Journal of Nonlinear Modeling and Analysis
Volume 5, Number 3, September 2023, 648–666
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.648

The Effect of Clone Template Parameters on the Spreading Speeds in CNNs*

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Abstract The aim of this paper is to investigate the effect of clone template parameters on the spreading speeds in cellular neural networks(CNNs). According to the property analysis of spreading speeds of monotone semiflows developed by Yu and Zhang [*European Journal of Applied Mathematics*, **31** (2020), 369-384], we investigate the sign of spreading speeds, continuity and limit cases with no propagation phenomena for CNNs with general output functions where each cell interacts with its 2-neighborhood cell.

Keywords Sign of spreading speeds, continuity, CNNs

MSC(2010) 35C07, 34A33, 94C99

1. Introduction

The aim of this paper is to investigate the effect of parameters on the spreading speeds in CNNs. Cellular neural network is a large-scale nonlinear simulation processor that is locally connected and capable of real-time signal processing, proposed by Chua and Yang [2] in 1988. Each of its basic circuit units is a cell neuron, which is regularly connected by the same cell neurons in space. These cell neurons only contact and interact with neighboring cell neurons, and each neuron has internal states related to input, output and dynamic rules. It has the characteristics of continuous real-time, high speed parallel computing and Very Large Scale Integration (VLSI). Over the past 20 years, the research results of CNNs have been widely applied in many fields, such as biomedicine, image processing, automatic control and pattern recognition. The circuit model of one-dimensional standard CNN without input is

$$\frac{dx_n(t)}{dt} = -x_n(t) + z + \sum_{k\in N_r(n)} A(n,k)f(x_k),\ n \in \mathbb{Z}. \tag{1.1}$$

In the above expression, the node voltage x_n at point n is called the state of the cell neuron at point n. The quantity z is called the threshold term or the offset term and is associated with an independent voltage source in the circuit. The output

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*Zhixian Yu is supported by Natural Science Foundation of China (No.12071297) and Shanghai Science and Technology Program (No. 20JC1414200).

function f(a nonlinear function) is given by

$$f(x) = \frac{1}{2}(|x+1| - |x-1|). \tag{1.2}$$

For a positive integer r, the r-neighborhood $N_r(n)$ of a cell at n is defined as

$$N_r(n) = \{k \in \mathbb{Z} : \ |k-n| \le r\}.$$

For each n and $k \in N_r(n)$, $A(n,k)$ constitutes the so-called clone template, which measures the coupling weights of the cells at n from the cells at k and specifies the interactions between each cell and all of its neighbors in terms of the state and output variables. When the template is the space-invariant, each cell is described by the simple identical cloning template, i.e. $A(n, n+k) \equiv A(0,k) := a_k$ $(k \in N_r(0))$ or $A(n,k) = A(n-k)$ $(k \in N_r(n))$. If $r = 1$, letting $a_k := A(0,k)$ $(k \in N_1(0))$, then these numbers can be arranged in a 1×3 matrix form $A := [a_{-1}, a_0, a_1]$ and (1.1) can be written by

$$\frac{dx_n(t)}{dt} = -x_n(t) + z + a_{-1}f(x_{n-1}) + a_0 f(x_n) + a_1 f(x_{n+1}), \ n \in \mathbb{Z}. \tag{1.3}$$

In CNNs, several experimental studies have revealed the propagation of activity in sections of excitable nerve tissue. The basic mechanism for the propagation of these waves (i.e., traveling waves) is thought to originate at synapses, rather than disperse like the propagation of action potentials. Since then,the study of the CNN equation has been extended to more general equations, and the propagation of these waves has also been widely investigated(see, e.g. [4–7, 10, 12, 14–16, 19, 20]). Wu and Hsu [14, 15] considered the existence of the entire solutions for CNNs. Yu and Zhao [17] investigated the propagation phenomena of monotone and non-monotone CNNs with asymmetric templates and distribution delays.

Recently, Yu and Zhang [18] have studied the properties of spreading speeds and have obtained a general method for analyzing the sign, continuity, and limit cases with no propagation phenomena for monotone semiflows. These results are applied to CNNs (1.3) with $z = 0$ where each cell interacts with its 1-neighborhood cell and the output function f satisfies (1.2). Moreover, three different propagation phenomena are determined according to the clone template parameters from 1-neighborhood cells. More recently, Bai and Yang [1] have studied the influence of parameters on the spreading speeds of CNNs ($r = 1$) with time delay. Therefore, motivated by the work of Yu and Zhang [18], we will further consider the influence of interaction parameters on the spreading speeds for the CNNs with general output functions where each cell interacts with its 2-neighborhood cells. More precisely, we investigate the following CNNs with a general output function

$$\begin{aligned} \frac{dx_i}{dt} =& - x_i(t) + \alpha_2 f(x_{i-2}(t)) + \alpha_1 f(x_{i-1}(t)) + a f(x_i(t)) \\ &+ \beta_1 f(x_{i+1}(t)) + \beta_2 f(x_{i+2}(t)), \end{aligned} \tag{1.4}$$

where $\alpha_2, \alpha_1, a, \beta_1, \beta_2$ are nonnegative and the output function f satisfies the following assumptions

