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Dynamics of a Degenerately Damped Stochastic Lorenz-Stenflo System*

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Abstract It seems that little has been known about the sensitivity of steady states in stochastic systems. This paper proves the conditions for the existence of an invariant measure in a degenerately damped stochastic Lorenz-Stenflo model. Precisely, the solution is proved to be a nice diffusion via the Lie bracket technique and non-trivial Lyapunov functions. The finiteness of the expected positive recurrence time entails the existence problem. On the other hand, a cut-off function is constructed to show the non-existence result through a proof by contradiction. For other interesting cases, the expected recurrence time is shown to be infinite.

Keywords Lorenz-Stenflo system, invariant measure, Lyapunov function, noise-induced stabilization.

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1. Introduction

To describe the low-frequency and short-wavelength acoustic-gravity perturbations in the atmosphere, Stenflo [22] derived a four-dimensional continuous-time dynamical system given by

$$\begin{cases} \dfrac{dx}{dt} = a(y-x) + rw, \\ \dfrac{dy}{dt} = cx - y - xz, \\ \dfrac{dz}{dt} = xy - bz, \\ \dfrac{dw}{dt} = -x - aw, \end{cases} \tag{1.1}$$

where x, y, z, w are state variables of the so-called Lorenz-Stenflo equation (1.1), and positive parameters a, c, r are the Prandtl, generalized Rayleigh and rotation numbers respectively, and b is the geometric parameter.

Obviously, one can reduce (1.1) to the usual Lorenz system in [15] with interesting mathematical properties, if the rotation of the earth is not considered. In

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the past decades, many scholars studied its complex dynamical behaviors such as boundedness [24, 33], periodicity [17, 28], bifurcation [26, 29, 30, 34], synchronization [6], chaotic and hyperchaotic dynamics [9, 19, 25, 27] as well as the influence of Lévy noise [12].

Notice that the geometric parameter b is strictly positive as shown in the derivation of (1.1). But it will tend to zero under the sufficiently large generalized Rayleigh number. On the other hand, the so-called Homogeneous Rayleigh-Bénard (HRB) system was established with $b \leq 0$ appearing in the temperature equation [3, 4]. Indeed, a similar degeneracy effect was observed in a certain zero Prandtl limit for modeling mantle convection [20, 23]. Therefore, it is natural to investigate the corresponding dynamics of (1.1), when $b \leq 0$. However, it is straightforward that the corresponding solution to (1.1) on the z-direction explodes in finite time under the initial conditions ($x_0 = y_0 = w_0 = 0, z_0 \neq 0$) provided that $b < 0$. As for the case $b = 0$, any point on the z-axis becomes an equilibrium. Thus, one can prove the existence of singularly degenerate heteroclinic cycles, even there is no compact global attractor in this situation. Therefore, both embarrassing cases motivate us to study the possibility of stabilizing the dynamics by adding external noise perturbations.

It is well-known that arbitrary small additive noise can stabilize an explosive ordinary differential equation (ODE) [16, 18]. If, in addition, the corresponding Markov process admits an invariant probability measure, it corresponds to the so-called noise-induced stabilization problem. In this respect, considerable interest has already been shown in studying stationary states, stable oscillations and the related work [1, 2, 5, 7, 8, 10, 11, 13, 14, 21, 31, 32].

Motivated by the aforementioned discussion, we are interested in the stochastic Lorenz-Stenflo system

$$\begin{cases} dx = (a(y-x) + rw)dt + \sqrt{2\kappa_1}dB_1, \\ dy = (cx - y - xz)dt + \sqrt{2\kappa_2}dB_2, \\ dz = (xy - bz)dt + \sqrt{2\kappa_3}dB_3, \\ dw = (-x - aw)dt + \sqrt{2\kappa_4}dB_4, \end{cases} \tag{1.2}$$

where $B_i, i = 1, 2, 3, 4$ are independent and standard Brownian motions, and $\kappa_i \geq 0, i = 1, 2, 3, 4$ represent the intensity of random noise and other parameters conform to the ones in system (1.1). To ensure system (1.2) is genuinely stochastic, we require that at least one κ_i is positive.

In the absence of noise, we know that the solutions are explosive or have no compact global attractor when $b \leq 0$. The interesting question here is whether the presence of noise induces the existence and the number of invariant probability measure for the generated Markov transition semigroup.

The paper aims to solve the noise-induced stabilization problem of (1.2) with additive Brownian noise by applying the way in [7, 32]. More precisely, we first state the philosophy of proving the existence of a unique invariant probability measure for the Markov transition semigroup generated by (1.2). The first step is to verify the non-explosion of the solution to (1.2) under a suitable Lyapunov function and Young's inequality. Then, Lie bracket over the vector fields shows that such a solution is a nice diffusion. The final step is to acquire the globally finite expected returns to some compact set by constructing another Lyapunov function. As for the non-existence under a highly degenerate noise, we construct a cut-off function and set forth the proof by contradiction. However, for the case $b < 0$, we shall directly prove that the expected recurrence time is infinite.

The rest of this paper is organized as follows: in Section 2, we collect necessary definitions, notations and criteria. Section 3 includes our main results, together with detailed proofs.

2. Preliminaries

Let M_{nk} be an $n \times k$ real matrix and consider the following Itô stochastic differential equation

$$dX_t = F(X_t)dt + G(X_t)dB_t, \tag{2.1}$$

where $F = (F_1, \dots, F_n) \in C^2(\mathbb{R}^n; \mathbb{R}^n)$, $G = (G_1, \dots, G_k) \in C^2(\mathbb{R}^n; M_{nk})$, and $B_t = (B_t^1, \dots, B_t^k)^T$ represents a standard k-dimensional Brownian motion in a filtered probability space $(w, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For a given function $V \in C^2(\mathbb{R}^n; \mathbb{R})$, the infinitesimal generator for equataion (2.1) is defined by

$$\begin{aligned}\mathcal{L}V(X) =& F(X)\nabla V(X) + \frac{1}{2}(GG^T)(X)\nabla^2 V(X) \\ =& \sum_{j=1}^{n} F_j(X)\partial_{X_j} V(X) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{l=1}^{k} G_{il}G_{jl}(X)\partial^2_{X_i X_j} V(X).\end{aligned} \tag{2.2}$$

As stated in [7], the smoothness of F and G cannot guarantee the existence of global solution to (2.1), but one can define a local unique pathwise solution, which is denoted by $X_t = X(0, x; t)$ under the initial condition $X_0 = x$. Next, we introduce a stopping time

$$\tau = \lim_{n\to\infty} \tau_n,$$

where $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\}$ for $n \in \mathbb{N}^+$. Thus, there is a unique solution X_t, for all times $t < \tau$, $\mathbb{P}$-almost surely. Herein, τ stands for the explosion time of the process X_t, and by which X_t is said to be non-explosive, if

$$\mathbb{P}_x\{\tau < \infty\} = 0 \quad \text{for all initial conditions } x \in \mathbb{R}^n.$$

Therefore, if X_t is non-explosive, it can generate a Markov process, and its transition probability measure is defined as $\mathcal{P}_t(x, \cdot) = \mathbb{P}_x\{X_t \in \cdot\}$. Denoted by $\mathcal{B}$ the Borel σ-field of subsets of $\mathbb{R}^n$, the Markov transition semigroup satisfies

$$\mathcal{P}_t V(X) = \mathbb{E}_X V(X_t) = \int_{\mathbb{R}^n} V(Y)\mathcal{P}_t(X, dY), \qquad X \in \mathbb{R}^n,$$

and

$$\pi\mathcal{P}_t(A) = \int_{\mathbb{R}^n} \pi(dX)\mathcal{P}_t(X, A), \quad A \in \mathcal{B},$$

for the bounded, $\mathcal{B}$-measurable functions $V : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{E}$ denotes the corresponding expectation. Indeed, a positive measure π is invariant for $\mathcal{P}_t$ if $\pi\mathcal{P}_t = \pi$ for all $t \geq 0$. An invariant measure π for $\mathcal{P}_t$ is an invariant probability measure for $\mathcal{P}_t$ provided that $\pi(\mathbb{R}^n) = 1$.

To realize our purpose, we next introduce two concepts to tell either the existence or the non-existence of an invariant probability measure and several notations related to the Lie bracket.

Definition 2.1. In an open set $U \subseteq \mathbb{R}^n$, the differential operator $\mathcal{A}$ is called hypoelliptic, if for any distribution $u \in V \subseteq U$, $\mathcal{A}u \in C^\infty(V)$ yields $u \in C^\infty(V)$.

Definition 2.2. Denote by X_t the solution to (2.1). Suppose that X_t is nonexplosive and satisfies

(1) $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $G \in C^\infty(\mathbb{R}^n; M_{nk})$;

(2) the operators $\mathcal{L}, \mathcal{L}^*, \mathcal{L} \pm \partial_t, \mathcal{L}^* \pm \partial_t$ are hypoelliptic on the respective domains $\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{R}^n \times \mathbb{R}^+$, $\mathbb{R}^n \times \mathbb{R}^+$, where $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$ with respect to the $L^2(\mathbb{R}^n; dx)$ inner product;

(3) the support $\operatorname{supp}(\mathcal{P}_t(X, \cdot)) = \mathbb{R}^n$, $\forall\ t > 0, X \in \mathbb{R}^n$.

