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Iterative Positive Solutions to Nonlinear *q*-Fractional Differential Equations with Integral Boundary Value Conditions*

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Abstract This paper is concerned with the existence of positive solutions to nonlinear q-fractional differential equations yielding to the integral boundary value conditions. Under sufficient conditions of the nonlinearity, by using some iterative techniques, we get that this problem has two positive solutions and a unique positive solution respectively. Our results improve some recent work.

Keywords Integral boundary value condition, q-fractional differential equation, iterative technique, positive solutions

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1. Introduction

The purpose of this paper is to obtain the existence of positive solutions to the following integral boundary value problem of the q-fractional differential equation (BVP)

$$\begin{cases} (D_q^\alpha u)(t) + f(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ D_q^i u(0) = 0, \quad 0 \leqslant i \leqslant n-2, \quad u(1) = \lambda \int_0^1 u(s) \mathrm{d}_q s, \end{cases} \tag{1.1}$$

where $\alpha \in (n-1, n]$, $n \geqslant 3$ is an integer, $f \in C([0,1] \times [0, +\infty) \times (0, +\infty), (0, +\infty))$, $\lambda \in (0, [\alpha]_q)$, and D_q^α denotes the α order fractional q-derivative operator of the Riemann-Liouville type.

Fractional q-difference equations have received considerable attention due to their ability to accurately describe various phenomena such as mathematical models, quantum calculus and engineering problems. More recently, various fractional q-difference systems have been reduced to the search of solutions by iterative methods [2, 8, 10, 12] and by other fixed point theories [7, 14, 15]. For example, by using the monotone iterative methods, the authors [8, 12] dealt with the existence of positive solutions to fractional q-difference equations, and among these results, f must be a monotonic function with respect to the only spatial variable. Especially, by iterative algorithm, Mao, Zhao and Wang [10] gained the unique positive solution to the

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fractional q-difference Schrödinger equation whose nonlinear term has two spatial variables, and the initial value of iterative series is a constant multiple of a point in the cone but not a fixed upper and lower solutions as [12]. Integral boundary value problems of fractional difference systems may respond to some special features such as the blood flow, chemical engineering and other issues. Therefore, much interest has been given to study the problems in this area (see [1, 2, 6, 8, 9, 17]). In fact, the authors [2, 8] have supplied a general form with respect to this kind of fractional q-difference boundary value problems. Moreover, in [2], Cui, Kang and Chen have given the corresponding expression to the Green's function. By carefully analyzing these works of [1, 8, 10, 12, 14, 16–18], the authors have found that those results in [8, 10, 12, 18] can be effectively promoted.

During our discussion, an inverse symmetry subset in the cone plays a fundamental role, which skillfully converts the existence of a positive solution of BVP (1.1) into the existence of a fixed point for the equivalent integral operator in this set. Compared with the results studied recently, these results presented here have improved in some aspects. First, here the fractional order $\alpha > 2$ and the nonlinear term f is mixed monotone, thus it includes much more types of functions. Second, we extend the ideas of [18] to establish richer conditions on f. In particular, when comparing with the preceding proof in Theorem 3.1 of [18], one can see that what we actually do reveals the characteristics of solutions for the similar conditions on the nonlinearity term f.

The main conditions on f are as follows:

(H) For each fixed $t \in (0,1)$, $f(t,u,v)$ is increasing on u and decreasing on v and

$$0 < \int_0^1 (1-qs)^{(\alpha-1)} f(s, s^{\alpha-1}, s^{\alpha-1}) \mathrm{d}_q s < +\infty.$$

(H_1) There exist constants $\sigma \in (0,1)$, $\rho \in (0,1]$ such that for all $(t,u,v) \in [0,1] \times [0,+\infty) \times (0,+\infty)$,

$$f(t, ru, r^{-1}v) \geqslant r[1 + \rho(r^{-\sigma} - 1)] f(t,u,v), \quad r \in (0,1]. \tag{1.2}$$

