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Dynamic Analysis of an Impulsive Chemostat Model with Microbial Competition and Nonlinear Perturbation*

Yue Dong[1] and Xinzhu Meng[1,†]

Abstract In this paper, we propose an impulsive chemostat model with microbial competition and nonlinear perturbation. First, thresholds for the extinction of both microoganisms are given. Second, we investigate the persistence in mean and boundedness of the chemostat system by constructing Lyapunov function. Moreover, we obtain the sufficient condition for the existence of an ergodic stationary distribution of the system. At last, numerical simulations are presented, and the results show that the competition between two species tends to make one species disappear from their common habitat, especially when the competition is concentrated in a single resource.

Keywords Impulsive chemostat model, microbial competition, ergodic stationary distribution, extinction, persistence in mean

MSC(2010) 37A50, 37H05, 37N25, 60G10.

1. Introduction

The chemostat is a device used for the continuous cultivation of microorganisms, as shown in Figure 1. It primarily consists of three parts, namely, the feeding device, the cultivation device and the collecting device. The three devices are connected by catheters, and nutrients flow into the culture device at a certain rate for the cultivation of microorganisms in the device. Then, the mixture in the culture device flows into the collection device at the same rate to complete the collection of the culture. As a complex system, it is difficult for scholars to study the natural ecosystem. However, if some minor factors are ignored, the complex system can be simplified to make the influence of research factors more prominent and facilitate the study. The chemostat can only control the velocity and concentration in order to achieve the purpose of simplifying the model.

†The corresponding author.
Email address: dydybest@163.com (Y. Dong), mxz721106@sdust.edu.cn (X. Meng)
[1]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China
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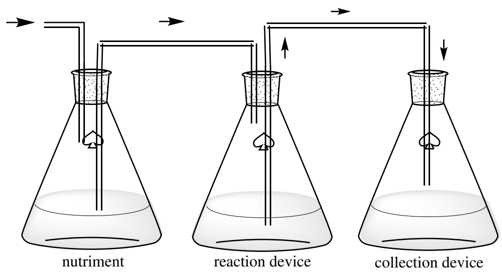

Figure 1. The principle diagram of the chemostat

Many scholars have done a lot of work in chemostat dynamics modeling and analysis, and abundant research results are obtained [3,4,11,14,15,17,19,21]. With the advancement of the research, the models studied by scholars become more complex, and the research results become more realistic. In [13], a deterministic impulsive model which incorporates both toxin input and saturation function is proposed. A stochastic differential equation with nonlinear function was established in [8]. Considering the influence of uncertain factors in the ecological environment [1, 7, 10, 20, 22], Lv, Meng and Wang [12] added random disturbances to the above model and proposed a corresponding stochastic chemostat model. There are complex relationships among microorganisms such as symbiosis, antagonism, predation, competition and parasitism. Among them, microbial competition is to compete for limited space or nutrients for growth, which drives development and evolution [2, 5, 16, 18]. For example, micellar bacteria compete with filamentous bacteria, resulting in the inhibition of growth on both sides. Therefore, based on the existing studies, and considering the microorganisms exposed to toxic environments, the stochastic impulsive chemostat model of competition between two microorganisms is considered as follows.

$$\begin{cases} \left.\begin{aligned} \mathrm{d}S(t) =& \left[D(S_0 - S(t)) - \frac{\mu_1 S(t) x_1(t)}{\delta_1(a_1 + x_1(t))} - \frac{\mu_2 S(t) x_2(t)}{\delta_2(a_2 + x_2(t))}\right]\mathrm{d}t \\ &+ S(t)\left(\sigma_{11} + \sigma_{12} S(t)\right)\mathrm{d}B_1(t), \\ \mathrm{d}x_1(t) =& \left[\frac{\mu_1 S(t) x_1(t)}{a_1 + x_1(t)} - Dx_1(t) - r_1 C_0(t) x_1(t)\right]\mathrm{d}t \\ &+ x_1(t)\left(\sigma_{21} + \sigma_{22} x_1(t)\right)\mathrm{d}B_2(t), \\ \mathrm{d}x_2(t) =& \left[\frac{\mu_2 S(t) x_2(t)}{a_2 + x_2(t)} - Dx_2(t) - r_2 C_0(t) x_2(t)\right]\mathrm{d}t \\ &+ x_2(t)\left(\sigma_{31} + \sigma_{32} x_2(t)\right)\mathrm{d}B_3(t), \\ \mathrm{d}C_0(t) =& \left[kC_e(t) - gC_0(t) - mC_0(t)\right]\mathrm{d}t, \\ \mathrm{d}C_e(t) =& -hC_e(t)\mathrm{d}t, \end{aligned}\right\} \quad t \neq n\tau, \\ \Delta S(t) = 0, \Delta x_1(t) = 0, \Delta x_2(t) = 0, \Delta C_0(t) = 0, \Delta C_e(t) = u, t = n\tau, n \in Z^+, \end{cases} \tag{1.1}$$

where $B_i(t)$ are independent standard Brownian motions defined on the complete probability space Ω with $B_i(0) = 0$ $(i = 1, 2, 3)$, and $\sigma_{i,j} > 0$ $(i, j = 1, 2, 3)$ represent for the intensities of the white noises on the $S(t)$, $x_1(t)$ and $x_2(t)$ respectively. Besides, the other parameters are defined in Table 1.

This paper is organized as follows. In Section 2, we provide the relevant prelim-

Table 1. Biological significance of each parameter

$S(t)$	the concentration of nutrient which is not consumed in the chemostat at time t
$x_i(t), i=1,2$	the biomass of microorganisms cultured in the chemostat at time t
D	the output rate
S_0	the initial concentration of nutrients
$\mu_i, i=1,2$	the maximum growth rates
$\delta_i, i=1,2$	the growth parameters
$a_i, i=1,2$	the half-saturation constants
$C_0(t)$	the level of toxins in the orgnism at time t
$C_e(t)$	the level of toxins in the environment at time t
$r_i, i=1,2$	the rates of decrease in the intrinsic growth rate
k	the environment toxicant uptake rate per unit mass organism
g	the intake rate of toxins by microoganisms
m	the purifying rate of toxicant
h	the loss rate of toxicant from the environment itself by volatilization
u	the amount of pulsed input concentration of the toxicant at each τ

inary knowledge and lemmas used in the proof process of this paper. We prove the existence and uniqueness of positive global solutions and the boundedness of solutions of system (1.1). Section 3 presents relevant results for the proposed system. First, sufficient conditions for microbial extinction are proved. Then, the conditions of persistence in mean of the system are given. Next, the existence of ergodic stationary distribution is established. Finally, numerical simulations are carried out and the conclusions are given in Section 4.

2. Preliminary results

Some definitions and lemmas are given in this section, which are used to prove the main results in the following section. In addition, in order to prove the process more concisely, we define some notations. We assume that $S(t)$, $x_1(t)$, $x_2(t)$ and $C_0(t)$ are continuous at $t = n\tau$, and at the same time, $C_e(t)$ is left continuous at $t = n\tau$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). If $f(t)$ is an intergrable function on $[0, +\infty)$, let $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)\mathrm{d}\theta$. In addition, define f^u as the maximum value of the function f, and define f^l as the minimum value of the function f. First, we consider the subsystem of the system

(1.1) as follows.

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}C_0(t) = [kC_e(t) - gC_0(t) - mC_0(t)]\,\mathrm{d}t, \\ &\mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{aligned}\right\} \quad t \neq n\tau, \\ \left.\Delta C_0(t) = 0, \Delta C_e(t) = u,\right\} \quad t = n\tau, n \in Z^+. \end{cases} \tag{2.1}$$

Lemma 2.1 ([13]). *System* (2.1) *has a unique positive periodic solution* $(C_0^*(t), C_e^*(t))$*, and for each solution* $(C_0(t), C_e(t))$ *of* (2.1), $C_0(t) \to C_0^*(t)$, $C_e(t) \to C_e^*(t)$ *as* $t \to +\infty$*, where*

$$\begin{cases} C_0^*(t) = C_0^*(0)e^{-(g+m)(t-n\tau)} + \frac{ku\left(e^{-(g+m)(t-n\tau)}\right) - e^{-h(t-n\tau)}}{(h-g-m)(1-e^{h\tau})}, \\ C_e^*(t) = \frac{ue^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ C_0^*(0) = \frac{ku\left(e^{-(g+m)\tau} - e^{-h\tau}\right)}{(h-g-m)\left(1-e^{-(g+m)\tau}\right)(1-e^{-h\tau})}, \\ C_e^*(0) = \frac{u}{1-e^{-h\tau}}, \end{cases} \tag{2.2}$$

for $t \in (n\tau, (n+1)\tau]$*. Furthermore, we have*

$$\lim_{t\to+\infty} \langle C_0(t)\rangle = \frac{ku}{h(g+m)\tau} \triangleq \widetilde{C_0}.$$

Lemma 2.2 (Stationary distribution [6]). *Let* $X(t)$ *be a homogenous Markov process in* E_l *(* E_l *denotes* l*-dimensional Euclidean space), and be described by the following stochastic differential equation*

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{k} g_r(X)\mathrm{d}B_r(t).$$

The diffusion matrix is defined as

$$A(x) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

If there is a bounded open set $U \subset R^n$ *with regular,*

(i) *for any* $x \in U, \varepsilon \in R^n$*, there is a constant* $\varsigma > 0$*, which satisfies that*

$$\textstyle\sum_{i,j=1}^{n} a_{ij}(x)\varepsilon_i\varepsilon_j \geq \varsigma\,|\varepsilon|^2.$$

(ii) *for any* $x \in R^n \setminus U$*, there is a* $C^{1,2}$ *function* V *such that* $LV < 0$*.*

Then, the Markov process $X(t)$ *exists a stationary distribution* $\psi(\cdot)$*.*

Next, we prove the existence and uniqueness of global positive solutions for system (1.1).

Lemma 2.3. *Assigned with any initial value, system* (1.1) *provides a unique positive solution* $(S(t), x_1(t), x_2(t))$ *for* $t \geq 0$*, and the solution will remain in* $\mathbb{R}_+^3$ *with probability one.*

Proof. As the coefficients of system (1.1) satisfy the local Lipschitz conditions, system (1.1) provides a unique local solution $(S(t), x_1(t), x_2(t))$ on $t \in [0, \tau_e)$, which requires to prove $\tau_e = \infty$ a.s. The method of proof resembles the other methods of that in the literature, and we only present the key component here. Define a C^2-function I: $\mathbb{R}^3_+ \to \mathbb{R}_+$ as follows.

$$I(S(t), x_1(t), x_2(t)) = (S\ 1 - \ln S) + (x_1 - 1 - \ln x_1) + (x_2 - 1 - \ln x_2) + \frac{x_1^p}{p} + \frac{x_2^p}{p}, \tag{2.3}$$

where $p \in (0, 1)$. Clearly, $I > 0$.