Then, X_t is called a nice diffusion.

Recall that for two smooth vector fields

$$\begin{cases} U(X) = \displaystyle\sum_{j=1}^n U^j(X) \frac{\partial}{\partial X_j}, \\ W(X) = \displaystyle\sum_{j=1}^n W^j(X) \frac{\partial}{\partial X_j}, \end{cases}$$

the Lie bracket of them is defined by

$$[U, W](X) = \sum_{j,k=1}^n \left(U^k(X) \frac{\partial W^j(X)}{\partial X_k} - W^k(X) \frac{\partial U^j(X)}{\partial X_k} \right) \frac{\partial}{\partial X_j}.$$

It allows us to introduce the following notations

$$\begin{aligned} \operatorname{ad}^0 U(W) &= W, \\ \operatorname{ad}^1 U(W) &= [U, W], \\ \operatorname{ad}^m U(W) &= \operatorname{ad}^1 U(\operatorname{ad}^{m-1} U(W)), \text{ for } m \geq 2. \end{aligned}$$

In particular, let us denote

$$\mathfrak{n}(X, W) := \max_{j=1,\dots,n} \deg(p_j) \quad \text{where } p_j(\lambda) := W_j(\lambda X),$$

when W polynomially depends on the components of X for any $X \in \mathbb{R}^n$. Thus, for any collection of vector fields $\mathcal{G}$ on $\mathbb{R}^n$, let

$$\operatorname{cone}_{\geq 0} \mathcal{G} = \left\{ \sum_{j=1}^N \lambda_j U_j : \{\lambda_1, \dots, \lambda_N\} \subset [0, \infty), \text{and } \{U_1, \dots, U_N\} \subset \mathcal{G} \right\}.$$

Since we are only interested in the situation that G is independent of X and F is a polynomial, we further denote

$$\begin{aligned} \mathcal{G}_0 &:= \operatorname{span}\{G_0, \dots, G_k\}, \\ \mathcal{G}_1^O &:= \mathcal{G}_0 \cup \left\{ \operatorname{ad}^{\mathfrak{n}(G,F)} G(F) : G \in \mathcal{G}_0,\ \mathfrak{n}(G,F) \text{ is odd} \right\}, \\ \overline{\mathcal{G}}_1^O &:= \left\{ G \in \mathcal{G}_1^O : G \text{ is a constant vector field} \right\}, \\ \mathcal{G}_1^E &:= \left\{ \operatorname{ad}^{\mathfrak{n}(G,F)} G(F) : G \in \mathcal{G}_0,\ \mathfrak{n}(G,F) \text{ is even} \right\}, \\ \mathcal{G}_1 &:= \operatorname{span}\left(\mathcal{G}_1^O\right) + \operatorname{cone}_{\geq 0}\left(\mathcal{G}_1^E\right). \end{aligned}$$

For $j > 1$, we pause to denote that

$$\tilde{\mathcal{G}}_j^O = \left\{ \mathrm{ad}^{\mathfrak{n}(G,F)} G(H) : G \in \overline{\mathcal{G}}_j^O,\ H \in \mathcal{G}_j,\ \mathfrak{n}(G,H) \text{ is odd} \right\},$$
$$\tilde{\mathcal{G}}_j^E = \left\{ \mathrm{ad}^{\mathfrak{n}(G,F)} G(H) : G \in \overline{\mathcal{G}}_j^O,\ H \in \mathcal{G}_j,\ \mathfrak{n}(G,H) \text{ is even} \right\},$$

by which, we let

$$\begin{aligned}
\mathcal{G}_{j+1}^O &:= \mathcal{G}_j^O \cup \tilde{\mathcal{G}}_j^O, \\
\overline{\mathcal{G}}_{j+1}^O &:= \left\{ G \in \mathcal{G}_{j+1}^O : G \text{ is a constant vector field} \right\}, \\
\mathcal{G}_{j+1}^E &:= \mathcal{G}_j^E \cup \tilde{\mathcal{G}}_j^E, \\
\mathcal{G}_{j+1} &:= \mathrm{span}\left(\mathcal{G}_{j+1}^O \right) + \mathrm{cone}_{\geq 0}\left(\mathcal{G}_{j+1}^E \right).
\end{aligned}$$

Next, we extract some criteria from [7], which are useful to conclude the existence or non-existence of an invariant probability measure.

Lemma 2.1 (Proposition 2.1, [7])**.** *Assume that $F, G \in C^2$ and let X_t be the solution to (2.1) with the corresponding infinitesimal generator $\mathcal{L}$ defined in (2.2).*

(i) *Suppose that there is a function $V \in C^2(\mathbb{R}^n; [0, +\infty))$ such that $V(X) \to \infty$ as $|X| \to \infty$ and*
$$\mathcal{L}V(X) \leq pV(X) + q,\ \forall X \in \mathbb{R}^n,$$
for $p, q > 0$. Then, X_t is non-explosive.

(ii) *Assume that X_t is non-explosive and there is a function $V \in C^2(\mathbb{R}^n; [0, +\infty))$, a compact set $\mathcal{K} \subseteq \mathbb{R}^n$ and constants $p, q > 0$ such that*
$$\mathcal{L}V(X) \leq -p + q\mathbb{1}_{\mathcal{K}}(X),\ \forall X \in \mathbb{R}^n.$$
Then,
$$\mathbb{E}_X \xi_{\mathcal{K}} \leq \frac{V(X)}{p},\ \forall X \in \mathbb{R}^n,$$
where $\xi_{\mathcal{K}} := \inf\{t \geq 0 : X_t \in \mathcal{K}\}$ represents the first hitting time of $\mathcal{K}$ by X_t.

Lemma 2.2 (Theorem 2.2, [7])**.** *Let $V_1, V_2 \in C^2(\mathbb{R}^n; \mathbb{R})$. If*

(i) $\limsup_{|X| \to \infty} V_1(X) = \infty$;

(ii) *V_2 is strictly positive outside of a compact set;*

(iii) $\limsup_{S \to \infty} \dfrac{\max_{|X|=S} V_1(X)}{\min_{|X|=S} V_2(X)} = 0$;

(iv) *there exists an $R > 0$ such that*
$$\mathcal{L}V_1(X) \geq 0 \quad \text{and} \quad \mathcal{L}V_2(X) \leq 1, \quad \forall\ |X| > R,$$
where $\xi_R = \inf\{t \geq 0 : |X_t| \leq R\}$.

Then, there exists $M \geq 0$ such that $\mathbb{E}_{X_*} \xi_R = \infty$, whenever $|X_*| \geq R$ and $V_1(X_*) \geq M$.

Lemma 2.3 (Theorem 2.6, [7]). *Let F be a polynomial and G be X-independent. If the solution X_t to (2.1) is non-explosive and*

$$\operatorname{span}\left\{H \in \bigcup_{j\geq 0} \mathcal{G}_j^O : \ H \text{ is a constant vector}\right\} = \mathbb{R}^n,$$

then X_t is a nice diffusion.

Lemma 2.4 (Proposition 2.5, [7]). *Suppose that X_t is a nice diffusion, the following statements hold.*

(1) There is at most one invariant probability measure for $\mathcal{P}_t$;

(2) $\mathcal{P}_t$ has an invariant probability measure, if and only if there exists $R > 0$ such that $\mathbb{E}_X\xi_R < \infty, \forall\, X \in \mathbb{R}^n$ and the mapping $X \mapsto \mathbb{E}_X\xi_R$ is bounded on compact subsets of $\mathbb{R}^n$.

3. Main results

First, we focus on the hypo-ellipticity and irreducibility. Namely, we prove that the solution is non-explosive by Lemma 2.1(i) and a nice diffusion with specific parameters via Lemma 2.3. Thanks to Lemma 2.1(ii), we turn to construct a suitable Lyapunov function and show its boundedness, by which we are able to prove the existence of a unique invariant probability measure based on Lemma 2.4. Nevertheless, we demonstrate the non-existence result through Lemma 2.2.

In the sequel, we will denote by $X_t = (x_t, y_t, z_t, w_t)$ the solution to (2.1). In order to guarantee that system (2.1) is genuinely stochastic, we always assume $\sum_{i=1}^4 \kappa_i^2 \neq 0$.