(F1) There is $K > 0$ such that $(\alpha_2 + \alpha_1 + a + \beta_1 + \beta_2)f(K) = K$ and $f(0) = 0$; $f \in C([0, K], [0, \frac{K}{\alpha_2+\alpha_1+a+\beta_1+\beta_2}])$, $(\alpha_2 + \alpha_1 + a + \beta_1 + \beta_2)f'(0) > 1$, $|f(u) - f(v)| \le f'(0)|u - v|$ for $u, v \in [0, K]$.

(F2) $f(u)$ is nondecreasing for $u \in (0, K]$ and $(\alpha_2 + \alpha_1 + a + \beta_1 + \beta_2)f(u) > u$ for $u \in (0, K)$.

The rest of the paper is organized as follows. In Section 2, we mainly discuss the existence, sign, continuity of the spreading speeds and the limit case of no propagation phenomenon for CNNs. In Section 3, we present some numerical simulations.

2. Properties of spreading speeds

2.1. Existence of spreading speeds

In this subsection, we discuss the existence of spreading speeds for CNNs.

Let Q_t be the solution map at time $t \geq 0$ of system (1.4). That is,

$$Q_t(x^0) = x(t, x^0), \quad \forall\, x^0 = \{x_i^0\}_{i\in\mathbb{Z}} \in \mathcal{X}_K,$$

where $\mathcal{X}_K = \{\varphi = \{\varphi_i\}_{i\in\mathbb{Z}} \,|\, \varphi_i \in [0, K],\, i \in \mathbb{Z}\}$. We can easily check that $Q := Q_1$ satisfies all hypotheses (A1)–(A6) in [9]. Thus, there exist c_+^* and c_-^* which are the rightward and leftward spreading speeds of Q, respectively.

Firstly, we estimate the rightward spreading speed. For this purpose, we consider the linearized equation of (1.4) at the zero solution, i.e.,

$$\begin{aligned}\frac{dx_i(t)}{dt} =& - x_i(t) + \alpha_2 f'(0)x_{i-2}(t) + \alpha_1 f'(0)x_{i-1}(t) + af'(0)x_i(t)\\ &+ \beta_1 f'(0)x_{i+1}(t) + \beta_2 f'(0)x_{i+2}(t),\ i \in \mathbb{Z}.\end{aligned} \tag{2.1}$$

Let $\{M_t\}_{t\geq 0}$ be the solution semiflow associated with (2.1). Thus, for each $t > 0$, the map M_t satisfies the assumptions (C1)–(C5) in [9]. By the comparison theorem, we have $Q_t(x^0) \leq M_t(x^0)$, $\forall\, x^0 \in \mathcal{X}_K,\ t \geq 0$. On the other hand, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $x^0 \in \mathcal{X}_K$ with $x^0 < \delta$, we can obtain $Q_t(x^0) \geq M_t^\epsilon(x^0)$ for all $t \in [0, 1]$, where M_t^ϵ is the solution semiflow of

$$\begin{aligned}\frac{dx_i(t)}{dt} =& - x_i(t) + (1-\epsilon)\alpha_2 f'(0)x_{i-2}(t) + (1-\epsilon)\alpha_1 f'(0)x_{i-1}(t)\\ &+ (1-\epsilon)af'(0)x_i(t) + (1-\epsilon)\beta_1 f'(0)x_{i+1}(t)\\ &+ (1-\epsilon)\beta_2 f'(0)x_{i+2}(t),\ i \in \mathbb{Z}.\end{aligned} \tag{2.2}$$

Let $x_i(t) = e^{-\mu i}v(t)$ be a solution of equation (2.1), then we can find that $v(t)$ satisfies the following differential equation:

$$\frac{dv(t)}{dt} = (af'(0) - 1 + \alpha_2 f'(0)e^{2\mu} + \alpha_1 f'(0)e^{\mu} + \beta_1 f'(0)e^{-\mu} + \beta_2 f'(0)e^{-2\mu})v(t). \tag{2.3}$$

Letting

$$B_\mu^t(v_0) := M_t[v_0 e^{-\mu i}](0) = v(t, v_0),\ \forall\ v(0) = v_0 \in [0, \infty),$$

it follows that B_μ^t is the solution map at time t of equation (2.3) and

$$B_\mu^t(v_0) = e^{(af'(0)-1+\alpha_2 f'(0)e^{2\mu}+\alpha_1 f'(0)e^{\mu}+\beta_1 f'(0)e^{-\mu}+\beta_2 f'(0)e^{-2\mu})t}v_0,\ \forall\ v_0 \in [0, \infty).$$

Thus, for any $\mu \geq 0$, $B_\mu := B_\mu^1$ is a compact and strongly positive linear operator on $[0, \infty)$, i.e., (C6) in [9] holds.