Applying Itô's formula yields

$$\begin{aligned}
LI =& \left(1 - \frac{1}{S}\right)\left[DS_0 - DS - \frac{\mu_1 S x_1}{\delta_1\,(a_1 + x_1)} - \frac{\mu_2 S x_2}{\delta_2 (a_2 + x_2)}\right] + \frac{(\sigma_{11} + \sigma_{12} S)^2}{2} \\
&+ \left(1 - \frac{1}{x_1} + x_1^{p-1}\right)\left[\frac{\mu_1 S x_1}{a_1 + x_1} - Dx_1 - r_1 C_0(t) x_1\right] + \frac{(\sigma_{21} + \sigma_{22} x_1)^2}{2} \\
&+ \left(1 - \frac{1}{x_2} + x_2^{p-1}\right)\left[\frac{\mu_2 S x_2}{a_2 + x_2} - Dx_2 - r_2 C_0(t) x_2\right] + \frac{(\sigma_{31} + \sigma_{32} x_2)^2}{2} \\
&+ \frac{(p-1) x_1^p}{2}(\sigma_{21} + \sigma_{22} x_1)^2 + \frac{(p-1) x_2^p}{2}(\sigma_{31} + \sigma_{32} x_2)^2 \\
=& DS_0 - DS - \frac{\mu_1 S x_1}{\delta_1\,(a_1 + x_1)} - \frac{\mu_2 S x_2}{\delta_2 (a_2 + x_2)} - \frac{DS_0}{S} + D + \frac{\mu_1 x_1}{\delta_1\,(a_1 + x_1)} \\
&+ \frac{\mu_2 x_2}{\delta_2 (a_2 + x_2)} + \frac{\mu_1 S x_1}{a_1 + x_1} - Dx_1 - r_1 C_0(t) x_1 - \frac{\mu_1 S}{a_1 + x_1} + D + r_1 C_0(t) \\
&+ \frac{\mu_2 S x_2}{a_2 + x_2} - Dx_2 - r_2 C_0(t) x_2 - \frac{\mu_2 S}{a_2 + x_2} + D + r_2 C_0(t) + \frac{\mu_1 S x_1^p}{a_1 + x_1} \\
&- Dx_1^p - r_1 C_0(t) x_1^p + \frac{\mu_2 S x_2^p}{a_2 + x_2} - Dx_2^p - r_2 C_0(t) x_2^p + \frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12} S \\
&+ \frac{\sigma_{12}^2 S^2}{2} + \frac{\sigma_{21}^2}{2} + \sigma_{21}\sigma_{22} x_1 + \frac{\sigma_{22}^2 x_1^2}{2} + \frac{\sigma_{31}^2}{2} + \sigma_{31}\sigma_{32} x_2 + \frac{\sigma_{32}^2 x_2^2}{2} \\
&- \frac{(1-p)\sigma_{21}^2 x_1^p}{2} - (1-p)\sigma_{21}\sigma_{22} x_1^{p+1} - \frac{(1-p)\sigma_{22}^2 x_1^{p+2}}{2} \\
&- \frac{(1-p)\sigma_{31}^2 x_2^p}{2} - (1-p)\sigma_{31}\sigma_{32} x_2^{p+1} - \frac{(1-p)\sigma_{32}^2 x_2^{p+2}}{2} \\
\leq& DS_0 + \left(1 - \frac{1}{\delta_1}\right)\frac{\mu_1 S x_1}{a_1 + x_1} + \left(1 - \frac{1}{\delta_2}\right)\frac{\mu_2 S x_2}{a_2 + x_2} + 3D + \frac{\mu_1}{\delta_1} + \frac{\mu_2}{\delta_2} \\
&+ (r_1 + r_2) C_0^u + \mu_1 S_0 x_1^{p-1} + \mu_2 S_0 x_2^{p-1} + \frac{\sigma_{11}^2}{2} + \frac{\sigma_{21}^2}{2} + \frac{\sigma_{31}^2}{2} \\
&+ \sigma_{11}\sigma_{12} S_0 + \frac{\sigma_{12}^2 S_0^2}{2} + \sigma_{21}\sigma_{22} x_1 + \frac{\sigma_{22}^2 x_1^2}{2} + \sigma_{31}\sigma_{32} x_2 + \frac{\sigma_{32}^2 x_2^2}{2} \\
&- \frac{(1-p)\sigma_{21}^2 x_1^p}{2} - (1-p)\sigma_{21}\sigma_{22} x_1^{p+1} - \frac{(1-p)\sigma_{22}^2 x_1^{p+2}}{2} \\
&- \frac{(1-p)\sigma_{31}^2 x_2^p}{2} - (1-p)\sigma_{31}\sigma_{32} x_2^{p+1} - \frac{(1-p)\sigma_{32}^2 x_2^{p+2}}{2} \\
\leq& DS_0 + 3D + \frac{\mu_1}{\delta_1} + \frac{\mu_2}{\delta_2} + (r_1 + r_2) C_0^u + \frac{\sigma_{11}^2}{2} + \frac{\sigma_{21}^2}{2} + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{12}^2 S_0^2}{2}
\end{aligned}$$

$$\begin{aligned}
&+\sigma_{11}\sigma_{12}S_0+\mu_1S_0x_1^{p-1}+\mu_2S_0x_2^{p-1}+\sigma_{21}\sigma_{22}x_1+\frac{\sigma_{22}^2x_1^2}{2}+\sigma_{31}\sigma_{32}x_2\\
&+\frac{\sigma_{32}^2x_2^2}{2}-\frac{(1-p)\sigma_{21}^2x_1^p}{2}-(1-p)\sigma_{21}\sigma_{22}x_1^{p+1}-\frac{(1-p)\sigma_{22}^2x_1^{p+2}}{2}\\
&-\frac{(1-p)\sigma_{31}^2x_2^p}{2}-(1-p)\sigma_{31}\sigma_{32}x_2^{p+1}-\frac{(1-p)\sigma_{32}^2x_2^{p+2}}{2}\\
\le{}& DS_0+3D+\frac{\mu_1}{\delta_1}+\frac{\mu_2}{\delta_2}+(r_1+r_2)C_0^u+\frac{\sigma_{11}^2}{2}+\frac{\sigma_{21}^2}{2}+\frac{\sigma_{31}^2}{2}+\frac{\sigma_{12}^2S_0^2}{2}\\
&+\sigma_{11}\sigma_{12}S_0+b,
\end{aligned}$$

where

$$\begin{aligned}
b=\sup_{(x_1,x_2)\in\mathbb{R}_+^2}&\left\{\mu_1S_0x_1^{p-1}+\mu_2S_0x_2^{p-1}+\sigma_{21}\sigma_{22}x_1+\frac{\sigma_{22}^2x_1^2}{2}+\sigma_{31}\sigma_{32}x_2\right.\\
&+\frac{\sigma_{32}^2x_2^2}{2}-\frac{(1-p)\sigma_{21}^2x_1^p}{2}-(1-p)\sigma_{21}\sigma_{22}x_1^{p+1}-\frac{(1-p)\sigma_{22}^2x_1^{p+2}}{2}\\
&\left.-\frac{(1-p)\sigma_{31}^2x_2^p}{2}-(1-p)\sigma_{31}\sigma_{32}x_2^{p+1}-\frac{(1-p)\sigma_{32}^2x_2^{p+2}}{2}\right\},
\end{aligned}$$

$$\begin{aligned}
LI\le{}& DS_0+3D+\frac{\mu_1}{\delta_1}+\frac{\mu_2}{\delta_2}+(r_1+r_2)C_0^u+\frac{\sigma_{11}^2}{2}+\frac{\sigma_{21}^2}{2}+\frac{\sigma_{31}^2}{2}+\frac{\sigma_{12}^2S_0^2}{2}\\
&+\sigma_{11}\sigma_{12}S_0+b:=\mathbb{K}.
\end{aligned}\tag{2.4}$$

Evidently, $\mathbb{K}$ is a positive constant. The proof is completed. □

Boundedness is an important property of biomathematics, which is of profound significance for the study of biological systems. Next, we prove the boundedness of system (1.1).

Lemma 2.4. *Assigned with any initial value $(S(0),x_1(0),x_2(0))\in\mathbb{R}_+^3$, the solution $(S(t),x_1(t),x_2(t))$ of the stochastic system* (1.1) *is bounded, and it satisfies*

$$\limsup_{t\to\infty}\left[S(t)+\frac{1}{\delta_1}x_1(t)+\frac{1}{\delta_2}x_2(t)\right]<\infty,\quad a.s.\tag{2.5}$$

Proof. Denote

$$Y(t)=S(t)+\frac{1}{\delta_1}x_1(t)+\frac{1}{\delta_2}x_2(t).$$

Then, from system (1.1), we obtain

$$\begin{aligned}
\mathrm{d}Y(t)={}&\left[DS_0-DS-\frac{\mu_1Sx_1}{\delta_1(a_1+x_1)}-\frac{\mu_2Sx_2}{\delta_2(a_2+x_2)}+\frac{\mu_1Sx_1}{\delta_1(a_1+x_1)}+\frac{\mu_2Sx_2}{\delta_2(a_2+x_2)}\right.\\
&\left.-\frac{D}{\delta_1}x_1-\frac{D}{\delta_2}x_2-\frac{r_1C_0(t)}{\delta_1}x_1-\frac{r_2C_0(t)}{\delta_2}x_2\right]\mathrm{d}t+S(\sigma_{11}+\sigma_{12}S)\mathrm{d}B_1(t)\\
&+\frac{1}{\delta_1}x_1(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t)+\frac{1}{\delta_2}x_2(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(t)\\
={}&\left[DS_0-DY(t)-C_0(t)\left(\frac{r_1}{\delta_1}x_1+\frac{r_2}{\delta_2}x_2\right)\right]\mathrm{d}t+S(\sigma_{11}+\sigma_{12}S)\mathrm{d}B_1(t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\delta_1}x_1(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t)+\frac{1}{\delta_2}x_2(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(t)\\
\leqslant{}& [DS_0-DY(t)]\,\mathrm{d}t+S(\sigma_{11}+\sigma_{12}S)\mathrm{d}B_1(t)+\frac{1}{\delta_1}x_1(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t)\\
&+\frac{1}{\delta_2}x_2(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(t).
\end{aligned}
$$

Define the following equation

$$
\begin{cases}
\mathrm{d}\widetilde{Y}(t)=\left[DS_0-D\widetilde{Y}(t)\right]\mathrm{d}t+S(\sigma_{11}+\sigma_{12}S)\mathrm{d}B_1(t)+\dfrac{1}{\delta_1}x_1(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t)\\
\qquad\qquad +\dfrac{1}{\delta_2}x_2(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(t),\\
\widetilde{Y}(0)=Y(0),
\end{cases}
\tag{2.6}
$$

where $\widetilde{Y}(t)$ is the solution of (2.6), and it has the following form

$$
\widetilde{Y}=S_0+\left[\widetilde{Y}(0)-S_0\right]e^{-Dt}+M(t),
$$

where $M(t)=\int_0^t e^{-D(t-s)}S(s)(\sigma_{11}+\sigma_{12}S(s))\mathrm{d}B_1(s)+\frac{1}{\delta_1}\int_0^t e^{-D(t-s)}x_1(s)(\sigma_{21}+\sigma_{22}x_1(s))\mathrm{d}B_2(s)+\frac{1}{\delta_2}\int_0^t e^{-D(t-s)}x_2(s)(\sigma_{31}+\sigma_{32}x_2(s))\mathrm{d}B_3(s)$ is a continuous local martingale with $M(0)=0$, a.s. On the other hand, from the Stochastic Comparison Theorem, we get $Y(t)\le\widetilde{Y}(t)$, a.s.