3.1. Nice diffusion

Theorem 3.1. *For $a, b, c \in \mathbb{R}, r \geq 0$ and $\kappa_1, \kappa_2,\ \kappa_3, \kappa_4 \geq 0$, solution X_t to (1.2) is non-explosive. Moreover, it is a nice diffusion provided that $\kappa_1, \kappa_4 > 0$ and $\kappa_2^2 + \kappa_3^2 \neq 0$.*

Proof. To prove the first assertion, let $r > 0$, and we take

$$H(x, y, z, w) = \frac{1}{2}\left[\frac{1}{r}x^2 + y^2 + \left(z - c - \frac{a}{r}\right)^2 + w^2\right],$$

and the corresponding infinitesimal generator reads

$$\begin{aligned}\mathcal{L} =&(a(y-x) + rw)\partial_x + (cx - y - xz)\partial_y + (xy - bz)\partial_z + (-x - aw)\partial_w \\ &+ \kappa_1\partial_x^2 + \kappa_2\partial_y^2 + \kappa_3\partial_z^2 + \kappa_4\partial_w^2.\end{aligned}$$

A direct computation leads to

$$\begin{aligned}\mathcal{L}H =& -\frac{a}{r}x^2 - y^2 - bz^2 + b\left(c + \frac{a}{r}\right)z - aw^2 \\ &+ \frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4.\end{aligned}$$

We need to find $p, q > 0$ such that $\mathcal{L}H \le pH + q$, i.e.,

$$\begin{aligned}\mathcal{L}H &= -\frac{a}{r}x^2 - y^2 - bz^2 + b\left(c + \frac{a}{r}\right)z - aw^2 + \frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4 \\ &\le \frac{p}{2}\left[\frac{1}{r}x^2 + y^2 + \left(z - c - \frac{a}{r}\right)^2 + w^2\right] + q.\end{aligned}$$

Therefore, one has

$$\begin{cases} -\dfrac{a}{r} < \dfrac{p}{2r}, \\ -a < \dfrac{p}{2}, \\ \left(\dfrac{1}{2}p + b\right)z^2 - (p+b)\left(c + \dfrac{a}{r}\right)z + \dfrac{1}{2}p\left(c + \dfrac{a}{r}\right)^2 + q - \left(\dfrac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4\right) > 0. \end{cases}$$

Let us fix $p \ge 0$ such that

$$p > -2a \quad \text{and} \quad p > -2b.$$

Then, we choose $q \ge 0$ such that

$$q > \frac{1}{2}\frac{(p+b)^2}{p+2b}\left(c + \frac{a}{r}\right)^2 - \frac{1}{2}p\left(c + \frac{a}{r}\right)^2 + \left(\frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4\right).$$

Thus, the required condition in Lemma 2.1(i) is reached when taking $V = H$. As for $r = 0$, let $V(x, y, z, w) = \frac{1}{2}\left[x^2 + y^2 + (z - c - a)^2\right]$. The required condition in Lemma 2.1(i) also holds.

Therefore, it remains to show that system (1.2) satisfies the spanning condition in Lemma 2.3. Then, it turns out that X_t is a nice diffusion. For this purpose, using the Lie bracket, we have

$$\begin{aligned} F &= [a(y-x) + rw]\partial_x + (cx - y - xz)\partial_y + (xy - bz)\partial_z + (-x - aw)\partial_w, \\ G_1 &= \sqrt{2\kappa_1}\partial_x, \\ G_2 &= \sqrt{2\kappa_2}\partial_y, \\ G_3 &= \sqrt{2\kappa_3}\partial_z, \\ G_4 &= \sqrt{2\kappa_4}\partial_w. \end{aligned}$$

By considering F, G_1, G_2, G_3, G_4 as vectors, we can get

$$F(\lambda G_1) = (-a\lambda\sqrt{2\kappa_1}, c\lambda\sqrt{2\kappa_1}, 0, -\lambda\sqrt{2\kappa_1})^T.$$

Therefore, we verify that $\mathfrak{n}(G_1, F) = 1$. Then, we get

$$\begin{aligned} G_1' &:= \mathrm{ad}^1 G_1(F) = [G_1, F] \\ &= -a\sqrt{2\kappa_1}\partial_x + (c - z)\sqrt{2\kappa_1}\partial_y + y\sqrt{2\kappa_1}\partial_z - \sqrt{2\kappa_1}\partial_w \\ &\in \mathcal{G}_1^O. \end{aligned}$$

Hence, by $\mathfrak{n}(G_2, G_1') = 1$, it holds that

$$\tilde{G}_3 := \mathrm{ad}^1 G_2(G_1') = [G_2, G_1'] = \sqrt{4\kappa_1\kappa_2}\partial_z \in \mathcal{G}_2^O.$$

On the other hand, from $\mathfrak{n}(G_3, G_1') = 1$, it follows

$$\tilde{G_2} := \mathrm{ad}^1 G_3(G_1^{'}) = [G_3, G_1^{'}] = -\sqrt{4\kappa_1\kappa_3}\partial_y \in \mathcal{G}_2^O.$$

There are only three cases:

(1) $G_1, G_2, \tilde{G_3}, G_4 \in \bigcup_{j\geq 1} \mathcal{G}_j^O$ if $\kappa_1, \kappa_2, \kappa_4 > 0$,

(2) $G_1, \tilde{G_2}, G_3, G_4 \in \bigcup_{j\geq 1} \mathcal{G}_j^O$ if $\kappa_1, \kappa_2, \kappa_4 > 0$,

(3) $G_1, G_2, G_3, G_4 \in \bigcup_{j\geq 1} \mathcal{G}_j^O$ if $\kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0$,

which satisfy the required spanning condition. Therefore, the proof of Theorem 3.1 is completed. □

3.2. Existence of invariant probability measure

We will be led in the sequel to consider the degenerately damped situation, where $b = 0$. Thus, (1.2) can be reduced as

$$\begin{cases} dx = (a(y-x) + rw)dt + \sqrt{2\kappa_1}dB_1, \\ dy = (cx - y - xz)dt + \sqrt{2\kappa_2}dB_2, \\ dz = xydt + \sqrt{2\kappa_3}dB_3, \\ dw = (-x - aw)dt + \sqrt{2\kappa_4}dB_4. \end{cases} \tag{3.1}$$

Our task here is to construct a suitable Lyapunov function that ensures in a globally finite expected return subject to some compact set. For this purpose, we take

$$\begin{aligned} V =& \frac{1}{2}\left[\frac{1}{r}x^2 + y^2 + z^2 - 2\left(c + \frac{a}{r}\right)z + w^2 + \kappa_0\right] \\ &+ \frac{n_1 y\theta_1(x,y,z,w)}{xz} + \frac{n_2\theta_2(x,y,z,w)}{2\kappa_1}\left(\frac{R_1^2}{|z|^{\frac{2}{3}}} - x^2\right), \end{aligned}$$

whose detailed construction is put in the Appendix A for the sake of readers' convenience.

Theorem 3.2. *Assume that $b = 0$. If $\kappa_1 > 0$ and $\kappa_2, \kappa_3, \kappa_4 \geq 0$, there exists an $R > 0$ such that for any $S > 0$,*

$$\sup_{|X|\leq S} \mathbb{E}_X \xi_R < \infty,$$

where ξ_R is the return time to the ball of radius R. Furthermore, system (3.1) possesses a unique invariant probability measure provided that $\kappa_1, \kappa_4 > 0$ and $\kappa_2^2 + \kappa_3^2 \neq 0$.

Proof. Before proceeding any further, we emphasize that $C > 0$ is independent of R_0, R_1, R_2, R_3 and κ_0, n_1, n_2 unless explicitly stated otherwise (more details about C are in the appendix A). Meanwhile, we let $X' = x^2 + y^2 + w^2$ and $X'' = |x||z|^{\frac{1}{3}}$ for the sake of simplicity.

Regarding Lemma 2.1(ii), we observe that

$$\mathcal{M}(V) = \mathcal{M}(\tilde{H}) + \sum_{i=1}^{2}\Big(\theta_i\mathcal{M}(\psi_i) + \psi_i\mathcal{M}(\theta_i) + 2\nabla_\kappa\theta_i \cdot \nabla_\kappa\psi_i\Big), \tag{3.2}$$

where $\nabla_\kappa = (\kappa_1\partial_x, \kappa_2\partial_y, \kappa_3\partial_z, \kappa_4\partial_w)$, and $\mathcal{M}, \tilde{H}, \psi_1, \psi_2$ are as in (A.1), (A.3), (A.6), (A.8) in the Appendix A. To estimate each term in (3.2), we proceed to their derivatives fall on the cut-off functions θ_1 and θ_2 (as in (A.11), (A.12)). In fact, it follows that for θ_1,

$$\begin{cases} \partial_x\theta_1 \leq \dfrac{C}{R_0^{\frac{1}{2}}}\mathbb{1}_{R_0\leq X'\leq 2R_0} + \dfrac{C|z|^{\frac{1}{3}}}{R_1}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1}, \\ \partial_y\theta_1 \leq \dfrac{C}{R_0^{\frac{1}{2}}}\mathbb{1}_{R_0\leq X'\leq 2R_0}, \\ \partial_z\theta_1 \leq \dfrac{C|x|}{R_1|z|^{\frac{2}{3}}}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1} + \dfrac{C}{R_3}\mathbb{1}_{\frac{R_3}{2}\leq |z|\leq R_3}, \\ \partial_w\theta_1 \leq \dfrac{C}{R_0^{\frac{1}{2}}}\mathbb{1}_{R_0\leq X'\leq 2R_0}, \end{cases}$$

and

$$\begin{cases} \partial_x^2\theta_1 \leq \dfrac{C(R_0+1)}{R_0}\mathbb{1}_{R_0\leq X'\leq 2R_0} \\ \qquad + \dfrac{C|z|^{\frac{2}{3}}}{R_1^2}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1} + \dfrac{C|z|^{\frac{1}{3}}}{R_1}\mathbb{1}_{R_0\leq X'\leq 2R_0,\frac{R_1}{2}\leq X''\leq R_1}, \\ \partial_y^2\theta_1 \leq \dfrac{C(R_0+1)}{R_0}\mathbb{1}_{R_0\leq X'\leq 2R_0}, \\ \partial_z^2\theta_1 \leq \dfrac{C|x|^2}{R_1^2|z|^{\frac{4}{3}}}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1} + \dfrac{C|x|}{R_1|z|^{\frac{5}{3}}}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1} \\ \qquad + \left(\dfrac{C|x|}{R_1R_3|z|^{\frac{2}{3}}}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1} + \dfrac{C}{R_3^2}\right)\mathbb{1}_{\frac{R_3}{2}\leq |z|\leq R_3}, \\ \partial_w^2\theta_1 \leq \dfrac{C(R_0+1)}{R_0}\mathbb{1}_{R_0\leq X'\leq 2R_0}. \end{cases}$$