It is obvious to see that

$$\lambda(\mu) = e^{(af'(0)-1+\alpha_2 f'(0)e^{2\mu}+\alpha_1 f'(0)e^{\mu}+\beta_1 f'(0)e^{-\mu}+\beta_2 f'(0)e^{-2\mu})}$$

is the principal eigenvalue of B_μ for any $\mu \geq 0$ and

$$\lambda(0) = e^{(af'(0)-1+\alpha_2 f'(0)+\alpha_1 f'(0)+\beta_1 f'(0)+\beta_2 f'(0))} > 1.$$

Let

$$\Phi(\mu) = \frac{\ln \lambda(\mu)}{\mu} = \frac{h(\mu)}{\mu},\ \mu \neq 0 \quad \text{and} \quad \Psi(\mu) = \frac{\lambda'(\mu)}{\lambda(\mu)},\ \mu \in (-\infty, +\infty),$$

where

$$h(\mu) = af'(0) - 1 + \alpha_2 f'(0)e^{2\mu} + \alpha_1 f'(0)e^{\mu} + \beta_1 f'(0)e^{-\mu} + \beta_2 f'(0)e^{-2\mu}.$$

It is obvious that Lemma 2.1 in [18] holds. We denote

$$\Phi(+\infty) := \lim_{\mu \to +\infty} \Phi(\mu) \qquad \Phi^+(\mu) = \Phi(\mu), \qquad \Phi^-(\mu) = -\Phi(-\mu),$$

$$\Psi(+\infty) := \lim_{\mu \to +\infty} \Psi(\mu) \qquad \Psi^+(\mu) = \Psi(\mu), \qquad \Psi^-(\mu) = -\Psi(-\mu),$$

where $\lambda^+(\mu) = \lambda(\mu)$, $\lambda^-(\mu) = \lambda(-\mu)$.

According to Proposition 3.9 and Theorem 3.10 in [8] and Lemma 4.6 in [3] (including that $\inf_{\mu>0} \Phi^{\pm}(\mu) = \Phi^{\pm}(+\infty)$), we have

$$\begin{aligned} c_+^* &= \inf_{\mu>0} \Phi^+(\mu) = \inf_{\mu>0} \frac{\ln \lambda(\mu)}{\mu} \\ &= \inf_{\mu>0} \frac{f'(0)(a + \alpha_2 e^{2\mu} + \alpha_1 e^{\mu} + \beta_1 e^{-\mu} + \beta_2 e^{-2\mu}) - 1}{\mu}. \end{aligned} \tag{2.4}$$

Similarly, it follows that the left spreading speed

$$\begin{aligned} c_-^* &= \inf_{\mu>0} \Phi^-(\mu) = \inf_{\mu>0} \frac{\ln \lambda(-\mu)}{\mu} \\ &= \inf_{\mu>0} \frac{f'(0)(a + \alpha_2 e^{-2\mu} + \alpha_1 e^{-\mu} + \beta_1 e^{\mu} + \beta_2 e^{2\mu}) - 1}{\mu}. \end{aligned} \tag{2.5}$$

According to Lemma 2.1 and Proposition 2.1 in [18], there exist $\mu_+^* \in (0, +\infty]$ and $\mu_-^* \in (0, +\infty]$ such that $c_+^* = \Phi^+(\mu_+^*)$ and $c_-^* = \Phi^-(\mu_-^*)$. Therefore, it follows from the monotonicity of $\Psi(\mu)$ that

$$\begin{aligned} c_+^* + c_-^* &= \Phi^+(\mu_+^*) + \Phi^-(\mu_-^*) \\ &= \Psi^+(\mu_+^*) + \Psi^-(\mu_-^*) \\ &= \Psi(\mu_+^*) - \Psi(-\mu_-^*) > 0. \end{aligned} \tag{2.6}$$

As a direct result of Theorem 2.12 in [3], Theorem 3.4 in [9] and Theorem 2.1 in [17], we have the following result.

Theorem 2.1. *Assume that* $(F1)$-$(F2)$ *hold. Let* $x(t)$ *be a solution of* (1.4) *with the initial condition* $x^0 \in \mathcal{X}_K$*. Then* c_+^* *and* c_-^* *defined by* (2.4) *and* (2.5) *are the rightward and leftward spreading speeds of* Q_1*, respectively, such that the following statements are valid:*

(i) For any $c > c_+^*$ *and* $c' > c_-^*$*, if* $x^0 \in \mathcal{X}_K$ *with* $x_i^0 = 0$ *for* i *outside of a bounded interval, then* $\lim_{t\to\infty, i\geq ct} x_i(t) = 0$ *and* $\lim_{t\to\infty, i\leq -c't} x_i(t) = 0$.

