsilon}_{Rt}-A^{\varepsilon'}_{Rt}\|^2_{L^2}+\|E^{\varepsilon}_{t}-E^{\varepsilon'}_{t}\|^2_{L^2}+\|n^{\varepsilon}-n^{\varepsilon'}\|^2_{H^1}+\|V^{\varepsilon}-V^{\varepsilon'}\|^2_{H^1}.
\end{aligned}
$$

Therefore, the desired bounds follow by Gronwall's inequality. □

Once Proposition 3.1 and Proposition 3.2 are proved, the existence part of Theorem 1.1 is then established by a limiting argument, while the uniqueness part can be proved by using the similar strategy as Proposition 3.2. For simplicity, we thus omit further details. This completes the proof of Theorem 1.1.

4. Global smooth solution for $d = 2$

In this section, we show that the smooth solution exists globally in the two-dimensional case provided that the L^2 norms of the initial data a_C, a_R, e are less than certain given constants. To this end, we introduce the following Gagliardo-Nirenberg inequality with the sharp constant ([17])

$$\|f\|_{L^4(\mathbb{R}^2)}^4 \le C\|f\|_{L^2(\mathbb{R}^2)}^2\|\nabla f\|_{L^2(\mathbb{R}^2)}^2, \tag{4.1}$$

where $C = \frac{2}{\|\psi\|_{L^2(\mathbb{R}^2)}^2}$ and ψ is the ground state solution of $\Delta\psi - \psi + \psi^3 = 0$. Also, one needs the following logarithmic embedding inequality ([3,4])

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le C\|u\|_{H^1(\mathbb{R}^2)}\left(1 + \ln\left(1 + \frac{\|u\|_{H^2(\mathbb{R}^2)}}{\|u\|_{H^1(\mathbb{R}^2)}}\right)\right)^{\frac{1}{2}}. \tag{4.2}$$

Proof of Theorem 1.2. Let T^* be the maximal existence time of the solution obtained by Theorem 1.1. Then in order to prove Theorem 1.2, we only need to show for every $t \in [0, T^*)$ there holds

$$\|A_C(t)\|_{H^m} + \|A_R(t)\|_{H^m} + \|E(t)\|_{H^m} + \|n(t)\|_{H^{m-1}} + \|n_t(t)\|_{H^{m-2}} \le C. \tag{4.3}$$

Firstly, the mass conservation law gives

$$\|A_C\|_{L^2} + \|A_R\|_{L^2} + \|E\|_{L^2} = \|a_C\|_{L^2} + \|a_R\|_{L^2} + \|e\|_{L^2} \le C. \tag{4.4}$$

Meanwhile, the energy conservation law (see Theorem 1.1) implies

$$\begin{aligned}
&\alpha\|\nabla A_C\|_{L^2}^2 + \beta\|\nabla A_R\|_{L^2}^2 + \gamma\|\nabla E\|_{L^2}^2 + \frac{b}{4a}v_s^2\|n\|_{L^2}^2 + \frac{b}{4a}\|V\|_{L^2}^2 \\
\le& |\Psi(0)| + \left|\frac{b^2}{2}\int_{\mathbb{R}^2} n(t)|A_C(t)|^2 dx\right| + \left|\frac{bc}{2}\int_{\mathbb{R}^2} n(t)|A_R(t)|^2 dx\right| + \left|\frac{b}{2}\int_{\mathbb{R}^2} n(t)|E(t)|^2 dx\right| \\
&+ \left|\int_{\mathbb{R}^2} v_C\partial_y A_C\overline{A_C}\right| + \left|\int_{\mathbb{R}^2} v_R\partial_y A_R\overline{A_R} dx\right| \\
\le& |\Psi(0)| + \frac{b^4 + b^2c^2 + b^2}{16}\varepsilon_1\|n\|_{L^2}^2 + \frac{1}{\varepsilon_1}\left(\|A_C\|_{L^4}^4 + \|A_R\|_{L^4}^4 + \|E\|_{L^4}^4\right) \\
&+ \frac{1}{4\varepsilon_2}\left(v_C\|A_C\|_{L^2}^2 + v_R\|A_R\|_{L^2}^2\right) + \varepsilon_2\left(\|\nabla A_C\|_{L^2}^2 + \|\nabla A_R\|_{L^2}^2\right) \\
\le& |\Psi(0)| + \frac{1}{\varepsilon_1}\frac{2}{\|\psi\|_{L^2}^2}\left(\|a_C\|_{L^2}^2\|\nabla A_C\|_{L^2}^2 + \|a_R\|_{L^2}^2\|\nabla A_R\|_{L^2}^2 + \|e\|_{L^2}^2\|\nabla E\|_{L^2}^2\right) \\
&+ \frac{b^4 + b^2c^2 + b^2}{16}\varepsilon_1\|n\|_{L^2}^2 + \frac{C}{4\varepsilon_2} + \varepsilon_2\left(\|\nabla A_C\|_{L^2}^2 + \|\nabla A_R\|_{L^2}^2\right),
\end{aligned} \tag{4.5}$$