Define $\widetilde{Y}(t)=\widetilde{Y}(0)+A(t)-U(t)+M(t)$, $A(t)=S_0\left(1-e^{-Dt}\right)$, and $U(t)=\widetilde{Y}(0)\left(1-e^{-Dt}\right)$. Obviously, $A(t)$ and $U(t)$ are continuous adapted increasing processes on $t\ge 0$ satisfying $A(0)=U(0)=0$. According to the nonnegative semi-martingale convergence theorem, we almost definitely get $\lim_{t\to\infty}\widetilde{Y}<\infty$. Thus,

$$
\limsup_{t\to\infty}Y(t)<\infty, a.s.
$$

Obviously, we can get $\lim_{t\to\infty}Y(t)<S_0$. Moreover, $x_1<S_0$ and $x_2<S_0$. The proof is completed. □

3. Main results

3.1. Extinction

This section explores the conditions which leads to the extinction of microorganisms in culture. The extinction of microorganisms means the failure of microbial culture in the chemostat.

Define parameters

$$
R_1=\frac{\mu_1\int_0^\infty x\pi(x)\mathrm{d}x}{a_1\left(D+\frac{\sigma_{21}^2}{2}+r_1\widetilde{C_0}\right)},
$$

$$
R_2=\frac{\mu_2\int_0^\infty x\pi(x)\mathrm{d}x}{a_2\left(D+\frac{\sigma_{31}^2}{2}+r_2\widetilde{C_0}\right)},
$$

where $x \in (0, \infty)$,

$$\pi(x) = \mathbb{Q}x^{-2-\frac{2(2DS_0\sigma_{12}+D\sigma 11)}{\sigma_{11}^3}}(\sigma_{11}+\sigma_{12}x)^{-2+\frac{2(2DS_0\sigma_{12}+D\sigma_{11})}{\sigma_{11}^3}} \times e^{-\frac{2}{\sigma_{11}(\sigma_{11}+\sigma_{12}x)}\left(\frac{DS_0}{x}+\frac{2DS_0\sigma_{12}+D\sigma_{11}}{\sigma_{11}}\right)},$$

$\mathbb{Q}$ is a constant satisfying that $\int_0^\infty x\pi(x)\mathrm{d}x = 1$.

Theorem 3.1. *Assume $R_1 < 1$. Then microbial x_1 population can be extinct. That is, $\lim_{t\to\infty} x_1(t) = 0$. Assume $R_2 < 1$, then microbial x_2 population can be extinct. That is, $\lim_{t\to\infty} x_2(t) = 0$.*

Proof. First, a stochastic differential equation is constructed as

$$\mathrm{d}X(t) = [DS_0 - DX]\,\mathrm{d}t + X(t)(\sigma_{11} + \sigma_{12}X(t))\mathrm{d}B_1(t), \tag{3.1}$$

where $X(0) = S(0) > 0$.

Applying the Stochastic Comparison Principle, $S(t) \le X(t)$ is obtained. Let

$$f(x) = DS_0 - Dx, \sigma(x) = x(t)(\sigma_{11} + \sigma_{12}x(t)), x \in (0, +\infty).$$

Compute the following indefinite integral

$$\begin{aligned}\int \frac{f(\tau)}{\sigma^2(\tau)}\mathrm{d}\tau &= \int \frac{DS_0 - D\tau}{(\sigma_{11}+\sigma_{12}\tau)^2\tau^2}\mathrm{d}\tau \\ &= \frac{2DS_0\sigma_{12} + D\sigma_{11}}{\sigma_{11}^3}\ln\frac{\sigma_{11}+\sigma_{12}\tau}{\tau} - \frac{DS_0}{\sigma_{11}\tau(\sigma_{11}+\sigma_{12}\tau)} \\ &\quad - \frac{2DS_0\sigma_{12}+D\sigma_{11}}{\sigma_{11}^2(\sigma_{11}+\sigma_{12}\tau)} + C.\end{aligned}$$

Then,

$$e^{\int \frac{f(\tau)}{\sigma^2(\tau)}d\tau} = e^C \cdot \left(\frac{\sigma_{11}+\sigma_{12}\tau}{\tau}\right)^{\frac{2DS_0\sigma_{12}+D\sigma_{11}}{\sigma_{11}^3}} \cdot e^{-\frac{1}{\sigma_{11}(\sigma_{11}+\sigma_{12}\tau)}\left(\frac{DS_0}{\tau}+\frac{2DS_0\sigma_{12}+D\sigma_{11}}{\sigma_{11}}\right)},$$

and we can get

$$\begin{aligned}\int_0^\infty \frac{1}{\sigma^2(x)}e^{\int_1^x \frac{2f(\tau)}{\sigma^2(\tau)}\mathrm{d}\tau}\mathrm{d}x &= \int_0^\infty x^{-2}(\sigma_{11}+\sigma_{12}x)^{-2}\left(\frac{\sigma_{11}+\sigma_{12}x}{x}\right)^{\frac{2(2DS_0\sigma_{12}+D\sigma_{11})}{\sigma_{11}^3}} \\ &\quad \times e^{-\frac{2}{\sigma_{11}(\sigma_{11}+\sigma_{12}x)}\left(\frac{DS_0}{x}+\frac{2DS_0\sigma_{12}+D\sigma_{11}}{\sigma_{11}}\right)}\mathrm{d}x < \infty.\end{aligned}$$

Therefore, equation (3.1) is ergodic. In addition, the ergodic theorem can be used to obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\mathrm{d}s = \int_0^\infty x\pi(x)\mathrm{d}x, a.s.$$

Applying Itô's formula, we get

$$\begin{aligned}\mathrm{d}\ln x_1(t) = &\left[\frac{\mu_1 S}{a_1+x_1} - D - r_1C_0 - \frac{(\sigma_{21}+\sigma_{22}x_1(t))^2}{2}\right]\mathrm{d}t \\ &+\sigma_{21}\mathrm{d}B_2(t) + \sigma_{22}x_1\mathrm{d}B_2(t)\end{aligned}$$

$$\leq \left[\frac{\mu_1 S}{a_1} - D - r_1 C_0 - \frac{\sigma_{21}^2}{2}\right] \mathrm{d}t + \sigma_{21}\mathrm{d}B_2(t) + \sigma_{22}x_1\mathrm{d}B_2(t). \tag{3.2}$$

Integrating both sides of equation (3.2) from 0 to t and dividing by t, we obtain

$$\begin{aligned}
\frac{\ln x_1(t)}{t} &\leq \frac{\mu_1}{a_1 t}\int_0^t S(s)\mathrm{d}s - D - \frac{\sigma_{21}^2}{2} - \frac{r_1}{t}\int_0^t C_0(s)\mathrm{d}s + \frac{\sigma_{21}B_2(t)}{t} \\
&\quad + \frac{1}{t}\int_0^t \sigma_{22}x_1(s)\mathrm{d}B_2(s) + \frac{\ln x_1(0)}{t} \\
&\leq \frac{\mu_1}{a_1 t}\int_0^t X(s)\mathrm{d}s - D - \frac{\sigma_{21}^2}{2} - \frac{r_1}{t}\int_0^t C_0(s)\mathrm{d}s + \frac{\sigma_{21}B_2(t)}{t} \qquad (3.3) \\
&\quad + \frac{1}{t}\int_0^t \sigma_{22}x_1(s)\mathrm{d}B_2(s) + \frac{\ln x_1(0)}{t}.
\end{aligned}$$

According to the strong law of large numbers [9], if $R_1 < 1$, the result is given as

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\ln x_1(t)}{t} &\leq \frac{\mu_1}{a_1}\int_0^\infty x\pi(x)\mathrm{d}x - \left(D + \frac{\sigma_{21}^2}{2} + r_1\widetilde{C_0}\right) \\
&\leq \left(D + \frac{\sigma_{21}^2}{2} + r_1\widetilde{C_0}\right)(R_1 - 1) < 0, \qquad (3.4)
\end{aligned}$$

which means $\lim_{t\to\infty} x_1(t) = 0$.