(ii) For $-c_-^* < -c' < c < c_+^*$*, if* $x^0 \in \mathcal{X}_K \backslash \{0\}$*, then* $\lim_{t\to\infty, -c't\leq i\leq ct} x_i(t) = K$.

2.2. The sign of spreading speeds

In this subsection, we begin to investigate the sign of spreading speeds for CNNs.

Proposition 2.1. *Assume that* $(F1)$-$(F2)$ *hold. Then the following statements hold.*

(i) If $\alpha_2 < \beta_2, \alpha_1 < \beta_1$*, then* $c_+^* < c_-^*$ *and* $c_-^* > 0$.

(ii) If $\alpha_2 = \beta_2, \alpha_1 = \beta_1$*, then* $c_+^* = c_-^* > 0$.

(iii) If $\alpha_2 > \beta_2, \alpha_1 > \beta_1$*, then* $c_+^* > c_-^*$ *and* $c_+^* > 0$.

Proof. (i) If $\alpha_2 < \beta_2, \alpha_1 < \beta_1$, then

$$\begin{aligned} c_-^* - c_+^* &= \inf_{\mu>0} \Phi^-(\mu) - \inf_{\mu>0} \Phi^+(\mu) \\ &\geq \inf_{\mu>0} [\Phi^-(\mu) - \Phi^+(\mu)] \\ &= \inf_{\mu>0} f'(0) \frac{(\beta_2 - \alpha_2)(e^{2\mu} - e^{-2\mu}) + (\beta_1 - \alpha_1)(e^{\mu} - e^{-\mu})}{\mu} > 0. \end{aligned}$$

It follows from (2.6) that $c_-^* > 0$.

In the similar way, we can prove the case where $\alpha_2 > \beta_2, \alpha_1 > \beta_1$. If $\alpha_2 = \beta_2, \alpha_1 = \beta_1$, it is obvious that $\Phi^+(\mu) = \Phi^-(\mu)$, which implies that $c_+^* = c_-^*$. This completes the proof. □

In order to discuss the sign of c_+^* (c_-^*), we only consider the sign of $h_0 := \inf_{\mu>0} h(\mu) = \min_{\mu\geq 0} h(\mu)$ by using Corollary **A** in the Appendix.

Lemma 2.1. *If* $0 < 2\alpha_2 + \alpha_1 < 2\beta_2 + \beta_1$*, then the equation* $2\alpha_2 e^{2\mu} + \alpha_1 e^{\mu} - \beta_1 e^{-\mu} - 2\beta_2 e^{-2\mu} = 0$ *has a uniquely positive root* μ_0.

Proof. Let $p(\mu) = 2\alpha_2 e^{2\mu} + \alpha_1 e^{\mu} - \beta_1 e^{-\mu} - 2\beta_2 e^{-2\mu}$. It is easily checked that $p(\mu)$ is increasing with respect to μ. And $p(0) = 2\alpha_2 + \alpha_1 - \beta_1 - 2\beta_2 < 0$, $p(+\infty) = +\infty$. Thus there is a unique constant $\mu_0 > 0$, s.t. $p(\mu_0) = 0$. This completes the proof. □

Theorem 2.2. *Assume that* $(F1)$-$(F2)$ *hold. Let* $0 < 2\alpha_2 + \alpha_1 < 2\beta_2 + \beta_1$ *and* μ_0 *be given in Lemma 2.1. Then* $h(\mu_0)$ *has the same sign as* c_+^*.

Proof. Since μ_0 is given by Lemma 2.1, we can have

$$h'(\mu_0) = f'(0)(2\alpha_2 e^{2\mu_0} + \alpha_1 e^{\mu_0} - \beta_1 e^{-\mu_0} - 2\beta_2 e^{-2\mu_0}) = 0.$$

Thus, we can obtain that

$$h_0 = h(\mu_0) = af'(0) - 1 + \alpha_2 f'(0)e^{2\mu_0} + \alpha_1 f'(0)e^{\mu_0} + \beta_1 f'(0)e^{-\mu_0} + \beta_2 f'(0)e^{-2\mu_0}.$$

Since $2\alpha_2+\alpha_1>0$, (L4′) ($\Phi(+\infty)=\lim_{\mu\to+\infty}\Phi(\mu)=+\infty$) is true. It follows from the Corollary **A** in the Appendix that these conclusions hold.

(i) If $h(\mu_0)>0$, then $c_+^*>0$.