However, for θ_2, one has

$$\begin{cases} \partial_x\theta_2 \leq \dfrac{C}{R_2^{\frac{1}{2}}}\mathbb{1}_{R_2\leq X'\leq 2R_2} + \dfrac{C|z|^{\frac{1}{3}}}{R_1}\mathbb{1}_{R_1\leq X''\leq 2R_1}, \\ \partial_y\theta_2 \leq \dfrac{C}{R_2^{\frac{1}{2}}}\mathbb{1}_{R_2\leq X'\leq 2R_2}, \\ \partial_z\theta_2 \leq \dfrac{C|x|}{R_1|z|^{\frac{2}{3}}}\mathbb{1}_{R_1\leq X''\leq 2R_1} + \dfrac{C}{R_3}\mathbb{1}_{\frac{R_3}{2}\leq |z|\leq R_3}, \\ \partial_w\theta_2 \leq \dfrac{C}{R_2^{\frac{1}{2}}}\mathbb{1}_{R_2\leq X'\leq 2R_2}, \end{cases}$$

and

$$\begin{cases} \partial_x^2\theta_2 \leq C\mathbb{1}_{R_2\leq X'\leq 2R_2}\left(1+\dfrac{1}{R_2}+\dfrac{|z|^{\frac{1}{3}}}{R_1}\mathbb{1}_{\frac{R_1}{2}\leq X''\leq R_1}\right) \\ \qquad\qquad + \dfrac{C|z|^{\frac{2}{3}}}{R_1^2}\mathbb{1}_{X'\leq 2R^2, R_1\leq X''\leq 2R_1, |z|\leq R_3}, \\ \partial_y^2\theta_2 \leq C\left(1+\dfrac{1}{R_2}\right)\mathbb{1}_{R_2\leq X'\leq 2R_2}, \\ \partial_z^2\theta_2 \leq C\mathbb{1}_{\frac{R_3}{2}\leq|z|\leq R_3}\left(\dfrac{|x|}{R_1R_3|z|^{\frac{2}{3}}}\mathbb{1}_{R_1\leq X''\leq 2R_1}+\dfrac{1}{R_3^2}\right) \\ \qquad\qquad + C\mathbb{1}_{R_1\leq X''\leq 2R_1}\left(\dfrac{|x|^2}{R_1^2|z|^{\frac{4}{3}}}+\dfrac{|x|}{R_1|z|^{\frac{5}{3}}}\right), \\ \partial_w^2\theta_2 \leq C\left(1+\dfrac{1}{R_2}\right)\mathbb{1}_{R_2\leq X'\leq 2R_2}, \end{cases}$$

where again $C > 0$ is independent of R_1, R_2 and R_3.

Now, we are ready to expand $\psi_1\mathcal{M}(\theta_1)$ as

$$\begin{aligned} \psi_1\mathcal{M}(\theta_1) =& \frac{n_1 y}{xz}[(ay-ax+rw)\partial_x\theta_1 + (-y-xz)\partial_y\theta_1 + xy\partial_z\theta_1 + (-x-aw)\partial_w\theta_1 \\ &+ \kappa_1\partial_x^2\theta_1 + \kappa_2\partial_y^2\theta_1 + \kappa_3\partial_z^2\theta_1 + \kappa_4\partial_w^2\theta_1]. \end{aligned} \tag{3.3}$$

On $\mathcal{R}_1$, one has

$$|\psi_1| = n_1\left|\frac{y}{xz}\right| \leq \frac{n_1 R_0^{\frac{1}{2}}}{R_1}\cdot\frac{1}{|z|^{\frac{2}{3}}}. \tag{3.4}$$

Using the estimates of derivatives on θ_i, we have

$$\begin{cases} |\psi_1 x\partial_x\theta_1| \leq \dfrac{n_1R_0^{\frac{1}{2}}}{R_1R_3^{\frac{2}{3}}}\left|\dfrac{C}{R_0^{\frac{1}{2}}}+\dfrac{C|z|^{\frac{1}{3}}}{R_1}\right| \leq Cn_1K_{R_3}, \\ |\psi_1 x\partial_y\theta_1| \leq \dfrac{n_1R_0^{\frac{1}{2}}}{R_1R_1R_3^{\frac{2}{3}}}\left|\dfrac{C}{R_0^{\frac{1}{2}}}\right| \leq Cn_1K_{R_3}, \\ |\psi_1 xz\partial_y\theta_1| = |n_1 y\partial_y\theta_1| \leq Cn_1\mathbb{1}_{R_0\leq X'\leq 2R_0}, \\ |\kappa_1\psi_1\partial_x^2\theta_1| \leq Cn_1\left(K_{R_3}+\dfrac{R_0^{\frac{1}{2}}}{R_1^3}\right), \\ |\psi_1(\kappa_2\partial_y^2\theta_1+\kappa_3\partial_z^2\theta_1+\kappa_4\partial_w^2\theta_1)| \leq Cn_1(K_{R_3}+\mathbb{1}_{\mathcal{R}_0}), \end{cases}$$

where K_{R_3} is a constant that might depend on R_0, R_1 and R_2 such that

$$\lim_{R_3\to\infty} K_{R_3} = 0. \tag{3.5}$$

Hence, we obtain

$$|\psi_1\mathcal{M}(\theta_1)| \leq Cn_1\left(\mathbb{1}_{\mathcal{R}_0}+K_{R_3}+\frac{R_0^{\frac{1}{2}}}{R_1^3}\right). \tag{3.6}$$

Next, we estimate

$$\begin{aligned}
&|\nabla_\kappa\theta_1\cdot\nabla_\kappa\psi_1|\\
&\le n_1\left(\left|\frac{y}{x^2z}\right||\partial_x\theta_1|+\left|\frac{1}{xz}\right||\partial_y\theta_1|+\left|\frac{y}{xz^2}\right||\partial_z\theta_1|\right)\\
&\le Cn_1\left(\frac{R_0^{\frac12}}{R_1^2|z|^{\frac13}}\left(\frac{1}{R_0^{\frac12}}+\frac{|z|^{\frac13}}{R_1}\right)+\frac{1}{R_1|z|^{\frac23}}\frac{1}{R_0^{\frac12}}+\frac{R_0^{1/2}}{R_1|z|^{\frac53}}\left(\frac{|x|}{R_1|z|^{\frac23}}+\frac{1}{R_3}\right)\right) \quad (3.7)\\
&\le Cn_1\left(\frac{R_0^{\frac12}}{R_1^3}+K_{R_3}\right).
\end{aligned}$$

In the sequel, we focus on the cut-off terms involving ψ_2. In this respect, we expand $\psi_2\mathcal{M}(\theta_2)$ as

$$\begin{aligned}
\psi_2\mathcal{M}(\theta_2)=&\frac{n_2}{2\kappa_1}\left(\frac{4R_1^2}{z^{\frac23}}-x^2\right)[(ay-ax+rw)\partial_x\theta_2+(cx-y-xz)\partial_y\theta_2+xy\partial_z\theta_2\\
&+(-x-aw)\partial_w\theta_2+\kappa_1\partial_x^2\theta_2+\kappa_2\partial_y^2\theta_2+\kappa_3\partial_z^2\theta_2+\kappa_4\partial_w^2\theta_2].
\end{aligned} \tag{3.8}$$