Applying Itô's formula gives

$$\begin{aligned}
\mathrm{d}\ln x_2(t) &= \left[\frac{\mu_2 S}{a_2 + x_2} - D - r_2 C_0 - \frac{(\sigma_{31} + \sigma_{32}x_2(t))^2}{2}\right]\mathrm{d}t \\
&\quad + \sigma_{31}\mathrm{d}B_3(t) + \sigma_{32}x_2\mathrm{d}B_3(t) \\
&\leq \left[\frac{\mu_2 S}{a_2} - D - r_2 C_0 - \frac{\sigma_{31}^2}{2}\right]\mathrm{d}t + \sigma_{31}\mathrm{d}B_3(t) + \sigma_{32}x_2\mathrm{d}B_3(t). \qquad (3.5)
\end{aligned}$$

Integrating both sides of this equation from 0 to t and dividing by t, we get

$$\begin{aligned}
\frac{\ln x_2(t)}{t} &\leq \frac{\mu_2}{a_2 t}\int_0^t S(s)\mathrm{d}s - D - \frac{\sigma_{31}^2}{2} - \frac{r_2}{t}\int_0^t C_0(s)\mathrm{d}s + \frac{\sigma_{31}B_3(t)}{t} \\
&\quad + \frac{1}{t}\int_0^t \sigma_{32}x_2(s)\mathrm{d}B_3(s) + \frac{\ln x_2(0)}{t} \\
&\leq \frac{\mu_2}{a_2 t}\int_0^t X(s)\mathrm{d}s - D - \frac{\sigma_{31}^2}{2} - \frac{r_2}{t}\int_0^t C_0(s)\mathrm{d}s + \frac{\sigma_{31}B_3(t)}{t} \qquad (3.6) \\
&\quad + \frac{1}{t}\int_0^t \sigma_{32}x_2(s)\mathrm{d}B_3(s) + \frac{\ln x_2(0)}{t}.
\end{aligned}$$

According to the strong law of large numbers, if $R_2 < 1$, we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\ln x_2(t)}{t} &\leq \frac{\mu_2}{a_2}\int_0^\infty x\pi(x)\mathrm{d}x - \left(D + \frac{\sigma_{31}^2}{2} + r_2\widetilde{C_0}\right) \\
&\leq \left(D + \frac{\sigma_{31}^2}{2} + r_2\widetilde{C_0}\right)(R_2 - 1) < 0, \qquad (3.7)
\end{aligned}$$

which means $\lim_{t\to\infty} x_2(t) = 0$. □

Remark 3.1. The intensities σ_{21} and σ_{31} gradually increase so that the corresponding threshold is less than zero. That is, with the increase of noise intensity, microorganisms gradually become extinct.

3.2. Persistence in mean

Define

$$R_3 = \frac{\mu_1 \int_0^\infty x\pi(x)\mathrm{d}x}{a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right)},$$

$$R_4 = \frac{\mu_2 \int_0^\infty x\pi(x)\mathrm{d}x}{a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)}.$$

Theorem 3.2. *Assigned with any given initial value* $(S(0), x_1(0), x_2(0)) \in \mathbb{R}_+^3$, *we have the following results of system* (1.1).

(i) *If* $R_2 < 1$ *and* $R_3 > 1$, *then the microorganism* x_1 *is persistence in mean, and the microorganism* x_2 *goes extinct. In addtion, we get*

$$\liminf_{t\to\infty} \langle x_1 \rangle \geq \frac{a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right)}{D + r_1 C_0^u + \frac{\mu_1^2 S_0}{D\delta_1 a_1}} (R_3 - 1).$$

(ii) *If* $R_1 < 1$ *and* $R_4 > 1$, *then the microorganism* x_2 *is persistence in mean, and the microorganism* x_1 *goes extinct. In addtion, we get*

$$\liminf_{t\to\infty} \langle x_2 \rangle \geq \frac{a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)}{D + r_2 C_0^u + \frac{\mu_2^2 S_0}{D\delta_2 a_2}} (R_4 - 1).$$

(iii) *If* $R_3 > 1$ *and* $R_4 > 1$, *then the microorganisms* x_1 *and* x_2 *are persistence in mean. In addtion, we get*

$$\begin{aligned}\liminf_{t\to\infty} [\langle x_1 \rangle + \langle x_2 \rangle] \geq{}& \frac{a_1}{Q}\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right)(R_3 - 1) \\ &+ \frac{a_2}{Q}\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)(R_4 - 1).\end{aligned}$$

Proof. Integrating the first equation of system (1.1) and equation (3.1) from 0 to t and dividing by t on both sides yield

$$\begin{aligned}\frac{S(t) - S(0)}{t} &= DS_0 - \frac{D}{t}\int_0^t S(s)\mathrm{d}s - \frac{1}{t}\int_0^t \frac{\mu_1 S(s)x_1(s)}{\delta_1(a_1 + x_1(s))}\mathrm{d}s \\ &\quad - \frac{1}{t}\int_0^t \frac{\mu_2 S(s)x_2(s)}{\delta_2(a_2 + x_2(s))}\mathrm{d}s + \frac{1}{t}\int_0^t S(s)(\sigma_{11} + \sigma_{12}S(s))\mathrm{d}B_1(s) \\ &\geq DS_0 - \frac{D}{t}\int_0^t S(s)\mathrm{d}s - \frac{1}{t}\int_0^t \frac{\mu_1 S_0 x_1(s)}{\delta_1 a_1}\mathrm{d}s - \frac{1}{t}\int_0^t \frac{\mu_2 S_0 x_2(s)}{\delta_2 a_2}\mathrm{d}s \\ &\quad + \frac{1}{t}\int_0^t S(s)(\sigma_{11} + \sigma_{12}S(s))\mathrm{d}B_1(s). \qquad (3.8)\end{aligned}$$

$$\frac{X(t)-X(0)}{t} = DS_0 - \frac{D}{t}\int_0^t X(s)\mathrm{d}s + \frac{1}{t}\int_0^t X(s)(\sigma_{11}+\sigma_{12}X(s))\mathrm{d}B_1(s).$$

By calculation, we obtain

$$\begin{aligned}0 \geq \frac{S(t)-X(t)}{t} &\geq -\frac{D}{t}\int_0^t (S(s)-X(s))\,\mathrm{d}s - \frac{\mu_1 S_0}{\delta_1 a_1}\langle x_1\rangle - \frac{\mu_2 S_0}{\delta_2 a_2}\langle x_2\rangle \\ &+\frac{1}{t}\int_0^t \left[\sigma_{11}(S(s)-X(s))+\sigma_{12}(S^2(s)-X^2(s))\right]\mathrm{d}B_1(s). \end{aligned} \tag{3.9}$$

Then, one can get

$$\begin{aligned}\frac{1}{t}\int_0^t (S(s)-X(s))\,\mathrm{d}s \geq -\frac{\mu_1 S_0}{D\delta_1 a_1}\langle x_1\rangle - \frac{\mu_2 S_0}{D\delta_2 a_2}\langle x_2\rangle \\ +\frac{1}{Dt}\int_0^t \left[\sigma_{11}(S-X)+\sigma_{12}(S^2-X^2)\right]\mathrm{d}B_1(s).\end{aligned}$$

Case (i). If $R_2 < 1$, the microoganism x_2 goes extinct, that is, for ε small enough and all t large enough, $0 < x_2 < \varepsilon$ holds. Combining Itô's formula gets

$$\begin{aligned}\mathrm{d}\left(a_1 \ln x_1 + x_1\right) &= \left[\mu_1 S - D(a_1+x_1) - r_1 C_0(t)(a_1+x_1) - \frac{a_1(\sigma_{21}+\sigma_{22}x_1)^2}{2}\right]\mathrm{d}t \\ &\quad +(a_1+x_1)(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t) \\ &\geq \left[\mu_1 S - a_1(D+r_1C_0(t)) - (D+r_1C_0^u)x_1 - a_1\sigma_{21}^2 - a_1\sigma_{22}^2S_0^2\right]\mathrm{d}t \\ &\quad +(a_1+x_1)(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t) \\ &\geq \left[\mu_1 X - a_1(D+r_1C_0(t)) - (D+r_1C_0^u)x_1 + \mu_1(S-X) - a_1\sigma_{21}^2 \right.\\ &\quad \left. -a_1\sigma_{22}^2S_0^2\right]\mathrm{d}t + (a_1+x_1)(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(t), \end{aligned} \tag{3.10}$$

from which we can obtain

$$\begin{aligned}&\frac{a_1\left[\ln x_1(t) - \ln x_1(0)\right]}{t} + \frac{x_1(t)-x_1(0)}{t} \\ \geq& \frac{1}{t}\int_0^t \mu_1 X(s)\mathrm{d}s - a_1\left(D + r_1\langle C_0(t)\rangle + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right) - (D+r_1C_0^u)\langle x_1\rangle \\ &-\frac{\mu_1^2 S_0}{D\delta_1 a_1}\langle x_1\rangle + \frac{\mu_1}{Dt}\int_0^t \left[\sigma_{11}(S(s)-X(s))+\sigma_{12}(S^2(s)-X^2(s))\right]\mathrm{d}B_1(s) \\ &-\frac{\mu_1\mu_2 S_0\varepsilon}{D\delta_2 a_2} + \frac{1}{t}\int_0^t (a_1+x_1(s))(\sigma_{21}+\sigma_{22}x_1(s))\mathrm{d}B_2(s).\end{aligned}$$

Taking the limit of the above formula, we can get

$$\begin{aligned}\liminf_{t\to\infty}\langle x_1\rangle &\geq \frac{1}{D+r_1C_0^u+\frac{\mu_1^2S_0}{D\delta_1a_1}}\left[\mu_1\int_0^\infty x\pi(x)\mathrm{d}x - a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2S_0^2\right)\right] \\ &\geq \frac{a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2S_0^2\right)}{D+r_1C_0^u+\frac{\mu_1^2S_0}{D\delta_1a_1}}\left[\frac{\mu_1\int_0^\infty x\pi(x)\mathrm{d}x}{a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2S_0^2\right)} - 1\right] \\ &\geq \frac{a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2S_0^2\right)}{D+r_1C_0^u+\frac{\mu_1^2S_0}{D\delta_1a_1}}\left(R_3-1\right). \end{aligned} \tag{3.11}$$