(H_2) There exists a constant $\sigma \in (0,1)$ such that for all $(t,u,v) \in [0,1] \times [0,+\infty) \times (0,+\infty)$,

$$f(t, ru, r^{-1}v) \geqslant r^{\sigma} f(t,u,v), \quad r \in (0,1]. \tag{1.3}$$

(H_3) There exist constants λ_i, $\mu_i (i = 1,2)$ satisfying $0 < \lambda_1 \leqslant \lambda_2 < 1$, $0 < \mu_1 \leqslant \mu_2 < 1$ and $\lambda_2 + \mu_2 < 1$ such that for all $(t,u,v) \in [0,1] \times [0,+\infty) \times (0,+\infty)$,

$$r^{\lambda_2} f(t,u,v) \leqslant f(t,ru,v) \leqslant r^{\lambda_1} f(t,u,v), \quad r \in (0,1]; \tag{1.4}$$

$$r^{\mu_2} f(t,u,v) \leqslant f(t,u,r^{-1}v) \leqslant r^{\mu_1} f(t,u,v), \quad r \in (0,1]. \tag{1.5}$$

Remark 1.1. For $r \in (1,+\infty)$, the condition (H_1) implies

$$f(t, ru, r^{-1}v) \leqslant r[1 + \rho(r^{\sigma} - 1)]^{-1} f(t,u,v), \tag{1.6}$$

and the condition (H_2) implies

$$f(t, ru, r^{-1}v) \leqslant r^{\sigma} f(t,u,v). \tag{1.7}$$

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2. Auxiliary lemmas

Referring to [8] and our later proofs in [2], we first show the related Green's function and its main properties.

Lemma 2.1 ([2], p.3, Lemma 2.4)**.** *Let* $\alpha \in (2,3]$, $\lambda \in (0,[\alpha]_q)$ *and* $h \in C[0,1]$. *Then, the unique solution of*

$$\begin{cases} (D_q^\alpha u)(t) + h(t) = 0, \quad t \in (0,1), \\ D_q^i u(0) = 0, \quad 0 \leqslant i \leqslant n-2, \quad u(1) = \lambda \int_0^1 u(s) d_q s \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,qs)h(s)d_q s, \quad t \in [0,1],$$

where
$G(t,qs)$

$$= \begin{cases} \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha) - ([\alpha]_q - \lambda)(t-qs)^{(\alpha-1)}}{([\alpha]_q - \lambda)\Gamma_q(\alpha)}, & 0 \leqslant qs \leqslant t \leqslant 1; \\ \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha)}{([\alpha]_q - \lambda)\Gamma_q(\alpha)}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases}$$

Lemma 2.2 ([2], p.4, Lemma 2.5)**.** *The Green's function* $G(t,qs)$ *given above satisfies*

(i) $G(t,qs) > 0$, $t,s \in (0,1)$;

(ii) $\frac{\lambda s q^\alpha (1-qs)^{(\alpha-1)}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} t^{\alpha-1} \leqslant G(t,qs) \leqslant \frac{[\alpha]_q (1-qs)^{(\alpha-1)}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} t^{\alpha-1}$; $t,s \in [0,1]$;

(iii) $G : [0,1]\times[0,1] \to [0,+\infty)$ *is continuous.*

As usual, we equip the space $E = C[0,1]$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$, $x \in E$. Let $P = \{x \in E : x(t) \geqslant 0,\ t \in [0,1]\}$. For all $x, y \in P$, the notation $x \sim y$ shows that there exist positive constants $\mu_1 \leqslant \mu_2$ satisfying $\mu_1 x \leqslant y \leqslant \mu_2 x$. Definitely, $\sim$ is an equivalence relation. For the sake of convenience, note that $e(t) = t^{\alpha-1}$, $t \in [0,1]$. Now, we construct an inverse symmetry set $P_e = \{x \in E : \exists\, c_x \in (0,1)$ such that

$$c_x e(t) \leqslant x(t) \leqslant (c_x)^{-1} e(t),\ t \in [0,1]\}.$$

It is true that $P_e \subset P$ is not a cone since $\theta \notin P_e$ and $(e,e) \in P_e \times P_e$.