(ii) If $h(\mu_0)=0$, then $c_+^*=0$.

(iii) If $h(\mu_0)<0$, then $c_+^*<0$.

On the other hand, when $c_+^*>0$, assuming that $h(\mu_0)\le 0$, we easily see that $c_+^*\le 0$, which is in contradiction with $c_+^*>0$. Thus we can obtain the following conclusions:

(i) If $c_+^*>0$, then $h(\mu_0)>0$.

(ii) If $c_+^*=0$, then $h(\mu_0)=0$.

(iii) If $c_+^*<0$, then $h(\mu_0)<0$.

This completes the proof. □

Theorem 2.3. *Assume that* $(F1)$*-*$(F2)$ *hold. Then the following conclusions hold:*

(1) If $2\alpha_2+\alpha_1\ge\beta_1+2\beta_2$*, then* $c_+^*>0$.

(2) If $\frac{\beta_2}{\alpha_2}=(\frac{\beta_1}{\alpha_1})^2=(\frac{\alpha_1}{\alpha_2})^4>1$*, then*

(i) $af'(0)+4f'(0)(\alpha_1\alpha_2\beta_1\beta_2)^{\frac{1}{4}}>1\Leftrightarrow c_+^*>0$.

(ii) $af'(0)+4f'(0)(\alpha_1\alpha_2\beta_1\beta_2)^{\frac{1}{4}}=1\Leftrightarrow c_+^*=0$.

(iii) $af'(0)+4f'(0)(\alpha_1\alpha_2\beta_1\beta_2)^{\frac{1}{4}}<1\Leftrightarrow c_+^*<0$.

Proof. (1) If $2\alpha_2+\alpha_1\ge\beta_1+2\beta_2$, we can easily obtain that $h(\mu)$ is nondecreasing in $\mu\ge 0$ and

$$h_0=\min_{\mu\ge 0}h(\mu)=h(0)=af'(0)-1+\alpha_2 f'(0)+\alpha_1 f'(0)+\beta_1 f'(0)+\beta_2 f'(0)>0.$$

Then it follows from the Corollary **A** in the Appendix that $c_+^*>0$.

(2) If $\frac{\beta_2}{\alpha_2}=\frac{\beta_1^2}{\alpha_1^2}>1$, it is easily known that $2\beta_2+\beta_1>2\alpha_2+\alpha_1>0$. According to Lemma 2.1, there is a unique constant μ_0, s.t. $h'(\mu_0)=0$. Notice that

$$\begin{aligned}h(\mu)=&af'(0)-1+\alpha_2 f'(0)e^{2\mu}+\alpha_1 f'(0)e^{\mu}+\beta_1 f'(0)e^{-\mu}+\beta_2 f'(0)e^{-2\mu}\\ \ge&af'(0)-1+4f'(0)(\alpha_1\alpha_2\beta_1\beta_2)^{\frac{1}{4}}.\end{aligned}$$

On the other hand, we can see that

$$h(\mu_0)=af'(0)-1+4f'(0)(\alpha_1\alpha_2\beta_1\beta_2)^{\frac{1}{4}},$$

where $\mu_0=\ln\frac{\alpha_2}{\alpha_1}=\frac{1}{2}\ln\frac{\beta_1}{\alpha_1}=\frac{1}{4}\ln\frac{\beta_2}{\alpha_2}$.

Thus, we can have

$$h_0=af'(0)-1+4f'(0)(\alpha_1\alpha_2\beta_1\beta_2)^{\frac{1}{4}}.$$

According to Theorem 2.2, conclusions (i)-(iii) are valid. This completes the proof. □

Theorem 2.4. *Assume that (F1)-(F2) hold. In addition,* $af'(0) + \beta_1 f'(0) + \beta_2 f'(0) > 1$ *and* $\beta_2, \beta_1 > \alpha_2 = \alpha_1 = 0$ *hold. Then the following conclusions hold.*

(i) If $0 \leq af'(0) < 1$*, then* $c_+^* < 0$*.*

(ii) If $af'(0) \geq 1$*, then* $c_+^* = 0$*.*

Proof. It is obvious that (L1)–(L3) in the Appendix hold under the condition that $af'(0) + \beta_1 f'(0) + \beta_2 f'(0) > 1$ and $\beta_2, \beta_1 > \alpha_2 = \alpha_1 = 0$.

Since $\Phi^+(\infty) = \lim_{\mu\to+\infty} \frac{af'(0)-1+\beta_1 f'(0)e^{-\mu}+\beta_2 f'(0)e^{-2\mu}}{\mu} = 0$, (L4) also holds. According to the definition of $\lambda^+(\mu)$ above, we have $\lambda^+(+\infty) = e^{f'(0)a-1}$. Thus, when $0 \leq af'(0) < 1$, it yields that $\lambda^+(+\infty) < 1$, which implies that $c_+^* < 0$ by Theorem **A** in the Appendix. When $af'(0) \geq 1$, then $\lambda^+(+\infty) \geq 1$ and $c_+^* = 0$. This completes the proof. □

Remark 2.1. For c_-^*, we can also give the assumptions and get some results similar to Theorems 2.2-2.4.