Notice that each term in (3.8) is supported on the set $\{X''\le 2R_1\}$, and therefore the estimate (A.9) applies. Then, it entails

$$\begin{cases}
|\psi_2x\partial_x\theta_2|\le Cn_2\dfrac{R_1^2}{|z|^{\frac23}}\left|\dfrac{C}{R_2^{\frac12}}+\dfrac{|z|^{\frac13}}{R_1}\right|\le Cn_2K_{R_3},\\
|\psi_2x\partial_y\theta_2|\le Cn_2\dfrac{R_1^2}{|z|^{\frac23}}\left|\dfrac{C}{R_2^{\frac12}}\right|\le Cn_2K_{R_3},\\
|\psi_2xz\partial_y\theta_2|\le|xz||n_2C\dfrac{R_1^2}{|z|^{\frac23}}\dfrac{1}{R_2^{\frac12}}\le Cn_2\dfrac{R_1^3}{R_2^{\frac12}},\\
|\psi_2(\kappa_2\partial_y^2\theta_2+\kappa_3\partial_z^2\theta_2+\kappa_4\partial_w^2\theta_2)|\le Cn_1K_{R_3},
\end{cases}$$

and

$$\begin{aligned}
|\kappa_1\psi_2\partial_x^2\theta_2|&\le Cn_2\left(K_{R_3}+\mathbb{1}_{X'\le 2R^2,R_1\le X''\le 2R_1,|z|\le R_3}\right)\\
&\le cn_2(K_{R_3}+\theta_1+\mathbb{1}_{\mathcal{R}_0}).
\end{aligned}$$

Hence, we have

$$|\psi_2\mathcal{M}(\theta_2)|\le Cn_2\left(\frac{R_1^3}{R_2^{\frac12}}+K_{R_3}+\theta_1+\mathbb{1}_{\mathcal{R}_0}\right), \tag{3.9}$$

where K_{R_3} is the same as that in (3.5). Recalling that $R_2\ge R_1$ and $\mathcal{R}_0=\{x^2+y^2+w^2\ge R_0\}$, we have

$$\begin{aligned}
|\nabla_\kappa\theta_2\cdot\nabla_\kappa\psi_2|\le& n_2\left(\left|\frac{x}{\kappa_1}\right||\partial_x\theta_2|+\left|\frac{4R_1^2}{3\kappa_1|z|^{\frac53}}\right||\partial_z\theta_2|\right)\\
\le& Cn_2\left(\frac{x^2}{R_2}\mathbb{1}_{R_2\le X'\le 2R_2}+K_{R_3}\right)\\
\le& Cn_2\left(\mathbb{1}_{\mathcal{R}_0}+K_{R_3}\right).
\end{aligned} \tag{3.10}$$

Combining the estimates (3.6), (3.7), (3.9), (3.10), (A.4), (A.7) with (A.10), one obtains that for $R_2 \geq R_0$,

$$\mathcal{M}(V) \leq -\frac{a}{r}x^2 - y^2 - aw^2 + \bar{\kappa} - n_1\theta_1\left(1 - C\frac{R_0^4}{R_1}\right) - n_2\theta_2\left(1 - C\frac{R_0R_1^2}{R_3^{\frac{1}{2}}}\right) + C(n_1+n_2)\mathbb{1}_{\mathcal{R}_0} + Cn_1\frac{R_0^{\frac{1}{2}}}{R_1^3} + Cn_2\frac{R_1^3}{R_2^{\frac{1}{2}}} + Cn_2\theta_1 + K_{R_3}(n_1+n_2),$$

where $\bar{\kappa} = 2\left(\frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4\right)$. Fixing $n_2 = 16\bar{\kappa}$, $n_1 \geq \max\{8\bar{\kappa}, 2Cn_2\}$ and $R_0 > 1$ such that in $\mathcal{R}_0 = \{x^2 + y^2 + w^2 \geq R_0\}$, it follows that

$$\frac{a}{r}x^2 + y^2 + aw^2 \geq 4\bar{\kappa} + 2C(n_1+n_2).$$

Then, we choose $R_1 > 1$ such that

$$Cn_1\frac{R_0^{\frac{1}{2}}}{R_1^3} \leq \frac{\bar{\kappa}}{3} \quad \text{and} \quad \frac{R_0^4}{R_1} \leq \frac{1}{2},$$

and $R_2 \geq R_0$ such that

$$Cn_2\frac{R_1^3}{R_2^{\frac{1}{2}}} \leq \frac{\bar{\kappa}}{3}.$$

Finally, we choose R_3 such that

$$K_{R_3}(n_1+n_2) \leq \frac{\bar{\kappa}}{3} \quad \text{and} \quad C\frac{R_0R_1^2}{R_3^{\frac{1}{2}}} \leq \frac{3}{4}.$$

With these parameter selections and referring back to (A.11), (A.12), we therefore have

$$\begin{aligned}\mathcal{M}(V) &\leq -4\bar{\kappa}\mathbb{1}_{\mathcal{R}_0} + 2\bar{\kappa} - \frac{1}{2}n_1\mathbb{1}_{\mathcal{R}_1} - \frac{1}{4}n_2\mathbb{1}_{\mathcal{R}_2}\\ &\leq -2\bar{\kappa} + 4\bar{\kappa}(1 - \mathbb{1}_{\mathcal{R}_0\cup\mathcal{R}_1\cup\mathcal{R}_2})\\ &\leq -2\bar{\kappa} + 4\bar{\kappa}\mathbb{1}_{\mathcal{K}}.\end{aligned}$$

Since $R_2 \geq R_0$, one has $\{x^2+y^2+w^2 \leq R_0,\ |z| \geq R_3\} \subset \mathcal{R}_1 \cup \mathcal{R}_2$. Therefore, $1 - \mathbb{1}_{\mathcal{R}_0\cup\mathcal{R}_1\cup\mathcal{R}_2} = \mathbb{1}_{\mathcal{K}}$, where $\mathcal{K} = \{x^2+y^2+w^2 \leq R_0,\ |z| \leq R_3\}$. Consequently, (A.2) follows with $p = 2\bar{\kappa}$ and $q = 4\bar{\kappa}$.

Finally, it remains to check the non-negativity of V. Notice that our selection of the parameters R_0, R_1, R_2, R_3 and n_1, n_2 is made independent of the value κ_0 (see (A.13)). Moreover, by (3.4) and (A.9), we have

$$|\theta_1\psi_1| \leq Cn_1\frac{R_0^{\frac{1}{2}}}{R_1R_3^{\frac{2}{3}}}$$

and

$$|\theta_2\psi_2| \leq Cn_2\frac{R_1^2}{R_3^{\frac{2}{3}}}$$

respectively. Thus, fixing $R_0, R_1, R_2, R_3, n_1, n_2$ and referring back to (A.13), we have

$$V \geq \frac{1}{2}\left[\frac{1}{r}x^2 + y^2 + z^2 - 2\left(c + \frac{a}{r}\right)z + w^2 + \kappa_0\right] - Cn_1\frac{R_0^{\frac{1}{2}}}{R_1 R_3^{\frac{2}{3}}} - Cn_2\frac{R_1^2}{R_3^{\frac{2}{3}}},$$

which can always be positive for every $(x, y, z, w) \in \mathbb{R}^4$ by choosing large enough κ_0. Therefore, the proof of Theorem 3.2 is now finished. □

3.3. Non-existence of invariant probability measure

In this subsection, we devote ourselves to the non-existence issue. Since our results are slightly different for $b = 0$ and $b < 0$, we state them separately.

Theorem 3.3. *When $b = \kappa_1 = \kappa_4 = 0$, and one of κ_2, κ_3 is positive, there is no invariant measure for system (3.1).*

Proof. Assume that there is an invariant probability measure μ of (3.1) and let (x, y, z, w) have law μ. Thus, there exists an increasing sequence of integers $(N_j)_{j=1}^{\infty}$ with $N_{j+1} - N_j \geq 2$ such that

$$\lim_{j\to\infty} \mathbb{P}\left(\left|2az - x^2 - rw^2\right| \in [N_j, N_j + 2]\right) = 0. \tag{3.11}$$

Based on the construction and properties of F_N as defined by (B.1) in Appendix B, we apply Itô's formula to $F_N\left(2az - x^2 - rw^2\right)$ to get

$$\begin{aligned}
&\mathbb{E}_\mu F_N(2az_t - x_t^2 - rw_t^2)\\
=&\mathbb{E}_\mu \int_0^t \big[(2ax^2 + 2arw^2)F_N'(2az - x^2 - rw^2)\\
&+ 4a^2\kappa_3 F_N''(2az - x^2 - rw^2)\big]ds + \mathbb{E}_\mu F_N(2az_0 - x_0^2 - rw_0^2),
\end{aligned}$$

which further implies

$$\mathbb{E}_\mu(x^2 + rw^2)F_N'(2az_t - x_t^2 - rw_t^2) = -2a\kappa_3\mathbb{E}_\mu F_N''\left(2az_t - x_t^2 - w_t^2\right).$$

The monotone convergence theorem further indicates

$$\begin{aligned}
\mathbb{E}[x^2 + rw^2] &= \lim_{j\to\infty} \mathbb{E}(x^2 + rw^2)F_{N_j}'\left(2az - x^2 - rw^2\right)\\
&= -2a\kappa_3 \lim_{j\to\infty} \mathbb{E}F_{N_j}''\left(2az - x^2 - rw^2\right).
\end{aligned} \tag{3.12}$$