Define an operator $A : E \times E \to E$ by

$$A(u,v)(t) = \int_0^1 G(t,qs) f(s,u(s),v(s)) \mathrm{d}_q s. \tag{2.2}$$

Lemma 2.3. *Suppose that (H) and (H$_1$) hold. Then,* $A : P_e \times P_e \to P_e$ *is completely continuous and* A *is increasing on* u *and decreasing on* v.

Proof. For each $(u,v) \in P_e \times P_e$, by the definition of P_e, there exist constants $c_u, c_v \in (0,1)$ such that

$$c_u e(t) \leqslant u(t) \leqslant (c_u)^{-1}e(t), \quad c_v e(t) \leqslant v(t) \leqslant (c_v)^{-1}e(t), \quad t \in [0,1].$$

Here, we suppose that $c_u \leqslant c_v$, then $c_u e(t) \leqslant v(t) \leqslant (c_u)^{-1}e(t)$, $t \in [0,1]$. By the condition (H), (1.2) and Lemma 2.2, we see that

$$\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,qs) f(s,u(s),v(s))\mathrm{d}_q s\\
&\geqslant \tfrac{\lambda q^\alpha e(t)}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s, c_u e(s), {c_u}^{-1} e(s))\mathrm{d}_q s\\
&\geqslant \tfrac{\lambda q^\alpha c_u [1+\rho({c_u}^{-\sigma}-1)] e(t)}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,e(s),e(s))\mathrm{d}_q s\\
&\geqslant c_A e(t),
\end{aligned}$$

and by (1.6), we have

$$\begin{aligned}
A(u,v)(t) &\leqslant \tfrac{[\alpha]_q e(t)}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} f(s, {c_u}^{-1} e(s), c_u e(s))\mathrm{d}_q s\\
&\leqslant \tfrac{[\alpha]_q {c_u}^{-1} [1+\rho({c_u}^{-\sigma}-1)]^{-1} e(t)}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} f(s,e(s),e(s))\mathrm{d}_q s\\
&\leqslant c_A^{-1} e(t).
\end{aligned}$$

Using the condition (H) again, we may choose the constant c_A satisfying

$$0 < c_A < \min\left\{1, \frac{\lambda q^\alpha c_u [1+\rho({c_u}^{-\sigma}-1)]}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,e(s),e(s))\mathrm{d}_q s\right\}$$

and

$$c_A^{-1} > \max\left\{1, \frac{[\alpha]_q {c_u}^{-1} [1+\rho({c_u}^{-\sigma}-1)]^{-1}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} f(s,e(s),e(s))\mathrm{d}_q s\right\}.$$

Thus, $c_A \in (0,1)$ and

$$c_A e(t) \leqslant A(u,v)(t) \leqslant c_A^{-1} e(t), \quad t \in [0,1].$$

Hence, $A : P_e \times P_e \to P_e$ is well defined. Notice the continuity of $G(t,qs)$ and f, by usual arguments, it is easy to infer that $A : P_e \times P_e \to P_e$ is completely continuous. Moreover, following (H) and Lemma 2.2, A is increasing on u and decreasing on v. □

Working as in Lemma 2.3, we can get the following lemma.

Lemma 2.4. *Suppose that (H) and (H_2) hold. Then, $A : P_e \times P_e \to P_e$ is continuous, and A is increasing on u and decreasing on v.*

In this case, we may choose the constant c_A satisfying

$$0 < c_A < \min\left\{1, \frac{\lambda q^\alpha {c_u}^\sigma}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,e(s),e(s))\mathrm{d}_q s\right\}$$

and

$$c_A^{-1} > \max\left\{1, \frac{[\alpha]_q {c_u}^{-\sigma}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} f(s,e(s),e(s))\mathrm{d}_q s\right\}.$$

An operator $B : P_e \times P_e \to P_e$ is said to be mixed monotone, if $B(x, y)$ is increasing on x and decreasing on y. If $B(x^*, x^*) = x^*$, then element $x^* \in P_e$ is called a fixed point of B, and if $B(x^*, y^*) = x^*$ and $B(y^*, x^*) = y^*$, then element $(x^*, y^*) \in P_e \times P_e$ is called a coupled fixed point of B. Obviously, if $A(x^*, x^*) = x^*$, then x^* is a positive solution of BVP (1.1), and if $(x^*, y^*) \in P_e \times P_e$ is a coupled fixed point of A, then x^*, y^* are two positive solutions of BVP (1.1).