2.3. Continuity of spreading speeds

In this subsection, we investigate CNNs with the variable templates

$$[\alpha_{2,n}, \alpha_{1,n}, a_n, \beta_{1,n}, \beta_{2,n}]$$

as follows:

$$\begin{aligned}\frac{dx_i(t)}{dt} = &-x_i(t) + \alpha_{2,n} f(x_{i-2}(t)) + \alpha_{1,n} f(x_{i-1}(t)) + a_n f(x_i(t)) \\ &+ \beta_{1,n} f(x_{i+1}(t)) + \beta_{2,n} f(x_{i+2}(t)),\ i \in \mathbb{Z},\end{aligned} \tag{2.7}$$

where the nonnegative parameters $\alpha_{2,n}, \alpha_{1,n}, a_n, \beta_{1,n}, \beta_{2,n}$ $(n \in \mathbb{N})$ satisfy

(P) $\lim_{n\to+\infty} \alpha_{2,n} = \alpha_2$, $\lim_{n\to+\infty} \alpha_{1,n} = \alpha_1$, $\lim_{n\to+\infty} a_n = a$, $\lim_{n\to+\infty} \beta_{1,n} = \beta_1$ and $\lim_{n\to+\infty} \beta_{2,n} = \beta_2$, where $\alpha_2, \alpha_1, a, \beta_1, \beta_2$ and $f(u)$ satisfy assumptions (F1)-(F2).

We mainly investigate the relation between the spreading speeds of CNNs with the templates $[\alpha_{2,n}, \alpha_{1,n}, a_n, \beta_{1,n}, \beta_{2,n}]$ and with the templates $[\alpha_2, \alpha_1, a, \beta_1, \beta_2]$.

According to the assumption (P), there exists a sufficiently large number $N_0 \in \mathbb{N}$ such that $\alpha_{2,n}, \alpha_{1,n}, a_n, \beta_{1,n}, \beta_{2,n}$ $(n \in \mathbb{N})$ and $f(u)$ also satisfies assumptions (F1)-(F2) for $n > N_0$. Thus it follows from Theorem 2.1 that, for any $n > N_0$, (2.7) admits the right and left spreading speeds

$$c_{n+}^* = \inf_{\mu>0} \frac{f'(0)\left[a_n - 1 + \alpha_{2,n}e^{2\mu} + \alpha_{1,n}e^{\mu} + \beta_{1,n}e^{-\mu} + \beta_{2,n}e^{-2\mu}\right]}{\mu} \tag{2.8}$$

and

$$c_{n-}^* = \inf_{\mu>0} \frac{f'(0)\left[a_n - 1 + \alpha_{2,n}e^{-2\mu} + \alpha_{1,n}e^{-\mu} + \beta_{1,n}e^{\mu} + \beta_{2,n}e^{2\mu}\right]}{\mu}. \tag{2.9}$$

Theorem 2.5. *Assume that (P) holds. Then* $\lim_{n\to+\infty} c_{n\pm}^* = c_\pm^*$.

Proof. We will verify $\lim_{n\to+\infty} c_{n+}^* = c_+^*$ by using Theorem 2.3 in [18]. The other case can be derived by the same method.

Define $\bar{\alpha}_{2,n} = \max_{k\geq n}\{\alpha, \alpha_{2,k}\}$ and $\underline{\alpha}_{2,n} = \min_{k\geq n}\{\alpha, \alpha_{2,k}\}$. It is not hard to verify that $\{\bar{\alpha}_{2,n}\}_{n\in\mathbb{N}}$ and $\{\underline{\alpha}_{2,n}\}_{n\in\mathbb{N}}$ are nonincreasing and nondecreasing sequences, respectively.

Moreover, $\bar{\alpha}_{2,n} \geq \alpha \geq \underline{\alpha}_{2,n}, \forall n \geq 1$. According to the assumption (P), we can obtain that

$$\lim_{n\to+\infty} \bar{\alpha}_{2,n} = \alpha_2 \text{ and } \lim_{n\to+\infty} \underline{\alpha}_{2,n} = \alpha_2.$$