Finally, it follows from $|F_N''| \leq c^*$, $F_N'' = 0$ on the complement of $[N, N + 2]$ and (3.11) that

$$\lim_{j\to\infty} \mathbb{E}\left|F_{N_j}''\left(2az - x^2 - rw^2\right)\right| \leq c^*\mathbb{P}\left((2az - x^2 - rw^2) \in [N_j, N_j + 2]\right) = 0. \tag{3.13}$$

Combining (3.12) with (3.13), one yields $\mathbb{E}[x^2 + rw^2] = 0$. Then $x, w = 0$ almost surely. Moreover, if $\kappa_3 > 0$, we have $z(t) = z(0) + \sqrt{2\kappa_3}B_3(t)$. This contradicts the invariance. However, if $\kappa_2 > 0$, we have

$$dx = aydt, \quad dy = -ydt + \sqrt{2\kappa_2}dB_3.$$

Using that $x = 0$ almost surely, we obtain

$$ay = \frac{dx}{dt} = 0.$$

Then $y = 0$, which contradicts to $\kappa_2 > 0$. Therefore, the proof of Theorem 3.3 is completed. $\square$

Theorem 3.4. *When $b < 0$, for any $\mathcal{K} \subseteq \mathbb{R}^3$ compact, there exists $(x, y, z, w) \notin \mathcal{K}$ such that*

$$\mathbb{E}_{(x,y,z,w)}\xi_{\mathcal{K}} = \infty,$$

where

$$\xi_{\mathcal{K}} = \inf\{t \geq 0 : (x_t, y_t, z_t, \omega_t) \in \mathcal{K}\}.$$

If we further let $\kappa_1, \kappa_4 > 0$ and $\kappa_2^2 + \kappa_3^2 \neq 0$, then (1.2) does not possess an invariant probability measure.

Proof. We proceed it in four steps to construct V_1 and V_2 satisfying the conditions in Lemma 2.2.

Step 1. Fix $R > 1$ such that $\tilde{H}(x, y, z, w) > 1$ for any $|(x, y, z, w)| > R$. Thus, we take $W_2 \in C^2\left(\mathbb{R}^4\right)$ via

$$W_2(x, y, z, w) = \ln \tilde{H}(x, y, z, w),$$

for $|(x, y, z, w)| > R$. Then, $W_2 > 0$ outside of a compact set. Moreover, standard calculations give that

$$\begin{aligned}
&\mathcal{L}W_2(x, y, z, w) \\
&= \frac{1}{\tilde{H}(x, y, z, w)}\left[|b|z^2 - y^2 - \frac{a}{r}x^2 - 2|b|(c + \frac{a}{r})z + \frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4\right] \\
&\quad - \frac{1}{\tilde{H}^2(x, y, z, w)}\left[\kappa_1\frac{x^2}{r^2} + \kappa_2 y^2 + \kappa_3\left(z - c - \frac{a}{r}\right)^2 + \kappa_4 w^2\right].
\end{aligned}$$

Consequently, there exists a constant $K > 0$ such that

$$\mathcal{L}W_2(x, y, z, w) \leq K \text{ for all } (x, y, z, w) \in \mathbb{R}^4,$$

which motivates us to define $V_2 = W_2/K$.

Step 2. Denote

$$A = \frac{2\kappa_1 + 2r\kappa_4 + 2}{|b|}, \quad m = \max\left\{\frac{2\kappa_1}{a}, 2a^2\kappa_3, \frac{2r\kappa_4}{a}\right\},$$

and let

$$f(\zeta) := (1 - \cos\zeta)^2.$$

One can check that $f(0) = f'(0) = f''(0) = 0$ and f is strictly increasing on $(0, \pi)$, convex on $\left(0, \frac{2}{3}\pi\right)$ and concave on $\left(\frac{2}{3}\pi, \pi\right)$. In particular, $f'\left(\frac{2}{3}\pi\right) > 0 = f''\left(\frac{2}{3}\pi\right)$. By continuity, fix $B > \frac{2}{3}\pi$ close to $\frac{2}{3}\pi$ such that $f' \geq -mf''$ on $\left(\frac{2}{3}\pi, B\right)$. Next, we define

$$\Psi(\zeta) = \begin{cases} 0 & \zeta < 0, \\ (1 - \cos\zeta)^2 = f(\zeta) & \zeta \in [0, B], \\ c_0 \ln\ln(\zeta + c_1) + c_2 & \zeta > B, \end{cases}$$

where constants c_0, c_1, c_2 are determined later. Now, we claim that Ψ is a C^2 function. Clearly, Ψ is a C^2 function at 0, and it remains to show that

$$\begin{cases} c_0 \ln\ln(B+c_1) + c_2 = f(B) > 0, \\ \dfrac{c_1}{(B+c_1)\ln(B+c_1)} = f'(B) > 0, \\ -\dfrac{c\,(1+\ln(B+c_1))}{[(B+c_1)\ln(B+c_1)]^2} = f''(B) < 0. \end{cases} \tag{3.14}$$

Substituting the second equation of (3.14) into the third one, we obtain

$$\frac{1+\ln(B+c_1)}{(B+c_1)\ln(B+c_1)} = -\frac{f''(B)}{f'(B)} > 0. \tag{3.15}$$

However, the function

$$z \mapsto \frac{1+\ln z}{z\ln z}$$

is positive and decreasing on $(1,\infty)$ with a vertical asymptote at $z=1$, which is decaying at infinity. Thus, there exists a unique c_1 such that $B+c_1>1$ and (3.15) holds true. Then, for the already fixed c_1, we set

$$c_0 = f'(B)\,(B+c_1)\ln(B+c_1) > 0$$

and

$$c_2 = f(B) - c_0 \ln\ln(B+c_1).$$

Thus, the claim is reached. Finally, we fix $\lambda \in (0,1)$ such that

$$1 \geq \lambda m \frac{(1+\ln(B+c_1))}{(B+c_1)\ln(B+c_1)}$$

and define V_1 by

$$V_1(x,y,z,w) = \Psi\left(\lambda\left(2az - x^2 - rw^2 - A\right)\right).$$

Then, V_1 is a $C^2\left(\mathbb{R}^4\right)$ function and

$$\begin{aligned} \mathcal{L}V_1 =& 2\left(a|b|z + ax^2 + arw^2 - \kappa_1 - r\kappa_4\right)\lambda\Psi' \\ &+ 4\left(\kappa_1 x^2 + a^2\kappa_3 + \kappa_4 r^2 w^2\right)\lambda^2\Psi''. \end{aligned} \tag{3.16}$$

Step 3. We claim that

$$\mathcal{L}V_1 \geq 0. \tag{3.17}$$

To this goal, we let

$$\zeta = \lambda\left(2az - x^2 - rw^2 - A\right).$$

First, if $\zeta \leq 0$, then $\Psi' = \Psi'' = 0$ and (3.17) follows. However, if $\zeta \geq 0$, it follows that

$$2az \geq 2az - x^2 - rw^2 \geq A = \frac{2\kappa_1 + 2r\kappa_4 + 2}{|b|},$$

and thus $a|b|z - \kappa_1 - r\kappa_4 \geq 1$. Hence,

$$\begin{aligned} &a|b|z + ax^2 + arw^2 - \kappa_1 - r\kappa_4 \geq \left(ax^2 + arw^2 + 1\right), \\ &\left(4\kappa_1 x^2 + 4a^2\kappa_3 + +\kappa_4 r^2 w^2\right) \leq 2m\left(ax^2 + arw^2 + 1\right). \end{aligned} \tag{3.18}$$

Therefore, if $\zeta \geq 0$, the coefficients of Ψ', Ψ'' in (3.16) are non-negative. We split the domain $\zeta \geq 0$ into three pieces, and finally conclude (3.17).

(1) If $\zeta \in \left[0, \frac{2}{3}\pi\right]$, $\Psi'(\zeta), \Psi''(\zeta) \geq 0$, and the non-negativity of coefficients of Ψ', Ψ'' in (3.16) implies (3.17).