3. Main results

In this section, by applying some iterative techniques, we present three main results of this paper and two related examples.

Theorem 3.1. *Under the assumptions (H) and* (H_1)*, BVP (1.1) has two positive solutions in* P_e.

Proof. For every point $x_0 \in P_e$, by Lemma 2.3, we have $A(x_0, x_0) \in P_e$. Therefore, there exists a constant $c_{x_0} \in (0, 1)$ small enough such that

$$c_{x_0} e(t) \leqslant x_0(t) \leqslant c_{x_0}^{-1} e(t), \quad t \in [0, 1]$$

and

$$c_{x_0} e(t) \leqslant A(x_0, x_0)(t) \leqslant c_{x_0}^{-1} e(t), \quad t \in [0, 1],$$

which show that

$$c_{x_0}^2 x_0(t) \leqslant A(x_0, x_0)(t) \leqslant c_{x_0}^{-2} x_0(t), \quad t \in [0, 1].$$

Let

$$\kappa = \left(\frac{\rho c_{x_0}^2}{1 - c_{x_0}^2 + \rho {c_{x_0}}^2} \right)^{\frac{1}{\sigma}}, \tag{3.1}$$

where ρ, σ are given in condition (H_1). Then $\kappa \in (0, 1)$, and by direct computation, we have

$$[1 + \rho(\kappa^{-\sigma} - 1)]^{-1} x_0(t) \leqslant A(x_0, x_0)(t) \leqslant [1 + \rho(\kappa^{-\sigma} - 1)] x_0(t), \quad t \in [0, 1]. \tag{3.2}$$

Take

$$u_0 = \kappa x_0, \quad v_0 = \kappa^{-1} x_0,$$

and define two sequences by

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n \in \mathbb{N}. \tag{3.3}$$

From (1.2) and (3.2), we have

$$\begin{aligned} u_1 &= \int_0^1 G(t, qs) f(s, \kappa x_0(s), \kappa^{-1} x_0(s)) \mathrm{d}_q s \\ &\geqslant \kappa[1 + \rho(\kappa^{-\sigma} - 1)] \int_0^1 G(t, qs) f(s, x_0(s), x_0(s)) \mathrm{d}_q s \\ &\geqslant u_0, \end{aligned}$$

and from (1.6) and (3.2), we get

$$\begin{aligned} v_1 &= \int_0^1 G(t,qs) f\left(s, \kappa^{-1} x_0(s), \kappa x_0(s)\right) \mathrm{d}_q s \\ &\leqslant \kappa^{-1}[1+\rho(\kappa^{-\sigma}-1)]^{-1} \int_0^1 G(t,qs) f(s, x_0(s), x_0(s)) \mathrm{d}_q s \\ &\leqslant v_0. \end{aligned}$$

Noticing that $u_0 \leqslant v_0$ and A is mixed monotone, we can deduce that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \cdots \leqslant v_1 \leqslant v_0. \tag{3.4}$$

By Lemma 2.3, using the compactness of A, one can see that $\{u_n\}, \{v_n\}$ are both sequence compact sets. Hence, there exist $u^*, v^* \in P_e$ such that $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$ and $u^* \leqslant v^*$. Since A is continuous, by (3.3),

$$A(u^*, v^*) = u^*, \quad A(v^*, u^*) = v^*.$$

Thus, A has a coupled fixed point in P_e, and then the BVP (1.1) has at least two positive solutions in P_e. Definitely, these two solutions may be identical. The proof is completed. □

Remark 3.1. An analogy condition as (H_1) is adopted by Zhang et al., [18] to prove the existence and uniqueness of the positive solution. In fact, as to the result (3.12) in [18], it is impossible to get such a similar result as

$$u_n \geqslant \left(\kappa^2 \rho^{\frac{1}{1-\sigma}}\right)^{(1-\sigma)^n} v_n, \quad n \in \mathbb{N},$$

which directly prevents us from gaining the uniqueness here and there. Though we give a precise and relatively weak existence result now, it is true that our results present a good extension of the problems, which have the corresponding conditions on the nonlinearity term f.