Similarly, for any $n \geq 1$, define

$$\bar{\alpha}_{1,n} = \max_{k\geq n}\{\alpha, \alpha_{1,k}\},\ \underline{\alpha}_{1,n} = \min_{k\geq n}\{\alpha, \alpha_{1,k}\},\ \bar{a}_n = \max_{k\geq n}\{a, a_k\},\ \underline{a}_n = \min_{k\geq n}\{a, a_k\},$$
$$\bar{\beta}_{1,n} = \max_{k\geq n}\{\beta, \beta_{1,k}\},\ \underline{\beta}_{1,n} = \min_{k\geq n}\{\beta, \beta_{1,k}\},\ \bar{\beta}_{2,n} = \max_{k\geq n}\{\beta, \beta_{2,k}\},$$
$$\underline{\beta}_{2,n} = \min_{k\geq n}\{\beta, \beta_{2,k}\}$$

with

$$\lim_{n\to+\infty} \bar{\alpha}_{1,n} = \alpha_1,\ \lim_{n\to+\infty} \underline{\alpha}_{1,n} = \alpha_1,\ \lim_{n\to+\infty} \bar{a}_n = a,\ \lim_{n\to+\infty} \underline{a}_n = a,$$
$$\lim_{n\to+\infty} \bar{\beta}_{1,n} = \beta_1,\ \lim_{n\to+\infty} \underline{\beta}_{1,n} = \beta_1,\ \lim_{n\to+\infty} \bar{\beta}_{2,n} = \beta_2,\ \lim_{n\to+\infty} \underline{\beta}_{2,n} = \beta_2,$$

and

$$\bar{\alpha}_{1,n} \geq \alpha_1 \geq \underline{\alpha}_{1,n},\ \bar{a}_n \geq a \geq \underline{a}_n,$$
$$\bar{\beta}_{1,n} \geq \beta_1 \geq \underline{\beta}_{1,n},\ \bar{\beta}_{2,n} \geq \beta_2 \geq \underline{\beta}_{1,n}.$$

According to (P), there exists a sufficiently large number $N_1 \in \mathbb{N}$ such that

$$\bar{\alpha}_{2,n} + \bar{\alpha}_{1,n} + \bar{a}_n + \bar{\beta}_{1,n} + \bar{\beta}_{2,n} > 1 \text{ and } \underline{\alpha}_{2,n} + \underline{\alpha}_{1,n} + \underline{a}_n + \underline{\beta}_{1,n} + \underline{\beta}_{2,n} > 1$$

for any $n \geq N_1$. Thus, for any $n > N_0$, (2.7) with the templates

$$[\bar{\alpha}_{2,n}, \bar{\alpha}_{1,n}, \bar{a}_n, \bar{\beta}_{1,n}, \bar{\beta}_{2,n}]$$

and $[\underline{\alpha}_{2,n}, \underline{\alpha}_{1,n}, \underline{a}_n, \underline{\beta}_{1,n}, \underline{\beta}_{2,n}]$ admits the right spreading speed $\bar{c}_{n+}^*$ and $\underline{c}_{n+}^*$, respectively. In view of Lemma 2.9 in [8], we have

$$\underline{c}_{n+}^* \leq c_{n+}^* \leq \bar{c}_{n+}^* \tag{2.10}$$

for all $n > \max\{N_0, N_1\}$.

On the other hand, we can verify that $\bar{\lambda}_n(\mu)$ and $\underline{\lambda}_n(\mu)$ corresponding to the definition of $\lambda_n(\mu)$ is nonincreasing and nondecreasing on $n \in \mathbb{N}$, respectively. Moreover, $\underline{\lambda}_n(\mu) \leq \lambda(\mu) \leq \bar{\lambda}_n(\mu)$ for any $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \underline{\lambda}_n(\mu) = \lim_{n\to+\infty} \bar{\lambda}_n(\mu) = \lambda(\mu)$ for any closed set on $(0, +\infty)$. According to Theorem **C** in the Appendix, we can obtain that

$$\lim_{n\to+\infty} \underline{c}_{n+}^* = \lim_{n\to+\infty} \bar{c}_{n+}^* = c_+^*. \tag{2.11}$$

Thus, it follows from (2.10) and (2.11) that

$$\lim_{n\to+\infty} c_{n+}^* = c_+^*.$$

□

2.4. Discussion about the limiting cases

In this subsection, we estimate the spreading speed of the limiting cases for

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -x_i(t) + \alpha_2(s)f(x_{i-2}(t)) + \alpha_1(s)f(x_{i-1}(t)) + a(s)f(x_i(t)) \\ &\quad + \beta_1(s)f(x_{i+1}(t)) + \beta_2(s)f(x_{i+2}(t)), \end{aligned} \tag{2.12}$$

$i \in \mathbb{Z}$, where the nonnegative parameters $\alpha_2(s), \alpha_1(s)$, $a(s)$ and $\beta_1(s), \beta_2(s)$ satisfy the following assumptions:

(S1) the continuous functions $\alpha_2(s), \alpha_1(s)$, $a(s)$ and $\beta_1(s), \beta_2(s)$ are strictly increasing on $s \in [0, +\infty)$.