(2) If $\zeta \in \left(\frac{2}{3}\pi, B\right)$, then $\Psi'(\zeta) > 0$ and $\Psi''(\zeta) < 0$. Thus, from (3.16) and (3.18), it follows

$$
\begin{aligned}
&\frac{1}{\lambda}\mathcal{L}V_1 \\
&\geq \left(2a|b|z + 2ax^2 + 2arw^2 - 2\kappa_1 - 2r\kappa_4\right)\Psi' + \lambda\left(4\kappa_1 x^2 + 4a^2\kappa_3 + \kappa_4 r^2 w^2\right)\Psi'' \\
&\geq 2\left(ax^2 + arw^2 + 1\right)\Psi' + 2\lambda m\left(ax^2 + arw^2 + 1\right)\Psi'' \\
&\geq 0.
\end{aligned}
$$

(3) If $\zeta \in [B, \infty)$, then $\Psi(\zeta) = c_0 \ln\ln(\zeta + c_1) + c_2$. Since $c_0 > 0$, one has $\Psi'(\zeta) > 0, \Psi''(\zeta) < 0$. Then,

$$
\begin{aligned}
\frac{\mathcal{L}V_1}{\lambda} &\geq 2\left(ax^2 + arw^2 + 1\right)\Psi' + 2\lambda m\left(ax^2 + arw^2 + 1\right)\Psi'' \\
&\geq \frac{2c_0\left(ax^2 + arw^2 + 1\right)}{(2\zeta + c_1)\ln(\zeta + c_1)}\left(1 - \lambda m\frac{(1 + \ln(\zeta + c_1))}{(\zeta + c_1)\ln(\zeta + c_1)}\right) \\
&\geq \frac{2c_0\left(ax^2 + arw^2 + 1\right)}{(2\zeta + c_1)\ln(\zeta + c_1)}\left(1 - \lambda m\frac{(1 + \ln(B + c_1))}{(B + c_1)\ln(B + c_1)}\right) \\
&\geq 0.
\end{aligned}
$$

Step 4. Let us verify that the assumptions of Lemma 2.2 are satisfied with V_1 and V_2. Clearly, (iv) follows from the construction of V_1 and V_2 , and it is definite that (ii) is due to the fact that $\lim_{|(x,y,z,w)|\to\infty} H(x,y,z,w) = \infty$. As for (i),

$$
\begin{aligned}
\limsup_{|(x,y,z,w)|\to\infty} V_1(x,y,z,w) &\geq \lim_{z\to\infty} V_1(0,0,z,0) \\
&= \lim_{z\to\infty} \Psi(\lambda(2az - A)) \\
&= \lim_{z\to\infty} c_0 \ln\ln(\lambda(2az - A) + c_1) + c_2 \\
&= \infty,
\end{aligned}
$$

while for (iii),

$$
\begin{aligned}
\lim_{R\to\infty}\sup\frac{\sup_{|x,y,z,w|=R} V_1(x,y,z,w)}{\inf_{|x,y,z,w|=R} V_2(x,y,z,w)} &\leq \limsup_{R\to\infty}\frac{V_1(0,0,R,0)}{\ln\left[\frac{1}{2}(R - c - \frac{a}{r})^2\right]} \\
&\leq \lim_{R\to\infty}\frac{c_0\ln\ln(\lambda(2\sigma R - A) + c_1) + c_2}{\ln\left[\frac{1}{2}(R - c - \frac{a}{r})^2\right]} \\
&= 0,
\end{aligned}
$$

we come to the fact that $z \mapsto V_1(x,y,z,w)$ is increasing for large z, and $(x,y,w) \mapsto V_1(x,y,z,w)$ is non-increasing. This finishes the proof based on Lemma 2.2. □

Appendix

A. Derivation of the Lyapunov function

It is notoriously difficult to check that according to (2.2), the infinitesimal generator of (3.1) leads to

$$\begin{aligned}\mathcal{M} =&(a(y-x)+rw)\partial_x + (-y-xz)\partial_y + xy\partial_z + (-x-aw)\partial_w \\ &+ \kappa_1\partial_x^2 + \kappa_2\partial_y^2 + \kappa_3\partial_z^2 + \kappa_4\partial_w^2,\end{aligned} \tag{A.1}$$

by which our immediate goal is to acquire the inequality

$$\mathcal{M} \leq -p + q\mathbb{1}_{\mathcal{K}} \tag{A.2}$$

for some constants $p, q > 0$ and some compact set $\mathcal{K} \subseteq \mathbb{R}^4$.

First, we choose the following Lyapunov function

$$\tilde{H}(x,y,z,w) = \frac{1}{2}\left[\frac{1}{r}x^2 + y^2 + z^2 - 2\left(c + \frac{a}{r}\right)z + w^2 + \kappa_0\right], \tag{A.3}$$

where $\kappa_0 > 0$ is large enough, so that $\tilde{H} \geq 0$. Since

$$\mathcal{M}(\tilde{H}) = -\frac{a}{r}x^2 - y^2 - aw^2 + 2\left(\frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4\right), \tag{A.4}$$

the required inequality (A.2) is sure on the set where $|(x,y,w)| := \sqrt{x^2+y^2+w^2}$ is large. More specifically, let the region

$$\mathcal{R}_0 = \{x^2 + y^2 + w^2 \geq R_0\}$$

be with a sufficiently large $R_0 \geq 0$. That is,

$$R_0 \geq \frac{2\bar{\kappa}}{\min\{\frac{c}{r}, 1, a\}} = \frac{2\left(\frac{\kappa_1}{r} + \kappa_2 + \kappa_3 + \kappa_4\right)}{\min\{\frac{c}{r}, 1, a\}},$$

Therefore, we have

$$\mathcal{M}(\tilde{H}) \leq -\bar{\kappa} \quad \text{in } \mathcal{R}_0.$$

Next, we pay attention to the situation that $x^2 + y^2 + w^2 \leq R_0$, and $|z|$ is large. For this purpose, we consider the scaling transformation

$$T_\lambda(x,y,z,w) = (\lambda^{-\alpha}x, y, \lambda z, \lambda^{-\alpha}w),$$

where $\lambda \gg 1$ and $\alpha \in (0,1)$. Applying T_λ to the generator, $\mathcal{M}$ yields

$$\begin{aligned}T_\lambda \circ \mathcal{M} =& (ay\lambda^\alpha - ax + rw)\partial_x + (c\lambda^{-\alpha}x - y - \lambda^{1-\alpha}xz)\partial_y + \lambda^{-1-\alpha}xy\partial_z \\ &+ (-x-aw)\partial_w + \kappa_1\lambda^{2\alpha}\partial_x^2 + \kappa_2\partial_y^2 + \kappa_3\lambda^{-2}\partial_z^2 + \kappa_4\lambda^{2\alpha}\partial_w^2 \\ &\sim \kappa_1\lambda^{2\alpha}\partial_x^2 - \lambda^{1-\alpha}xz\partial_y + \kappa_4\lambda^{2\alpha}\partial_w^2.\end{aligned} \tag{A.5}$$

Obviously, the dynamics of (A.5) is twofold. First, if $\alpha \in (0, \frac{1}{3})$, the dominant term in (A.5) is $-\lambda^{1-\alpha}xz\partial_y$, which allows us to consider the dominant equation

$$\dot{X} = 0,\ \dot{Y} = -XZ,\ \dot{Z} = 0,\ \dot{W} = 0.$$

Indeed, it suggests that we should seek a function ψ_1 such that

$$-xz\partial_y\psi_1 = -n_1,$$

where the constant $n_1 > 2\bar{\kappa}$. Thus,

$$\psi_1 = n_1\frac{y}{xz}. \tag{A.6}$$

Now, we pause to define a region

$$\mathcal{R}_1 := \{x^2+y^2+w^2 \le R_0,\ |x||z|^{1/3} \ge R_1,\ |z| \ge R_3\},$$

where $R_0, R_1, R_3 \ge 1$ are large constants to be determined below. Note that, on $\mathcal{R}_1$, the following estimates

$$\begin{aligned}
\left|\frac{y^2}{x^2z}\right| &\le \left|\frac{y^2}{x^2z^{\frac{2}{3}}}\right|\left|\frac{1}{z^{\frac{1}{3}}}\right| \le \frac{R_0}{R_1^2R_3^{\frac{1}{3}}} \le \frac{R_0}{R_1^2},\\
\left|\frac{y}{x^3z}\right| &\le \frac{R_0^{\frac{1}{2}}}{R_1^3},\\
\left|\frac{y}{xz}\right| &\le \frac{R_0^{\frac{1}{2}}}{R_1}\frac{1}{|z|^{\frac{2}{3}}},\\
\left|\frac{y^2}{z^2}\right| &\le \left|\frac{x^6y^2}{x^6z^2}\right| \le \frac{R_0^4}{R_1^6} \le \frac{R_0^4}{R_1}
\end{aligned}$$

are satisfied. Therefore, it is obvious that

$$\begin{aligned}
\frac{1}{n_1}\mathcal{M}(\psi_1) &= (ay-ax+rw)\left(\frac{-y}{x^2z}\right) + (cx-y-xz)\frac{1}{xz}\\
&\quad + xy\left(\frac{-y}{xz^2}\right) + \frac{2\kappa_1 y}{x^3z} + \frac{2\kappa_3 y}{xz^3}\\
&\le \frac{CR_0^4}{R_1} - 1,
\end{aligned} \tag{A.7}$$

where $C = C(a,c,r,\kappa_1,\kappa_3)$ is independent of R_0, R_1, R_2 and n_1. Thus, for sufficiently large R_1 depending on R_0, we obtain

$$\mathcal{M}(\psi_1) \le -\frac{1}{2}n_1 \quad \text{on the region } \mathcal{R}_1.$$

Consequently, for any fixed $R_0 \ge 1$, we can choose suitably large $n_1 \ge 1 \vee 4\bar{\kappa}$ and $R_1 \ge 1$, deriving

$$\mathcal{M}(\tilde{H}+\psi_1) \le -\frac{1}{2}n_1 \quad \text{on the region } \mathcal{R}_1.$$