Case (ii). If $R_1 < 1$, the microoganism x_1 goes extinct, that is, for ε small enough and all t large enough, $0 < x_1 < \varepsilon$ holds. Combining Itô's formula gets

$$\begin{aligned}
\mathrm{d}\left(a_2 \ln x_2 + x_2\right) &= \left[\mu_2 S - D(a_2 + x_2) - r_2 C_0(t)(a_2 + x_2) - \frac{a_2(\sigma_{31} + \sigma_{32} x_2)^2}{2}\right] \mathrm{d}t \\
&\quad + (a_2 + x_2)(\sigma_{31} + \sigma_{32} x_2) \mathrm{d}B_3(t) \\
&\geq \left[\mu_2 S - a_2(D + r_2 C_0(t)) - (D + r_2 C_0^u) x_2 - a_2 \sigma_{31}^2 - a_2 \sigma_{32}^2 S_0^2\right] \mathrm{d}t \\
&\quad + (a_2 + x_2)(\sigma_{31} + \sigma_{32} x_2) \mathrm{d}B_3(t) \\
&\geq \left[\mu_2 X - a_2(D + r_2 C_0(t)) - (D + r_2 C_0^u) x_2 + \mu_2(S - X) - a_2 \sigma_{31}^2\right. \\
&\quad \left. - a_2 \sigma_{32}^2 S_0^2\right] \mathrm{d}t + (a_2 + x_2)(\sigma_{31} + \sigma_{32} x_2) \mathrm{d}B_3(t),
\end{aligned} \tag{3.12}$$

from which we can obtain

$$\begin{aligned}
&\frac{a_2\left[\ln x_2(t) - \ln x_2(0)\right]}{t} + \frac{x_2(t) - x_2(0)}{t} \\
&\geq \frac{1}{t}\int_0^t \mu_2 X(s) \mathrm{d}s - a_2\left(D + r_2\langle C_0(t)\rangle + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right) - \left(D + r_2 C_0^u\right)\langle x_2\rangle \\
&\quad - \frac{\mu_2^2 S_0}{D\delta_2 a_2}\langle x_2\rangle + \frac{\mu_2}{Dt}\int_0^t \left[\sigma_{11}(S(s) - X(s)) + \sigma_{12}(S^2(s) - X^2(s))\right] \mathrm{d}B_1(s) \\
&\quad - \frac{\mu_1 \mu_2 S_0 \varepsilon}{D\delta_1 a_1} + \frac{1}{t}\int_0^t (a_2 + x_2(s))(\sigma_{31} + \sigma_{32} x_2(s)) \mathrm{d}B_3(s).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\liminf_{t\to\infty}\langle x_2\rangle &\geq \frac{1}{D + r_2 C_0^u + \frac{\mu_2^2 S_0}{D\delta_2 a_2}}\left[\mu_2\int_0^\infty x\pi(x)\mathrm{d}x - a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)\right] \\
&\geq \frac{a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)}{D + r_2 C_0^u + \frac{\mu_2^2 S_0}{D\delta_2 a_2}}\left[\frac{\mu_2\int_0^\infty x\pi(x)\mathrm{d}x}{a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)} - 1\right] \\
&\geq \frac{a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)}{D + r_2 C_0^u + \frac{\mu_2^2 S_0}{D\delta_2 a_2}}\left(R_4 - 1\right).
\end{aligned} \tag{3.13}$$

Case (iii). Define

$$V(t) = a_1 \ln x_1 + x_1 + a_2 \ln x_2 + x_2.$$

Applying Itô's formula gives

$$\begin{aligned}
\mathrm{d}V(t) &= \left[\mu_1 S - D(a_1 + x_1) - r_1 C_0(t)(a_1 + x_1) - \frac{a_1(\sigma_{21} + \sigma_{22} x_1)^2}{2}\right. \\
&\quad \left. + \mu_2 S - D(a_2 + x_2) - r_2 C_0(t)(a_2 + x_2) - \frac{a_2(\sigma_{31} + \sigma_{32} x_2)^2}{2}\right] \mathrm{d}t \\
&\quad + (a_1 + x_1)(\sigma_{21} + \sigma_{22} x_1) \mathrm{d}B_2(t) + (a_2 + x_2)(\sigma_{31} + \sigma_{32} x_2) \mathrm{d}B_3(t) \\
&\geq \left[(\mu_1 + \mu_2) X - a_1\left(D + r_1 C_0(t) + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right) - (D + r_1 C_0^u) x_1\right. \\
&\quad \left. + (\mu_1 + \mu_2)(S - X) - a_2\left(D + r_2 C_0(t) + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right) - (D + r_2 C_0^u) x_2\right] \mathrm{d}t \\
&\quad + (a_1 + x_1)(\sigma_{21} + \sigma_{22} x_1) \mathrm{d}B_2(t) + (a_2 + x_2)(\sigma_{31} + \sigma_{32} x_2) \mathrm{d}B_3(t).
\end{aligned}$$

Integrating both sides of this equation from 0 to t and dividing by t, we get

$$\begin{aligned}
&\frac{V(t)-V(0)}{t}\\
\geq&\frac{1}{t}\int_0^t(\mu_1+\mu_2)X(s)\mathrm{d}s-a_1\left(D+r_1\left\langle C_0(t)\right\rangle+\sigma_{21}^2+\sigma_{22}^2S_0^2\right)-\left(D+r_1C_0^u\right)\langle x_1\rangle\\
&-a_2\left(D+r_2\left\langle C_0(t)\right\rangle\sigma_{31}^2+\sigma_{32}^2S_0^2\right)-\left(D+r_2C_0^u\right)\langle x_2\rangle+\frac{\mu_1+\mu_2}{t}\int_0^t(S-X)\mathrm{d}s\\
&+\frac{1}{t}\int_0^t(a_1+x_1)(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(s)+\frac{1}{t}\int_0^t(a_2+x_2)(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(s)\\
\geq&\frac{1}{t}\int_0^t(\mu_1+\mu_2)X(s)\mathrm{d}s-a_1\left(D+r_1\left\langle C_0(t)\right\rangle+\sigma_{21}^2+\sigma_{22}^2S_0^2\right)\\
&-a_2\left(D+r_2\left\langle C_0(t)\right\rangle\sigma_{31}^2+\sigma_{32}^2S_0^2\right)-\left(D+r_1C_0^u+\frac{(\mu_1+\mu_2)\mu_1S_0}{D\delta_1a_1}\right)\langle x_1\rangle\\
&-\left(D+r_2C_0^u+\frac{(\mu_1+\mu_2)\mu_2S_0}{D\delta_2a_2}\right)\langle x_2\rangle+\frac{1}{t}\int_0^t(a_1+x_1)(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(s)\\
&+\frac{1}{t}\int_0^t(a_2+x_2)(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(s)\\
&+\frac{\mu_1+\mu_2}{Dt}\int_0^t\left[\sigma_{11}(S-X)+\sigma_{12}(S^2-X^2)\right]\mathrm{d}B_1(s)\\
\geq&\frac{1}{t}\int_0^t(\mu_1+\mu_2)X(s)\mathrm{d}s-a_1\left(D+r_1\left\langle C_0(t)\right\rangle+\sigma_{21}^2+\sigma_{22}^2S_0^2\right)\\
&-a_2\left(D+r_2\left\langle C_0(t)\right\rangle\sigma_{31}^2+\sigma_{32}^2S_0^2\right)+\frac{1}{t}\int_0^t(a_1+x_1)(\sigma_{21}+\sigma_{22}x_1)\mathrm{d}B_2(s)\\
&+\frac{1}{t}\int_0^t(a_2+x_2)(\sigma_{31}+\sigma_{32}x_2)\mathrm{d}B_3(s)-Q\left(\langle x_1\rangle+\langle x_2\rangle\right)\\
&+\frac{\mu_1+\mu_2}{Dt}\int_0^t\left[\sigma_{11}(S-X)+\sigma_{12}(S^2-X^2)\right]\mathrm{d}B_1(s),
\end{aligned}$$

where $Q=\max\{D+r_1C_0^u+\frac{(\mu_1+\mu_2)\mu_1S_0}{D\delta_1a_1},D+r_2C_0^u+\frac{(\mu_1+\mu_2)\mu_2S_0}{D\delta_2a_2}\}$. Besides, by further calculation, we can get

$$\begin{aligned}
&\liminf_{t\to\infty}\left[\langle x_1\rangle+\langle x_2\rangle\right]\\
\geq&\frac{\mu_1}{Q}\int_0^\infty x\pi(x)\mathrm{d}x-\frac{a_1}{Q}\left(D+r_1\left\langle C_0(t)\right\rangle\sigma_{21}^2+\sigma_{22}^2S_0^2\right)+\frac{\mu_2}{Q}\int_0^\infty x\pi(x)\mathrm{d}x\\
&-\frac{a_2}{Q}\left(D+r_2\left\langle C_0(t)\right\rangle+\sigma_{31}^2+\sigma_{32}^2S_0^2\right)\\
\geq&\frac{a_1}{Q}\left(D+r_1\widetilde{C_0}+\sigma_{21}^2+\sigma_{22}^2S_0^2\right)(R_3-1)+\frac{a_2}{Q}\left(D+r_2\widetilde{C_0}+\sigma_{31}^2+\sigma_{32}^2S_0^2\right)(R_4-1).
\end{aligned}$$

This is the end of the proof. □

3.3. Existence of ergodic stationary distribution

This section explores the existence of the ergodic stationary distribution of system (1.1).

Define

$$R_0^l = \frac{D^2 S_0}{\left(D+\sigma_{11}^2+\frac{2\sigma_{21}^2 DS_0}{\sigma_{11}}\right)\left(D+r_1C_0^u+\frac{\sigma_{21}^2}{2}\right)\left(2D+r_2C_0^u+\frac{\sigma_{31}^2}{2}\right)}.$$

Theorem 3.3. *Assume $R_0^l > 1$. Then system* (1.1) *exists a stationary Markov process.*

Proof. In order to prove this theorem, it is necessary to prove that system (1.1) satisfies the conditions in Lemma 2.2. Construct a C^2-function $V : \mathbb{R}_+^3 \to \mathbb{R}_+$:

$$\begin{aligned} V_1(S,x_1,x_2) \;&=\; M\left[-c_1\ln S - c_2\ln x_1 - c_3\ln x_2 + \frac{2c_1(\sigma_{11}+\sigma_{12}S)^\theta}{\theta(1-\theta)\sigma_{11}^\theta}\right] \\ &\quad+\left(\frac{1}{p}S^p+\frac{1}{p}x_1^p+\frac{1}{p}x_2^p\right) \\ &:= MV_2+V_3, \end{aligned}$$

where $\theta, p \in (0,1)$, and M is a sufficiently large constant such that

$$M\left[-3DS_0\left(\sqrt[3]{R_0^l}-1\right)\right]+\max\{A,B,C,E,F,G\} \le -2, \tag{3.14}$$

where A,B,C,E,F,G are defined in equations (3.21), (3.22), (3.23), (3.24), (3.25), (3.26).