Remark 3.2. In general, the coupled fixed point (u^*, v^*) is associated to the choice of κ. Specifically, u^* is increasing on κ, while v^* is decreasing on κ.

Remark 3.3. If there exists a point $x^* \in P_e$ such that $A(x^*, x^*) = x^*$, then for any coupled fixed point $(u^*, v^*) \in P_e \times P_e$, we have $u^* = v^* = x^*$.

In fact, let

$$\bar{r} = \sup\{\ 0 < r < 1 \mid rx^* \leqslant u^* \leqslant r^{-1}x^*,\ rx^* \leqslant v^* \leqslant r^{-1}x^*\}.$$

Then $\bar{r} \in (0,1]$ and

$$\bar{r}x^* \leqslant u^* \leqslant \bar{r}^{-1}x^*, \quad \bar{r}x^* \leqslant v^* \leqslant \bar{r}^{-1}x^*.$$

By (1.2) and (1.6), we have

$$\begin{aligned} u^* = A(u^*, v^*) &\geqslant A(\bar{r}x^*, \bar{r}^{-1}x^*) \\ &\geqslant \bar{r}[1+\rho(\bar{r}^{-\sigma}-1)]T(x^*, x^*) \\ &= \bar{r}[1+\rho(\bar{r}^{-\sigma}-1)]x^*, \end{aligned}$$

$$\begin{aligned} v^* = A(v^*, u^*) &\leqslant A(\bar{r}^{-1}x^*, \bar{r}x^*) \\ &\leqslant \bar{r}^{-1}[1+\rho(\bar{r}^{-\sigma}-1)]^{-1}T(x^*, x^*) \\ &= \bar{r}^{-1}[1+\rho(\bar{r}^{-\sigma}-1)]^{-1}x^*. \end{aligned}$$

Thus,

$$\bar{r}[1+\rho(\bar{r}^{-\sigma}-1)]x^* \leqslant u^* \leqslant v^* \leqslant \bar{r}^{-1}[1+\rho(\bar{r}^{-\sigma}-1)]^{-1}x^*. \tag{3.5}$$

If $\bar{r} \neq 1$, then (3.5) is contradictory with the definition of $\bar{r}$, since $\bar{r} < \bar{r}[1+\rho(\bar{r}^{-\sigma}-1)]$. Hence, we get $u^* = v^* = x^*$.

Theorem 3.2. *Under the assumptions (H_0) and (H_2), BVP (1.1) has a unique positive solution $u^* \in P_e$. Further, for any initial point $\bar{u} \in P_e$, there exists a sequence $\{\bar{u}_n\}$, which uniformly converges to u^* with the convergence rate*

$$||\bar{u}_n - u^*|| = o\left(1-\tau^{2\sigma^n}\right),$$

where τ is associated with the initial value $\bar{u}$.

Proof. According to Lemma 2.4 that $A(e,e) \in P_e$, there exists a constant $c_{Ae} \in (0,1)$ such that

$$c_{Ae}e(t) \leqslant A(e,e)(t) \leqslant (c_{Ae})^{-1}e(t), \quad t \in [0,1]. \tag{3.6}$$

Take a fixed number τ satisfying

$$0 < \tau < (c_{Ae})^{\frac{1}{1-\sigma}} < 1, \tag{3.7}$$

where σ is defined in (H_2). Here, we choose

$$u_0 = \tau e(t), \ v_0 = \tau^{-1}e(t), \quad t \in [0,1].$$

Define two iterative sequences u_n, v_n as

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n \in \mathbb{N}. \tag{3.8}$$