where we have used (4.1) in the last step. If the sizes of $\|a_C\|_{L^2}$, $\|a_R\|_{L^2}$ and $\|e\|_{L^2}$ satisfy

$$\begin{aligned}
\alpha &> \frac{ab(b^2 + c^2 + 1)}{2v_s^2\|\psi\|_{L^2}^2}\|a_C\|_{L^2}^2, \\
\beta &> \frac{ab(b^2 + c^2 + 1)}{2v_s^2\|\psi\|_{L^2}^2}\|a_R\|_{L^2}^2,
\end{aligned}$$

$$\gamma > \frac{ab(b^2+c^2+1)}{2v_s^2\|\psi\|_{L^2}^2}\|e\|_{L^2}^2,$$

then we can take $\varepsilon_1 < \frac{4v_s^2}{ab(b^2+c^2+1)}$ (close to $\frac{4v_s^2}{ab(b^2+c^2+1)}$) and ε_2 sufficiently small. Thus, from (4.4) and (4.5), we can get

$$\|A_C\|_{H^1} + \|A_R\|_{H^1} + \|E\|_{H^1} + \|n\|_{L^2} + \|V\|_{L^2} \le C. \tag{4.6}$$

Secondly, we apply energy estimate for equation (1.4) with n_t to obtain

$$\begin{aligned}
\frac{d}{dt}(\|n_t\|_{L^2}^2 + v_s^2\|\nabla n\|_{L^2}^2) =& 2a\int_{\mathbb{R}^2}(bn_t\Delta|A_C|^2 + cn_t\Delta|A_R|^2 + n_t\Delta|E|^2)dx\\
\le& 4ab\int_{\mathbb{R}^2}|n_t|\left(|A_C|\cdot|\Delta A_C| + |\nabla A_C|^2\right)dx\\
&+ 4ac\int_{\mathbb{R}^2}|n_t|\left(|A_R|\cdot|\Delta A_R| + |\nabla A_C|^2\right)dx\\
&+ 4a\int_{\mathbb{R}^2}|n_t|\left(|E|\cdot|\Delta E| + |\nabla E|^2\right)dx\\
\le& C\|n_t\|_{L^2}\left(\|A_C\|_{L^\infty}\|\Delta A_C\|_{L^2} + \|\nabla A_C\|_{L^4}^2\right)\\
&+ C\|n_t\|_{L^2}\left(\|A_R\|_{L^\infty}\|\Delta A_R\|_{L^2} + \|\nabla A_R\|_{L^4}^2\right)\\
&+ C\|n_t\|_{L^2}\left(\|E\|_{L^\infty}\|\Delta E\|_{L^2} + \|\nabla E\|_{L^4}^2\right)\\
\le& C\|n_t\|_{L^2}\|\Delta A_C\|_{L^2}(\|A_C\|_{L^\infty}+1)\\
&+ C\|n_t\|_{L^2}\|\Delta A_R\|_{L^2}(\|A_R\|_{L^\infty}+1)\\
&+ C\|n_t\|_{L^2}\|\Delta E\|_{L^2}(\|E\|_{L^\infty}+1).
\end{aligned}\tag{4.7}$$

Notice that equation (1.1) implies

$$\begin{aligned}
\|\Delta A_C\|_{L^2} &\le C\left(\|A_{Ct}\|_{L^2} + \|n\|_{L^4}\|A_C\|_{L^4} + \|\partial_y A_C\|_{L^2}\right)\\
&\le C\left(\|A_{Ct}\|_{L^2} + \|n\|_{L^2}^{\frac{1}{2}}\|\nabla n\|_{L^2}^{\frac{1}{2}}\|A_C\|_{H^1} + \|\nabla A_C\|_{L^2}\right)\\
&\le C\left(\|A_{Ct}\|_{L^2} + \|\nabla n\|_{L^2} + 1\right).
\end{aligned}\tag{4.8}$$

Similarly, we can deduce from (1.2) and (1.3) that

$$\|\Delta A_R\|_{L^2} \le C\left(\|A_{Rt}\|_{L^2} + \|\nabla n\|_{L^2} + 1\right), \quad \|\Delta E\|_{L^2} \le C\left(\|E_t\|_{L^2} + \|\nabla n\|_{L^2} + 1\right). \tag{4.9}$$