Since $V(S,x_1,x_2)$ is continuous, there exists a minimum value $V_{\min}$ in the interior of R_+^3. Define a C^2-function $\bar V : \mathbb{R}_+^3 \to \mathbb{R}_+$

$$\bar V(S,x_1,x_2) = V_1(S,x_1,x_2) - V_{1\min}.$$

Applying Itô's formula gives

$$L\bar V = LV_1 = MLV_2 + LV_3.$$

$$\begin{aligned} LV_2 = &-\frac{c_1DS_0}{S}+c_1D+\frac{c_1\mu_1x_1}{\delta_1(a_1+x_1)}+\frac{c_1\mu_2x_2}{\delta_2(a_2+x_2)}+\frac{c_1(\sigma_{11}+\sigma_{12}S)^2}{2}-\frac{c_2\mu_1S}{a_1+x_1} \\ &+Dc_2+c_2r_1C_0(t)+\frac{c_2(\sigma_{21}+\sigma_{22}x_1)^2}{2}-\frac{c_3\mu_2S}{a_2+x_2}+Dc_3+c_3r_2C_0(t) \\ &+\frac{c_3(\sigma_{31}+\sigma_{32}x_2)^2}{2}+\frac{2c_1\sigma_{12}DS_0}{(1-\theta)\sigma_{11}^\theta}(\sigma_{11}+\sigma_{12}S)^{\theta-1}-\frac{c_1\sigma_{12}^2S^2}{\sigma_{11}^\theta}(\sigma_{11}+\sigma_{12}S)^\theta \\ &-\frac{2c_1\sigma_{12}DS}{(1-\theta)\sigma_{11}^\theta}(\sigma_{11}+\sigma_{12}S)^{\theta-1}-\frac{2c_1\sigma_{12}\mu_1Sx_1}{\delta_1(1-\theta)\sigma_{11}^\theta(a_1+x_1)}(\sigma_{11}+\sigma_{12}S)^{\theta-1} \\ &-\frac{2c_1\sigma_{12}\mu_2Sx_2}{\delta_1(1-\theta)\sigma_{11}^\theta(a_2+x_2)}(\sigma_{11}+\sigma_{12}S)^{\theta-1} \\ \le &-\frac{c_1DS_0}{S}+D(c_1+c_2+c_3)+\frac{c_1\mu_1}{\delta_1a_1}x_1+\frac{c_2\mu_2}{\delta_2a_2}x_2+\frac{c_1(\sigma_{11}+\sigma_{12}S)^2}{2} \\ &+C_0^u(c_2r_1+c_3r_2)+\frac{c_2(\sigma_{21}+\sigma_{22}x_1)^2}{2}+\frac{c_3(\sigma_{31}+\sigma_{32}x_2)^2}{2}+\frac{2c_1\sigma_{21}DS_0}{(1-\theta)\sigma_{11}} \\ &-\frac{c_1\sigma_{12}^2S^2}{\sigma_{11}^\theta}(\sigma_{11}+\sigma_{12}S)^\theta \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{c_1DS_0}{S}+D(c_1+c_2+c_3)+\frac{c_1\mu_1}{\delta_1 a_1}x_1+\frac{c_2\mu_2}{\delta_2 a_2}x_2+C_0^u(c_2r_1+c_3r_2)+c_1\sigma_{11}^2\\
&\quad+\frac{c_2(\sigma_{21}+\sigma_{22}x_1)^2}{2}+\frac{c_3(\sigma_{31}+\sigma_{32}x_2)^2}{2}+\frac{2c_1\sigma_{21}DS_0}{(1-\theta)\sigma_{11}}-\frac{c_1}{2}(\sigma_{11}-\sigma_{12}S)^2\\
&\leq -\frac{c_1DS_0}{S}+D(c_1+c_2+c_3)+C_0^u(c_2r_1+c_3r_2)+\frac{2c_1\sigma_{21}DS_0}{(1-\theta)\sigma_{11}}+\frac{c_2\sigma_{21}^2}{2}\\
&\quad+\frac{c_3\sigma_{31}^2}{2}+c_1\sigma_{11}^2+\frac{c_2\sigma_{22}^2}{2}x_1^2+\frac{c_3\sigma_{32}^2}{2}x_2^2+\left(\frac{c_1\mu_1}{\delta_1 a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1\\
&\quad+\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2\\
&\leq -\frac{c_1DS_0}{S}-c_2S-c_3D+c_2S+c_1\left(D+\sigma_{11}^2+\frac{2\sigma_{21}DS_0}{(1-\theta)\sigma_{11}}\right)\\
&\quad+c_2\left(D+r_1C_0^u+\frac{\sigma_{21}^2}{2}\right)+c_3\left(2D+r_2C_0^u+\frac{\sigma_{31}^2}{2}\right)+\left(\frac{c_1\mu_1}{\delta_1 a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1\\
&\quad+\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{c_2\sigma_{22}^2}{2}x_1^2+\frac{c_3\sigma_{32}^2}{2}x_2^2\\
&\leq -3\sqrt[3]{c_1c_2c_3D^2S_0}+c_1\left(D+\sigma_{11}^2+\frac{2c_1\sigma_{21}DS_0}{(1-\theta)\sigma_{11}}\right)+c_2\left(D+r_1C_0^u+\frac{\sigma_{21}^2}{2}\right)\\
&\quad+c_3\left(2D+r_2C_0^u+\frac{\sigma_{31}^2}{2}\right)+\left(\frac{c_1\mu_1}{\delta_1 a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1+\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2\\
&\quad+c_2S+\frac{c_2\sigma_{22}^2}{2}x_1^2+\frac{c_3\sigma_{32}^2}{2}x_2^2.
\end{aligned}$$

Let

$$c_1=\frac{DS_0}{D+\sigma_{11}^2+\frac{2\sigma_{21}DS_0}{(1-\theta)\sigma_{11}}},c_2=\frac{DS_0}{D+r_1C_0^u+\frac{\sigma_{21}^2}{2}},c_3=\frac{DS_0}{2D+r_2C_0^u+\frac{\sigma_{31}^2}{2}}.$$

We can get

$$\begin{aligned}
LV_2\leq& -3DS_0\left(\sqrt[3]{R_0^l(\theta)}-1\right)+\left(\frac{c_1\mu_1}{\delta_1 a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1+\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2\\
&+\frac{c_2\sigma_{22}^2}{2}x_1^2+\frac{c_3\sigma_{32}^2}{2}x_2^2+c_2S.
\end{aligned}$$

Clearly, $R_0^l(\theta)=\frac{D^2S_0}{\left(D+\sigma_{11}^2+\frac{2\sigma_{21}^2DS_0}{(1-\theta)\sigma_{11}}\right)\left(D+r_1C_0^u+\frac{\sigma_{21}^2}{2}\right)\left(2D+r_2C_0^u+\frac{\sigma_{31}^2}{2}\right)}$ is continuous, choosing θ small enough such that

$$\begin{aligned}
LV_2\leq& -3DS_0\left(\sqrt[3]{R_0^l}-1\right)+\left(\frac{c_1\mu_1}{\delta_1 a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1+\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2\\
&+\frac{c_2\sigma_{22}^2}{2}x_1^2+\frac{c_3\sigma_{32}^2}{2}x_2^2+c_2S.
\end{aligned}$$

Then,

$$LV_3=DS_0S^{p-1}-DS^p-\frac{\mu_1S^px_1}{\delta_1(a_1+x_1)}-\frac{\mu_2S^px_2}{\delta_2(a_2+x_2)}-\frac{1-p}{2}S^p(\sigma_{11}+\sigma_{12}S)^2$$

$$
\begin{aligned}
&+\frac{\mu_1 S x_1^p}{a_1+x_1}-Dx_1^p-r_1C_0(t)x_1^p-\frac{1-p}{2}x_1^p(\sigma_{21}+\sigma_{22}x_1)^2\\
&+\frac{\mu_2 S x_2^p}{a_2+x_2}-Dx_2^p-r_2C_0(t)x_2^p-\frac{1-p}{2}x_2^p(\sigma_{31}+\sigma_{32}x_2)^2\\
\le\ & DS_0S^{p-1}-DS^p-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-Dx_1^p-r_1C_0^lx_1^p-\frac{1-p}{2}\sigma_{22}^2x_1^{2+p}\\
&-Dx_2^p-r_2C_0^lx_2^p-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}+\frac{\mu_1 S x_1^p}{a_1}+\frac{\mu_2 S x_2^p}{a_2}.
\end{aligned}
$$

We get

$$
\begin{aligned}
L\bar{V}\le\ & -3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1\\
&+M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2+Mc_2S+DS_0S^{p-1}\\
&-DS^p-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-Dx_1^p-r_1C_0^lx_1^p-\frac{1-p}{2}\sigma_{22}^2x_1^{2+p}-Dx_2^p\\
&-r_2C_0^lx_2^p-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}+\frac{\mu_1 S x_1^p}{a_1}+\frac{\mu_2 S x_2^p}{a_2}.
\end{aligned}
$$

Define a compact bounded subset U ,

$$
U=\left\{(S,x_1,x_2)\in\mathbb{R}_+^3:\epsilon\le S\le\frac{1}{\epsilon},\epsilon\le x_1\le\frac{1}{\epsilon},\epsilon\le x_2\le\frac{1}{\epsilon}\right\}.
$$

In the set $\mathbb{R}_+^3\setminus U$, choosing ϵ small enough such that

$$
-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+Mc_2\epsilon+A<-1, \tag{3.15}
$$

$$
\begin{aligned}
-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+M\left(\tfrac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)\epsilon+\tfrac{Mc_2\sigma_{22}^2}{2}\epsilon^2&\\
+\tfrac{\mu_1S_0\epsilon^p}{a_1}+B<-1,&
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+M\left(\tfrac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)\epsilon+\tfrac{Mc_3\sigma_{32}^2}{2}\epsilon^2&\\
+\tfrac{\mu_2S_0\epsilon^p}{a_2}+C<-1,&
\end{aligned} \tag{3.17}
$$

$$
-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)-\frac{1-p}{4\epsilon^{p+2}}\sigma_{12}^2-\frac{D^p}{\epsilon}+E<-1, \tag{3.18}
$$

$$
-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)-\frac{1-p}{4\epsilon^{p+2}}\sigma_{22}^2+F<-1, \tag{3.19}
$$

$$
-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)-\frac{1-p}{4\epsilon^{p+2}}\sigma_{32}^2+G<-1. \tag{3.20}
$$

Next, six domains are given as

$$U_1=\left\{(S,x_1,x_2)\in\mathbb{R}^3_+:0<S<\epsilon\right\},U_2=\left\{(S,x_1,x_2)\in\mathbb{R}^3_+:0<x_1<\epsilon\right\},$$

$$U_3=\left\{(S,x_1,x_2)\in\mathbb{R}^3_+:0<x_2<\epsilon\right\},U_4=\left\{(S,x_1,x_2)\in\mathbb{R}^3_+:S>\frac{1}{\epsilon}\right\},$$

$$U_5=\left\{(S,x_1,x_2)\in\mathbb{R}^3_+:x_1>\frac{1}{\epsilon}\right\},U_6=\left\{(S,x_1,x_2)\in\mathbb{R}^3_+:x_2>\frac{1}{\epsilon}\right\}.$$

Case 1. If $(S,x_1,x_2)\in U_1$,

$$\begin{aligned}L\bar{V}\leq&-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1\\&+M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2+Mc_2\epsilon+DS_0S^{p-1}\\&-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-Dx_1^p-r_1C_0^l(t)x_1^p-\frac{1-p}{2}\sigma_{22}^2x_1^{2+p}-Dx_2^p\\&-r_2C_0^l(t)x_2^p-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}+\frac{\mu_1\epsilon x_1^p}{a_1}+\frac{\mu_2\epsilon x_2^p}{a_2}\\\leq&-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+Mc_2\epsilon+A,\end{aligned}$$

where

$$\begin{aligned}A=&\sup_{(S,x_1,x_2)\in\mathbb{R}^3_+}\left\{M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1+M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2\right.\\&+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2+DS_0S^{p-1}-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-Dx_1^p\\&-r_1C_0^l(t)x_1^p-Dx_2^p-\frac{1-p}{2}\sigma_{22}^2x_1^{2+p}-r_2C_0^l(t)x_2^p-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}\\&\left.+\frac{\mu_1\epsilon x_1^p}{a_1}+\frac{\mu_2\epsilon x_2^p}{a_2}\right\}.\end{aligned}\tag{3.21}$$

Together with the defination of A, we get $L\bar{V}<-1$.