The second situation is that $\alpha \in (\frac{1}{3}, 1)$, where the dominant term in (A.5) becomes $\kappa_1\lambda^{2\alpha}\partial_x^2 + \kappa_4\lambda^{2\alpha}\partial_w^2$. This allows us to consider

$$dX = \sqrt{2\kappa_1}dB_1,\ \dot{Y} = 0,\ \dot{Z} = 0,\ dW = \sqrt{2\kappa_4}dB_4.$$

To reach our goal, we define the region

$$\mathcal{R}_2 := \{x^2+y^2+w^2 \le R_2,\ |x||z|^{\frac{1}{3}} \le R_1,\ |z| \ge R_3\}.$$

Similarly, our focus lies in identifying a function ψ_2 that solves

$$(\kappa_1\partial_x^2 + \kappa_4\partial_w^2)\psi_2 = -n_2.$$

Clearly, a particular solution to the above equation is

$$\psi_2 = \frac{n_2}{2\kappa_1}\left(\frac{4R_1^2}{z^{\frac{2}{3}}} - x^2\right), \tag{A.8}$$

implying the estimate

$$|\psi_2| \leq C\frac{n_2R_1^2}{|z|^{\frac{2}{3}}}, \qquad \text{whenever } |x||z|^{\frac{1}{3}} \leq 2R_1. \tag{A.9}$$

Thus, we obtain

$$\begin{aligned}\frac{1}{n_2}\mathcal{M}(\psi_2) &= (ay - ax + rw)\left(\frac{-x}{\kappa_1}\right) + xy\left(\frac{-4R_1^2}{3\kappa_1|z|^{\frac{5}{3}}}\right) - 1 + \frac{20\kappa_3R_1^2}{9\kappa_1|z|^{\frac{8}{3}}} \\ &\leq \frac{CR_1^2R_0}{R_3^{\frac{1}{3}}} - 1,\end{aligned} \tag{A.10}$$

since

$$|(y-x)x| \leq \frac{2R_1R_2^{\frac{1}{2}}}{|z|^{\frac{1}{3}}} \leq \frac{2R_1R_2^{\frac{1}{2}}}{R_3^{\frac{1}{3}}},$$

$$|wx| \leq \frac{R_1|w|}{|z|^{\frac{1}{3}}} \leq \frac{R_1R_2^{\frac{1}{2}}}{R_3^{\frac{1}{3}}},$$

where $C = C(a, c, r, \kappa_1, \kappa_3)$ is independent of R_1, R_2, R_3 and n_2. Hence, we get

$$\mathcal{M}(\tilde{H} + \psi_2) \leq -\frac{1}{2}n_2 \quad \text{on the region } \mathcal{R}_2$$

by choosing large $R_3 \geq 1$ and $n_2 \geq 1 \vee 4\bar{\kappa}$. For the critical situation $\alpha = \frac{1}{3}$, the dominant dynamics is

$$\lambda^{\frac{2}{3}}\partial_x^2 - \lambda^{\frac{2}{3}}xz\partial_y + \lambda^{\frac{2}{3}}\partial_w^2.$$

Whereas, ψ_2 is also valid because it is independent of y.

Besides, it is easy to check that

$$\limsup_{|X|\to\infty}\frac{\psi_i(X)}{\tilde{H}(X)} = 0$$

for $i = 1, 2$, so that the inequality (A.2) is sure, where

$$\mathcal{K} := \{x^2 + y^2 + w^2 \leq R_0,\ |z| \leq R_3\}.$$

Based on the above discussion, we arrive at a preliminary candidate

$$V := \tilde{H} + \mathbb{1}_{\mathcal{R}_1}\psi_1 + \mathbb{1}_{\mathcal{R}_2}\psi_2,$$

where $\mathbb{1}$ stands for the indicator function. Therefore, all that remains is to smooth this Lyapunov function. For this purpose, we introduce

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1\ , \\ 0 & \text{if } |x| \geq 2\ , \end{cases} \quad \text{and} \quad \tilde{\chi}(x) = \begin{cases} 1 & \text{if } |x| \geq 1\ , \\ 0 & \text{if } |x| \leq 1/2\ , \end{cases}$$

by which, we define

$$\theta_1 := \chi\left(\frac{x^2+y^2+w^2}{R_0}\right)\tilde{\chi}\left(\frac{|x||z|^{\frac{1}{3}}}{R_1}\right)\tilde{\chi}\left(\frac{|z|}{R_3}\right), \tag{A.11}$$

$$\theta_2 := \chi\left(\frac{x^2+y^2+w^2}{R_2}\right)\chi\left(\frac{|x||z|^{\frac{1}{3}}}{R_1}\right)\tilde{\chi}\left(\frac{|z|}{R_3}\right). \tag{A.12}$$

Therefore, we obtain

$$\begin{aligned} V :=& \tilde{H} + \theta_1\psi_1 + \theta_2\psi_2 \\ =& \frac{1}{2}\left[\frac{1}{r}x^2 + y^2 + z^2 - 2\left(c + \frac{a}{r}\right)z + w^2 + \kappa_0\right] + \frac{n_1 y\theta_1(x,y,z,w)}{xz} \\ &+ \frac{n_2\theta_2(x,y,z,w)}{2\kappa_1}\left(\frac{R_1^2}{|z|^{\frac{2}{3}}} - x^2\right) \end{aligned} \tag{A.13}$$

with specific parameters $R_0, R_1, R_2, R_3 \geq 1$ and $\kappa_0, n_1, n_2 > 0$.

B. Construction of cut-off function F_N

For each $N \geq 1$, we define a C^2 function $F_N : \mathbb{R} \to \mathbb{R}$ as

$$F_N(x) = \begin{cases} x & x \in [0, N), \\ h(x-N) + N & x \in [N, N+2), \\ N+1 & x \geq N+2, \end{cases} \tag{B.1}$$

where $h : [0,2] \to \mathbb{R}$ is a non-decreasing C^2 function such that

$$h(0) = h''(0) = h'(2) = h''(2) = 0, h'(0) = 1, h(2) = 1, \max_{[0,2]} |h'| \leq 1.$$

Denote $c^* = \max_{[0,2]} |h''|$. Clearly,

$$F_N' \geq 0, \ \max_{[0,2]} |F_N'| \leq 1, \text{ and } \max_{[0,2]} |F_N''| = c^*.$$

Next, we claim $F'_{N_{j+1}} \geq F'_{N_j}$ for any j. In fact, for $|\xi| \leq N_j$ one has

$$1 = F'_{N_j}(\xi) = F'_{N_{j+1}}(\xi),$$

and for $|\xi| \geq N_{j+2}$ one has

$$F'_{N_j}(\xi) = 0 \leq F'_{N_{j+1}}(\xi).$$

Finally, since $N_{j+1} \geq N_j + 2$, for any $|\xi| \in [N_j, N_j + 2]$, we have

$$F'_{N_j}(\xi) \leq 1 = F'_{N_{j+1}}(\xi).$$

Thus, (F'_{N_j}) is a non-decreasing sequence of non-negative functions that converge pointwise to 1 on $\mathbb{R}$. In order to facilitate the readers, we carry out the following simulation. To this aim, we take

$$h(x) := \frac{1}{16}x^4 - \frac{1}{4}x^3 + x,$$

for any $x \in [0,2]$. Obviously, the required conditions for $h(x)$ are satisfied, and the phase diagram is shown in Figure 1.

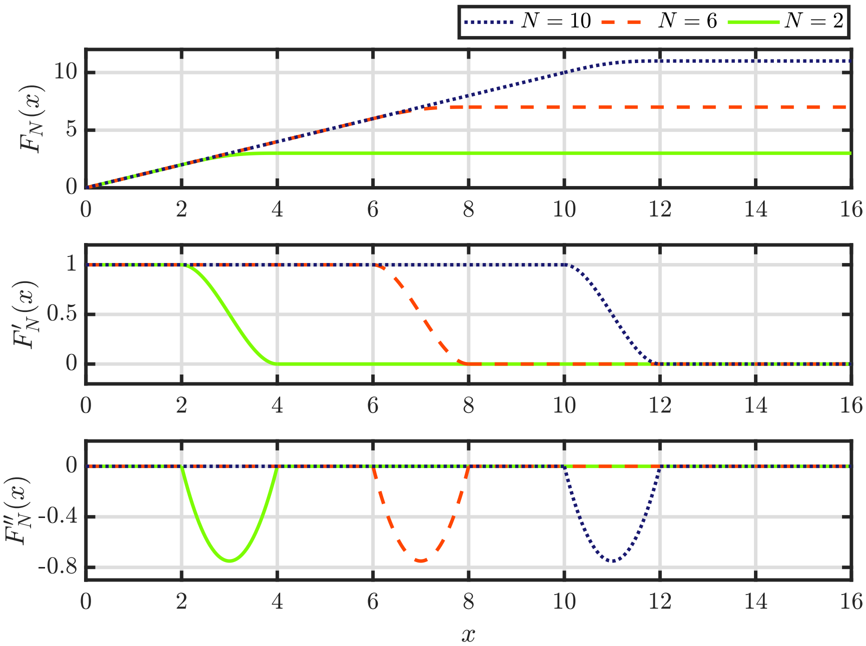

Figure 1. Phase diagram of F_N and its derivatives

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