Now differentiating (1.1)-(1.3) with respect to t gives

$$(\mathrm{i}(\partial_t + v_C\partial_y) + \alpha\Delta)A_{Ct} - \frac{b^2}{2}(n_tA_C + nA_{Ct}) = 0, \tag{4.10}$$

$$(\mathrm{i}(\partial_t + v_R\partial_y) + \beta\Delta)A_{Rt} - \frac{bc}{2}(n_tA_R + nA_{Rt}) = 0, \tag{4.11}$$

$$(\mathrm{i}\partial_t + \gamma\Delta)E_t - \frac{b}{2}(n_tE + nE_t) = 0, \tag{4.12}$$

from which we can obtain

$$\frac{d}{dt}(\|A_{Ct}\|_{L^2}^2 + \|A_{Rt}\|_{L^2}^2 + \|E_t\|_{L^2}^2)$$

$$\leq C \left\| n_t \right\|_{L^2} \left(\left\| A_{Ct} \right\|_{L^2} \left\| A_C \right\|_{L^\infty} + \left\| A_{Rt} \right\|_{L^2} \left\| A_R \right\|_{L^\infty} + \left\| E_t \right\|_{L^2} \left\| E \right\|_{L^\infty} \right). \tag{4.13}$$

Letting

$$G_1(t) := \|n_t\|_{L^2}^2 + v_s^2 \|\nabla n\|_{L^2}^2 + \|A_{Ct}\|_{L^2}^2 + \|A_{Rt}\|_{L^2}^2 + \|E_t\|_{L^2}^2 + 1,$$

then it can be obtained by combining (4.6)-(4.9) and (4.13) that

$$\begin{aligned}
\frac{d}{dt} G_1(t) &\leq C G_1(t) \left(1 + \|A_C\|_{L^\infty}^2 + \|A_R\|_{L^\infty}^2 + \|E\|_{L^\infty}^2\right) \\
&\leq C G_1(t) \left(1 + \ln\left(1 + \|\Delta A_C\|_{L^2}\right) + \ln\left(1 + \|\Delta A_R\|_{L^2}\right) + \ln\left(1 + \|\Delta E\|_{L^2}\right)\right) \\
&\leq C G_1(t) \left(1 + \ln G_1(t)\right),
\end{aligned}$$

where we have used (4.2) in the second step. By Gronwall's inequality, there holds $G_1(t) \leq C$ which implies

$$\|A_C(t)\|_{H^2} + \|A_R(t)\|_{H^2} + \|E(t)\|_{H^2} + \|n(t)\|_{H^1} + \|n_t\|_{L^2} \leq C, \qquad \forall t \in [0, T^*). \tag{4.14}$$

Next, we will show

$$\|A_C(t)\|_{H^3} + \|A_R(t)\|_{H^3} + \|E(t)\|_{H^3} + \|n(t)\|_{H^2} + \|n_t\|_{H^1} \leq C, \qquad \forall t \in [0, T^*). \tag{4.15}$$

Since the L^∞ estimates of A_C, A_R and E have already been established, it is not hard to obtain the bound (4.15). Indeed, from the equation (1.4), we have

$$\begin{aligned}
&\frac{d}{dt} \left(\|\nabla n_t\|_{L^2}^2 + v_s^2 \|\Delta n\|_{L^2}^2\right) \\
=& 2a \int \left(-\Delta n_t\right) \Delta \left(|E|^2 + b|A_C|^2 + c\,|A_R|^2\right) dx \\
\leq& 2a \int |\nabla n_t| \left(\left|\nabla \Delta |E|^2\right| + b\left|\nabla \Delta |A_C|^2\right| + c\left|\nabla \Delta |A_R|^2\right|\right) dx \\
\leq& C \left\|\nabla n_t\right\|_{L^2} \left(\left\|\nabla \Delta |E|^2\right\|_{L^2} + \left\|\nabla \Delta |A_C|^2\right\|_{L^2} + \|\nabla \Delta |A_R|^2\|_{L^2}\right) \\
\leq& C \left\|\nabla n_t\right\|_{L^2} \left(\|\nabla \Delta E\|_{L^2} + \|\nabla \Delta A_C\|_{L^2} + \|\nabla \Delta A_R\|_{L^2} + 1\right),
\end{aligned} \tag{4.16}$$

and by (1.1)-(1.3) and (4.14),

$$\begin{aligned}
\|\nabla \Delta A_C\|_{L^2} &\leq C \left(\|\nabla A_{Ct}\|_{L^2} + \|\nabla n\|_{L^2} \|A_C\|_{L^\infty} + \|n\|_{L^4} \|\nabla A_C\|_{L^4} + \|\nabla(\partial_y A_C)\|_{L^2}\right) \\
&\leq C \left(\|\nabla A_{Ct}\|_{L^2} + 1\right),
\end{aligned} \tag{4.17}$$

and

$$\|\nabla \Delta A_R\|_{L^2} \leq C \left(\|\nabla A_{Rt}\|_{L^2} + 1\right), \quad \|\nabla \Delta E\|_{L^2} \leq C \left(\|\nabla E_t\|_{L^2} + 1\right). \tag{4.18}$$