Case 2. If $(S,x_1,x_2)\in U_2$,

$$\begin{aligned}L\bar{V}\leq&-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)\epsilon\\&+M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{Mc_2\sigma_{22}^2}{2}\epsilon^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2+Mc_2S+DS_0S^{p-1}\\&-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-Dx_2^p-r_2C_0^l(t)x_2^p-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}+\frac{\mu_1S_0\epsilon^p}{a_1}+\frac{\mu_2S_0x_2^p}{a_2}\\\leq&-3MDS_0\left(\sqrt[3]{R_0^l}-1\right)+M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)\epsilon+\frac{Mc_2\sigma_{22}^2}{2}\epsilon^2\\&+\frac{\mu_1S_0\epsilon^p}{a_1}+B,\end{aligned}$$

where

$$B=\sup_{(S,x_1,x_2)\in\mathbb{R}^3_+}\left\{M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2+Mc_2S+DS_0S^{p-1}\right.$$

$$-\frac{1-p}{2}\sigma_{12}^2 S^{p+2} - Dx_2^p - r_2 C_0^l(t)x_2^p - \frac{1-p}{2}\sigma_{32}^2 x_2^{p+2} + \frac{\mu_2 S_0 x_2^p}{a_2}\Bigg\}. \tag{3.22}$$

On account of (3.14) and (3.16), we get $L\bar{V} < -1$.

Case 3. If $(S, x_1, x_2) \in U_3$,

$$\begin{aligned}
L\bar{V} \le & -3MDS_0\left(\sqrt[3]{R_0^l} - 1\right) + M\left(\frac{c_1\mu_1}{\delta_1 a_1} + c_2\sigma_{21}\sigma_{22}\right)x_1 \\
& + M\left(\frac{c_2\mu_2}{\delta_2 a_2} + c_3\sigma_{31}\sigma_{32}\right)\epsilon + \frac{Mc_2\sigma_{22}^2}{2}x_1^2 + \frac{Mc_3\sigma_{32}^2}{2}\epsilon^2 + Mc_2 S + DS_0 S^{p-1} \\
& - \frac{1-p}{2}\sigma_{12}^2 S^{p+2} - \frac{1-p}{2}\sigma_{22}^2 x_1^{p+2} + \frac{\mu_1 S_0 x_1^p}{a_1} + \frac{\mu_2 S_0 \epsilon^p}{a_2} \\
\le & -3MDS_0\left(\sqrt[3]{R_0^l} - 1\right) + M\left(\frac{c_2\mu_2}{\delta_2 a_2} + c_3\sigma_{31}\sigma_{32}\right)\epsilon + \frac{Mc_3\sigma_{32}^2}{2}\epsilon^2 \\
& + \frac{\mu_2 S_0 \epsilon^p}{a_2} + C,
\end{aligned}$$

where

$$\begin{aligned}
C = \sup_{(S,x_1,x_2)\in\mathbb{R}_+^3} & \Bigg\{M\left(\frac{c_1\mu_1}{\delta_1 a_1} + c_2\sigma_{21}\sigma_{22}\right)x_1 + \frac{Mc_2\sigma_{22}^2}{2}x_1^2 + Mc_2 S + DS_0 S^{p-1} \\
& - \frac{1-p}{2}\sigma_{12}^2 S^{p+2} - \frac{1-p}{2}\sigma_{22}^2 x_1^{p+2} + \frac{\mu_1 S_0 x_1^p}{a_1}\Bigg\}.
\end{aligned} \tag{3.23}$$

In view of (3.14) and (3.17), we get $L\bar{V} < -1$.

Case 4. If $(S, x_1, x_2) \in U_4$,

$$\begin{aligned}
L\bar{V} \le & -3MDS_0\left(\sqrt[3]{R_0^l} - 1\right) - \frac{1-p}{4\epsilon^{p+2}}\sigma_{12}^2 - \frac{D}{\epsilon^p} + M\left(\frac{c_1\mu_1}{\delta_1 a_1} + c_2\sigma_{21}\sigma_{22}\right)x_1 \\
& + M\left(\frac{c_2\mu_2}{\delta_2 a_2} + c_3\sigma_{31}\sigma_{32}\right)x_2 + \frac{Mc_2\sigma_{22}^2}{2}x_1^2 + \frac{Mc_3\sigma_{32}^2}{2}x_2^2 - \frac{1-p}{4}\sigma_{12}^2 S^{p+2} \\
& + Mc_2 S - \frac{1-p}{2}\sigma_{22}^2 x_1^{p+2} - \frac{1-p}{2}\sigma_{32}^2 x_2^{p+2} + DS_0 S^{p-1} + \frac{\mu_1 S_0 x_1^p}{a_1} + \frac{\mu_2 S_0 x_2^p}{a_2} \\
\le & -3MDS_0\left(\sqrt[3]{R_0^l} - 1\right) - \frac{1-p}{4\epsilon^{p+2}}\sigma_{12}^2 - \frac{D}{\epsilon^p} + E,
\end{aligned}$$

where

$$\begin{aligned}
E = \sup_{(S,x_1,x_2)\in\mathbb{R}_+^3} & \Bigg\{M\left(\frac{c_1\mu_1}{\delta_1 a_1} + c_2\sigma_{21}\sigma_{22}\right)x_1 + M\left(\frac{c_2\mu_2}{\delta_2 a_2} + c_3\sigma_{31}\sigma_{32}\right)x_2 \\
& + \frac{Mc_2\sigma_{22}^2}{2}x_1^2 + \frac{Mc_3\sigma_{32}^2}{2}x_2^2 + Mc_2 S - \frac{1-p}{4}\sigma_{12}^2 S^{p+2} - \frac{1-p}{2}\sigma_{22}^2 x_1^{p+2} \\
& - \frac{1-p}{2}\sigma_{32}^2 x_2^{p+2} + DS_0 S^{p-1} + \frac{\mu_1 S_0 x_1^p}{a_1} + \frac{\mu_2 S_0 x_2^p}{a_2}\Bigg\}.
\end{aligned} \tag{3.24}$$

Considering (3.14) and (3.18), $L\bar{V} < -1$ is obtained.

Case 5. If $(S, x_1, x_2) \in U_5$,

$$L\bar{V} \le -3MDS_0\left(\sqrt[3]{R_0^l} - 1\right) - \frac{1-p}{4\epsilon^{p+2}}\sigma_{22}^2 + M\left(\frac{c_1\mu_1}{\delta_1 a_1} + c_2\sigma_{21}\sigma_{22}\right)x_1$$

$$+M\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2-\frac{1-p}{2}\sigma_{12}^2S^{p+2}$$
$$+Mc_2S-\frac{1-p}{4}\sigma_{22}^2x_1^{p+2}-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}+DS_0S^{p-1}+\frac{\mu_1S_0x_1^p}{a_1}+\frac{\mu_2S_0x_2^p}{a_2}$$
$$\leq -3MDS_0\left(\sqrt[3]{R_0^l}-1\right)-\frac{1-p}{4\epsilon^{p+2}}\sigma_{22}^2+F,$$

where

$$F=\sup_{(S,x_1,x_2)\in\mathbb{R}_+^3}\left\{M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1+M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+Mc_2S\right.$$
$$+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-\frac{1-p}{4}\sigma_{22}^2x_1^{p+2}-\frac{1-p}{2}\sigma_{32}^2x_2^{p+2}$$
$$\left.+DS_0S^{p-1}+\frac{\mu_1S_0x_1^p}{a_1}+\frac{\mu_2S_0x_2^p}{a_2}\right\}. \tag{3.25}$$

Combining (3.14) and (3.19), $L\bar{V}<-1$ is given.

Case 6. If $(S,x_1,x_2)\in U_6$,

$$L\bar{V}\leq -3MDS_0\left(\sqrt[3]{R_0^l}-1\right)-\frac{1-p}{4\epsilon^{p+2}}\sigma_{32}^2+M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1$$
$$+M\left(\frac{c_2\mu_2}{\delta_2 a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2-\frac{1-p}{2}\sigma_{12}^2S^{p+2}$$
$$+Mc_2S-\frac{1-p}{2}\sigma_{22}^2x_1^{p+2}-\frac{1-p}{4}\sigma_{32}^2x_2^{p+2}+DS_0S^{p-1}+\frac{\mu_1S_0x_1^p}{a_1}+\frac{\mu_2S_0x_2^p}{a_2}$$
$$\leq -3MDS_0\left(\sqrt[3]{R_0^l}-1\right)-\frac{1-p}{4\epsilon^{p+2}}\sigma_{32}^2+G,$$

where

$$G=\sup_{(S,x_1,x_2)\in\mathbb{R}_+^3}\left\{M\left(\frac{c_1\mu_1}{\delta_1a_1}+c_2\sigma_{21}\sigma_{22}\right)x_1+M\left(\frac{c_2\mu_2}{\delta_2a_2}+c_3\sigma_{31}\sigma_{32}\right)x_2+Mc_2S\right.$$
$$+\frac{Mc_2\sigma_{22}^2}{2}x_1^2+\frac{Mc_3\sigma_{32}^2}{2}x_2^2-\frac{1-p}{2}\sigma_{12}^2S^{p+2}-\frac{1-p}{2}\sigma_{22}^2x_1^{p+2}-\frac{1-p}{4}\sigma_{32}^2x_2^{p+2}$$
$$\left.+DS_0S^{p-1}+\frac{\mu_1S_0x_1^p}{a_1}+\frac{\mu_2S_0x_2^p}{a_2}\right\}. \tag{3.26}$$

Combined with the condition (3.14) and (3.20), $L\bar{V}<-1$ is given.