Hence, by (1.3), (3.6) and (3.7), we get

$$\begin{aligned} u_1 &\geqslant \tau^\sigma \int_0^1 G(t,qs)f(s,e(s),e(s))\mathrm{d}_q s \\ &\geqslant \tau^\sigma c_{Ae}e(t) \\ &\geqslant u_0, \end{aligned}$$

and by (1.7), (3.6) and (3.7), we also get

$$\begin{aligned} v_1 &\leqslant \tau^{-\sigma} \int_0^1 G(t,qs)f(s,e(s),e(s))\mathrm{d}_q s \\ &\leqslant \tau^{-\sigma}(c_{Ae})^{-1}e(t) \\ &\leqslant v_0. \end{aligned}$$

Since $u_0 \leqslant v_0$ and A is a mixed monotone operator, we deduce that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \cdots \leqslant v_1 \leqslant v_0. \tag{3.9}$$

On the other hand, analogous to (1.3), we may get

$$u_1 = A(\tau^2\tau^{-1}e(t), \tau^{-2}\tau e(t)) \geqslant \tau^{2\sigma}A(\tau^{-1}e(t), \tau e(t)) = \tau^{2\sigma}v_1.$$

By induction, for each natural number n, we deduce that

$$u_n \geqslant \tau^{2\sigma^n}v_n.$$

Thus, by (3.9), for any positive integers n and p, we obtain

$$0 \leqslant u_{n+p} - u_n \leqslant v_n - u_n \leqslant (1-\tau^{2\sigma^n})v_n \leqslant (1-\tau^{2\sigma^n})v_0. \tag{3.10}$$

Clearly, (3.10) implies that $\{u_n\}, \{v_n\}$ are two Cauchy sequences. Hence, there exists $u^* \in P_e$ such that $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = u^*$. Applying the continuity of A, let $n \to +\infty$ in $u_n = A(u_{n-1}, v_{n-1})$, then we obtain $u^* = A(u^*, u^*)$. Therefore, A has a fixed point in P_e, and then BVP (1.1) has a positive solution.

Now, we turn to the uniqueness problem. Suppose that there exists another point $v^* \in P_e$ and $v^* = A(v^*, v^*)$. It is obvious that there exists $c_v \in (0,1)$ such that

$$c_v e(t) \leqslant v^*(t) \leqslant {c_v}^{-1}e(t), \quad t \in [0,1].$$

Let τ be defined in (3.7) which is sufficiently small such that $\tau < c_v$. Therefore, $\tau^{-1} > {c_v}^{-1}$. Then,

$$u_0 \leqslant v^* \leqslant v_0.$$

Notice that $v^* = A(v^*, v^*)$ and A is mixed monotone. Thus the following relation holds

$$u_n \leqslant v^* \leqslant v_n, \quad n \in \mathbb{N}.$$

Let $n \to +\infty$, and we get $v^* = u^*$. That is, the uniqueness is verified. Hence, BVP (1.1) has a unique positive solution in P_e. Moreover, one deduces immediately that the existence of u^* has nothing to do with the size of $\tau \in (0,1)$.

Lastly, for any initial point $\bar{u} \in P_e$, which is the same as the discussion above, we may choose $\tau \in (0,1)$ small enough such that $\tau e(t) \leqslant \bar{u}(t) \leqslant \tau^{-1}e(t),\ t \in [0,1]$. Let $\bar{u}_n = A(\bar{u}_{n-1}, \bar{u}_{n-1}),\ n \in \mathbb{N}$. Then,

$$u_n \leqslant \bar{u}_n \leqslant v_n, \tag{3.11}$$

where u_n, v_n are defined in (3.8). Finally from (3.11), it is easy to know that the sequence $\{\bar{u}_n\}$ uniformly converges to the positive solution u^*. Furthermore, by (3.10), one may get the error estimation

$$||\bar{u}_n - u^*|| \leqslant |1-\tau^{2\sigma^n}| \cdot \|v_0\|,$$

and the convergence rate

$$||\bar{u}_n - u^*|| = o\left(1-\tau^{2\sigma^n}\right),$$

where τ is related to the initial point $\bar{u}$. The proof is completed. □

Remark 3.4. Actually, if $\rho = 1$ is taken, then one may check immediately that the condition (H_2) is a special case of (H_1).