On the other hand, it follows from (4.10)-(4.12) that

$$\begin{aligned}
&\frac{d}{dt} \left(\|\nabla A_{Ct}\|_{L^2}^2 + \|\nabla A_{Rt}\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2\right) \\
\leq& C \left(\|\nabla A_{Ct}\|_{L^2}^2 + \|\nabla A_{Rt}\|_{L^2}^2 + \|\nabla n_t\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2 + \|\Delta n\|_{L^2}^2 + 1\right).
\end{aligned} \tag{4.19}$$

Let

$$G_2(t) := \|\nabla n_t\|_{L^2}^2 + v_s^2 \|\Delta n\|_{L^2}^2 + \|\nabla A_{Ct}\|_{L^2}^2 + \|\nabla A_{Rt}\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2 + 1.$$

Now combining the estimates (4.16)-(4.19) gives

$$\frac{d}{dt}G_2(t) \leq CG_2(t),$$

which proves (4.15) as desired.

Finally, we can apply similar strategy to obtain (4.3). Since the proof is similar, we omit further details. This ends the proof of Theorem 1.2. □

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References

[1] M. Colin and T. Colin, *On a quasilinear Zakharov system describing laser-plasma interactions*, Differential Integral Equations, 2004, 17(3-4), 297–330.

[2] T. Cazenave and F.B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Analysis: Theory, Methods and Applications, 1990, 14(10), 807–836.

[3] H. Brezis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Analysis: Theory, Methods and Applications, 1980, 4(4), 677–681.

[4] H. Brezis and S. Wainger, *A note on limiting cases of sobolev embeddings*, Communications in Partial Differential Equations, 1980, 5(7), 773–789.

[5] B. Guo, Z. Gan, L. Kong and J. Zhang, *The Zakharov system and its soliton solutions*, Science Press, Beijing and Springer, 2016.

[6] B. Guo and L. Shen, *The existence and uniqueness of the classical solution on the periodic initial value problem for Zakharov equations (in Chinese)*, Acta Mathematicae Applicatae Sinica, 1982, 5(3), 310–324.

[7] B. Guo, J. Zhang and X. Pu, *On the existence and uniqueness of smooth solution for a generalized Zakharov equation*, Journal of Mathematical Analysis and Applications, 2010, 365(1), 238–253.

[8] N. Hayashi and P.I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, American Journal of Mathematics Johns Hopkins University Press, 1998, 120(2), 369–389.

[9] H. Hirayama, *Well-posedness and scattering for a system of quadratic derivative nonlinear schrödinger equations with low regularity initial data*, Communications on pure and applied analysis, 2014, 13(4), 1563–1591.

[10] H. Hirayama and S. Kinoshita, *Sharp bilinear estimates and its application to a system of quadratic derivative nonlinear Schrödinger equations*, Nonlinear Analysis: Theory Methods and Applications, 2019, 178, 205–226.

[11] H. Hirayama, S. Kinoshita and M. Okamoto, *Well-posedness for a system of quadratic derivative nonlinear Schrödinger equations with radial initial data*, Annales Henri Poincaré, 2020, 21(8), 2611–2636.

[12] J. Kato and F. Pusateri, *A new proof of long range scattering for critical nonlinear Schrödinger equations*, Differential Integral Equations, 2011, 24(9-10), 923–940.

[13] H. Liang, L. Han and Y. Huang, *Solitary Waves for the Generalized Nonautonomous Dual-power Nonlinear Schrödinger Equations with Variable Coeffcients*, Journal of Nonlinear Modeling and Analysis, 2019, 1(2), 251–260.

[14] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge University Press, 2002.

[15] C. Sulem and P. L. Sulem, *Quelques résulatats de régularité pour les équation de la turbulencede Langmuir*, Comptes Rendus de l'Académie des Sciences, 1979, 289, 173–176.

[16] T. Terence, *Nonlinear dispersive equations. Local and global analysis*, CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, American Mathematical Society, Providence, RI, Washington, DC, 2006.

[17] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Communications in Mathematical Physics, 1983, 87(4), 567–576.

[18] V. E. Zakharov, *Collapse of Langmuir waves*, Journal of Experimental and Theoretical Physics, 1972, 35(5), 908–914.