Clearly, there exists a sufficiently small ε such that $L\bar{V}\leq -1$, for any $(S,x_1,x_2)\in R_+^3\setminus U$.

In addition, the diffusion matrix of system (1.1) is

$$\begin{pmatrix} S^2(\sigma_{11}+\sigma_{12}S)^2 & 0 & 0\\ 0 & x_1^2(\sigma_{21}+\sigma_{22}x_1)^2 & 0\\ 0 & 0 & x_2^2(\sigma_{31}+\sigma_{32}x_2)^2\end{pmatrix}.$$

Then, there exists a positive

$$N=\min_{(S,x_1,x_2)\in U_\varepsilon}\left\{S^2(\sigma_{11}+\sigma_{12}S)^2,x_1^2(\sigma_{21}+\sigma_{22}x_1)^2,x_2^2(\sigma_{31}+\sigma_{32}x_2)^2\right\}$$

such that

$$\sum_{i,j=1}^{3} a_{i,j}\xi_i\xi_j = S^2(\sigma_{11}+\sigma_{12}S)^2\xi_1^2 + x_1^2(\sigma_{21}+\sigma_{22}x_1)^2\xi_2^2 + x_2^2(\sigma_{31}+\sigma_{32}x_2)^2\xi_3^2$$
$$\geq N\left|\xi\right|^2,$$

where $(S, x_1, x_2) \in R_+^3 \setminus U$, $\xi = (\xi_1, \xi_2, \xi_3) \in R_+^3$. Obviously, this satisfies the conditions in Lemma 2.2. Hence, system (1.1) exists a stationary Markov process. □

4. Conclusion and numerical simulations

The microbial competition system in the chemostat model with nonlinear random perturbation is studied in this paper. Considering the input of toxins such as environmental pollution, the system simulates the real environment by pulse toxin input. By constructing a suitable Lyapunov function, in a chemostat system, the thresholds of microbial culture extinction and persistence in mean are given. In addition, it is proved that system (1.1) has an ergodic stationary distribution under certain conditions. At the same time, through the analysis of the threshold value, the strong noise will accelerate the extinction of microorganisms, which is not conducive to the survival of microorganisms, and also has an adverse impact on the existence of ergodic stationary distribution. Now, we sum up the main results as follows.

(1) If $R_1 < 1$, then microbial x_1 population can be extinct. That is, $\lim_{t\to\infty} x_1(t) = 0$. If $R_2 < 1$, then microbial x_2 population can be extinct. That is, $\lim_{t\to\infty} x_2(t) = 0$.

(2) For any given initial value $(S(0), x_1(0), x_2(0)) \in \mathbb{R}_+^3$, we have the following results.

   (i) If $R_2 < 1$ and $R_3 > 1$, then the microorganism x_1 is persistence in mean, and the microorganism x_2 goes extinct. In addtion, we get

$$\liminf_{t\to\infty}\langle x_1\rangle \geq \frac{a_1\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right)}{D + r_1 C_0^u + \frac{\mu_1^2 S_0}{D\delta_1 a_1}}\left(R_3 - 1\right).$$

   (ii) If $R_1 < 1$ and $R_4 > 1$, then the microorganism x_2 is persistence in mean, and the microorganism x_1 goes extinct. In addtion, we get

$$\liminf_{t\to\infty}\langle x_2\rangle \geq \frac{a_2\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)}{D + r_2 C_0^u + \frac{\mu_2^2 S_0}{D\delta_2 a_2}}\left(R_4 - 1\right).$$

   (iii) If $R_3 > 1$ and $R_4 > 1$, then the microorganisms x_1 and x_2 are persistence in mean. In addtion, we get

$$\liminf_{t\to\infty}\left[\langle x_1\rangle + \langle x_2\rangle\right] \geq \frac{a_1}{Q}\left(D + r_1\widetilde{C_0} + \sigma_{21}^2 + \sigma_{22}^2 S_0^2\right)\left(R_3 - 1\right)$$
$$+\frac{a_2}{Q}\left(D + r_2\widetilde{C_0} + \sigma_{31}^2 + \sigma_{32}^2 S_0^2\right)\left(R_4 - 1\right).$$

(3) If $R_0^l > 1$, then system (1.1) exists a stationary Markov process.

Next, numerical simulations are used for further illustrating the results.

Example 4.1. In our simulations, first, we set

$$D = 0.2, S_0 = 0.3, \mu_1 = 0.3, \mu_2 = 0.6, \delta_1 = 0.1, \delta_2 = 0.2, a_1 = 2, a_2 = 1, r_1 = 0.05,$$
$$r_2 = 0.01, k = 0.2, g = 0.1, m = 0.05, u = 0.01, h = 0.1, \sigma_{11} = 0.15, \sigma_{12} = 0.02,$$
$$\sigma_{21} = 0.01, \sigma_{22} = 0.02, \sigma_{31} = 0.01, \sigma_{32} = 0.02, \tau = 1,$$

and the result is as follows.

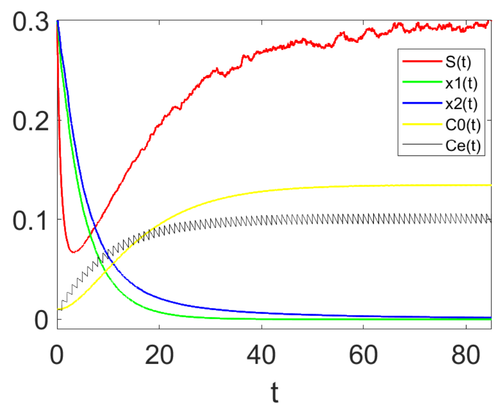

Figure 2. The trajectories of the solution of the stochastic model

As shown in Figure 2, the blue track and the green track gradually approach zero. Under this parameter setting, $R_1 = 0.1968 < 1$, $R_2 = 0.8080 < 1$, and the threshold condition of microbial extinction is met. Therefore, the theoretical results are consistent with the numerical simulation results.

Example 4.2. The parameters of the Figure 3(I) are set as follows.

$$D = 0.2, S_0 = 0.3, \mu_1 = 0.3, \mu_2 = 1, \delta_1 = 0.1, \delta_2 = 0.2, a_1 = 2, a_2 = 1, r_1 = 0.05,$$
$$r_2 = 0.01, k = 0.2, g = 0.1, m = 0.05, u = 0.01, h = 0.1, \sigma_{11} = 0.15, \sigma_{12} = 0.02,$$
$$\sigma_{21} = 0.01, \sigma_{22} = 0.02, \sigma_{31} = 0.01, \sigma_{32} = 0.02, \tau = 1.$$

The parameters of the Figure 3 (II) are set as follows.

$$D = 0.2, S_0 = 0.3, \mu_1 = 1, \mu_2 = 1, \delta_1 = 0.1, \delta_2 = 0.2, a_1 = 1, a_2 = 1, r_1 = 0.05,$$
$$r_2 = 0.01, k = 0.2, g = 0.1, m = 0.05, u = 0.01, h = 0.1, \sigma_{11} = 0.2, \sigma_{12} = 0.02,$$
$$\sigma_{21} = 0.01, \sigma_{22} = 0.02, \sigma_{31} = 0.01, \sigma_{32} = 0.02, \tau = 1.$$

In Figure 3(I), the trajectory of x_2 eventually tends to be stable, and that of x_1 eventually approaches zero. At the same time, under this parameter setting, $R_3 = 0.1968 < 1$, $R_4 = 1.3464 > 1$, the threshold satisfies the conditions of the Theorem 3.2, and the conclusion is valid and consistent with the numerical simulation results. In Figure 3(II), as time goes by, the two tracks gradually stabilize. This means that the two microbes compete with each other but live in harmony. Meanwhile, $R_3 = 1.1617 < 1$, $R_4 = 1.1924 > 1$, which is consistent with the obtained theorem.

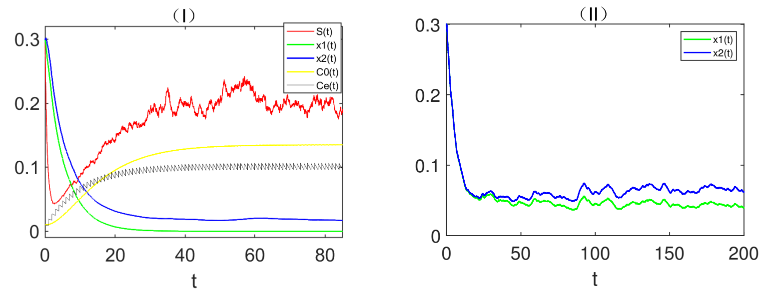

Figure 3. Figure 3(I) is the trajectory of the solution of model (1.1), and Figure 3(II) is the trajectories of $x_1(t)$ and $x_2(t)$.

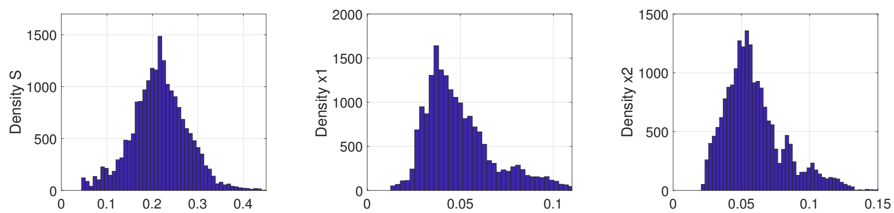

Figure 4. The probability density of $S(t)$, $x_1(t)$ and $x_2(t)$

Example 4.3. Finally, we verify the existence of ergodic stationary distribution through numerical simulation. The numerical simulation results are shown in Figure 4. Set the parameters as follows.

$$D = 0.2, S_0 = 0.3, \mu_1 = 1, \mu_2 = 1, \delta_1 = 0.1, \delta_2 = 0.2, a_1 = 1, a_2 = 1, r_1 = 0.05,$$

$$r_2 = 0.01, k = 0.2, g = 0.1, m = 0.05, u = 0.01, h = 0.1, \sigma_{11} = 0.2, \sigma_{12} = 0.02,$$

$$\sigma_{21} = 0.2, \sigma_{22} = 0.02, \sigma_{31} = 0.2, \sigma_{32} = 0.02, \tau = 1.$$

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