Theorem 3.3. *If (H_3) holds and*

$$0 < \int_0^1 (1-qs)^{(\alpha-1)} f(s, s^{\alpha-1}, s^{\alpha-1})d_q s < +\infty,$$

then BVP (1.1) has a unique positive solution in P_e.

Proof. **Case 1.** Suppose that $0 \leqslant u_1 \leqslant u_2$. If $u_1 = u_2 = 0$, it is clear that $f(t, u_1, v) = f(t, u_2, v)$; if $u_2 \neq 0$ and $u_1 = ru_2$, then $r \in (0, 1]$. By (1.4), we have

$$f(t, u_1, v) = f(t, ru_2, v) \leqslant r^{\lambda_1} f(t, u_2, v),$$

which shows that the function $f(t, u, v)$ is increasing on u in $[0, +\infty)$.

Case 2. Suppose that $0 < v_1 \leqslant v_2$. If $v_1 = rv_2$, then $r \in (0, 1]$. By (1.5), we have

$$f(t, u, v_2) = f(t, u, r^{-1}v_1) \leqslant r^{\mu_1} f(t, u, v_1),$$

which shows that $f(t, u, v)$ is decreasing on v in $(0, +\infty)$.

Meanwhile, for all $(t, u, v) \in [0, 1] \times [0, +\infty) \times (0, +\infty)$, we see that

$$f(t, ru, r^{-1}v) \geqslant r^{\lambda_2} f(t, u, r^{-1}v) \geqslant r^{\lambda_2+\mu_2} f(t, u, v), \quad r \in (0, 1].$$

Then, Theorem 3.2 implies that Theorem 3.3 holds. The proof is completed. □

Next, we present two simple examples. Consider the following integral boundary value problem

$$\begin{cases} \left(D_{\frac{1}{2}}^{\frac{5}{2}} u\right)(t) + f(t, u(t), v(t)) = 0, \quad t \in (0, 1), \\ u(0) = D_{\frac{1}{2}} u(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) \mathrm{d}_q s, \end{cases} \tag{3.12}$$

where $\lambda \in (0, 2 - \frac{\sqrt{2}}{4})$.

Example 3.1. Let $f(t, u, v) = tu^{\frac{1}{2}} + v^{-\frac{1}{2}}$, $(t, u, v) \in [0, 1] \times [0, +\infty) \times (0, +\infty)$. It is easy to see that f satisfies all the hypotheses of Theorem 3.1, for $\rho = \sigma = \frac{1}{2}$. Meantime, f satisfies all the hypotheses of Theorem 3.2, for $\sigma = \frac{1}{2}$. Hence, BVP (3.12) has a unique positive solution.

Example 3.2. Let $f(t, u, v) = u^{\frac{1}{3}} v^{-\frac{1}{2}}$, $(t, u, v) \in [0, 1] \times [0, +\infty) \times (0, +\infty)$. Then f satisfies all the hypotheses of Theorem 3.2, for $\sigma = \frac{5}{6}$. Meantime, f satisfies all the hypotheses of Theorem 3.3, for $\lambda_i = \frac{1}{3}$, $\mu_i = \frac{1}{2}$, $i = 1, 2$. Hence, BVP(3.12) has a unique positive solution.

Remark 3.5. Especially, either $f(t, u, u) = tu^{\frac{1}{2}} + u^{-\frac{1}{2}}$ or $f(t, u, u) = u^{\frac{1}{3}} u^{-\frac{1}{2}}$, $(t, u) \in [0, 1] \times (0, +\infty)$. Here, v vanishes, and f is not monotonic with respect to u. This reveals that our results generalize the conclusions obtained in [8, 10, 12, 18].

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