(S2) $\lim_{s\to 0^+} \alpha_2(s) = \alpha_2$, $\lim_{s\to 0^+} \alpha_1(s) = \alpha_1$, $\lim_{s\to 0^+} a(s) = a$, $\lim_{s\to 0^+} \beta_1(s) = \beta_1$, $\lim_{s\to 0^+} \beta_2(s) = \beta_2$, where $\alpha_2, \alpha_1, a, \beta_1, \beta_2$ satisfy the following condition (P′): $\alpha_2, \alpha_1, a, \beta_1, \beta_2 \geq 0$, $f'(0)(\alpha_2 + \alpha_1 + \beta_1 + \beta_2) > 0$ and $f'(0)(\alpha_2 + \alpha_1 + a + \beta_1 + \beta_2) = 1$.

Notice that Theorem 2.1 does not hold for $s = 0$. According to (S1) and (S2), it is easily seen that for all $s > 0$,

$$f'(0)[\alpha_2(s)+\alpha_1(s)+a(s)+\beta_1(s)+\beta_2(s)] > 1 \quad \text{and} \quad f'(0)[a(s)+\beta_1(s)+\beta_2(s)] > 0.$$

Therefore, it follows from Theorem 2.1 that for any $s > 0$, (2.12) admits the right and left spreading speeds

$$c(s)^*_+ = \inf_{\mu>0} \Phi^+(s,\mu)$$

and

$$c(s)^*_- = \inf_{\mu>0} \Phi^-(s,\mu),$$

where

$$\Phi^{\pm}(s,\mu) = \frac{\ln \Lambda^{\pm}(s,\mu)}{\mu},$$

$$\Lambda^{\pm}_{\mu}(s,\mu) = e^{(a(s)f'(0)-1+\alpha_2(s)f'(0)e^{\pm 2\mu}+\alpha_1(s)f'(0)e^{\pm\mu}+\beta_1(s)f'(0)e^{\mp\mu}+\beta_2(s)f'(0)e^{\mp 2\mu})}$$

and

$$\Psi^{\pm}(s,\mu) = \frac{\Lambda^{\pm}_{\mu}(s,\mu)}{\Lambda^{\pm}(s,\mu)} = f'(0)(\pm 2\alpha_2(s)e^{\pm 2\mu} \pm \alpha_1(s)e^{\pm\mu} \mp \beta_1(s)e^{\mp\mu} \mp 2\beta_2(s)e^{\mp 2\mu}).$$

That is, we investigate where $c^*_{\pm}(s)$ will go as s approaches 0. It is easily verified that $\Lambda^+(s,\mu)$ and $\Lambda^-(s,\mu)$ satisfy (K1)–(K4) in [18]. The following conclusions hold from Theorem 2.4 in [18].

Theorem 2.6. *Assume that (S1) and (S2) hold. Then* $\lim_{s\to 0^+} c(s)^*_+ = f'(0)(2\alpha_2 + \alpha_1 - \beta_1 - 2\beta_2)$ *and* $\lim_{s\to 0^+} c(s)^*_- = f'(0)(-2\alpha_2 - \alpha_1 + \beta_1 + 2\beta_2)$.

3. Numerical analysis

In the above discussion, we investigated some properties of the spreading speeds in CNNs when the output function f is general. Now we use a numerical analysis to estimate some properties of spreading speeds when the output function in (1.4)is expressed in the special form $f(x) = xe^{-0.5x}$.

According to the calculation method above, we can obtain

$$\lambda(\mu) = e^{a-1+\alpha_2 e^{2\mu}+\alpha_1 e^{\mu}+\beta_1 e^{-\mu}+\beta_2 e^{-2\mu}}$$

for any $\mu \geq 0$ and $\lambda(0) = e^{a-1+\alpha_2+\alpha_1+\beta_1+\beta_2} > 1$.

And we can get

$$c_+^* = \inf_{\mu>0} \frac{a-1+\alpha_2 e^{2\mu}+\alpha_1 e^{\mu}+\beta_1 e^{-\mu}+\beta_2 e^{-2\mu}}{\mu} \tag{3.1}$$

and

$$c_-^* = \inf_{\mu>0} \frac{a-1+\alpha_2 e^{-2\mu}+\alpha_1 e^{-\mu}+\beta_1 e^{\mu}+\beta_2 e^{2\mu}}{\mu}. \tag{3.2}$$

Next, we will approximate the spreading speeds of the differential equation (1.4) by investigating the difference equation

$$\begin{aligned}\frac{\Delta x_i(t)}{\Delta t} =& \frac{x_i(t)-x_i(t-\Delta)}{\Delta t}\\ =& -x_i(t)+\alpha_2 f(x_{i-2}(t))+\alpha_1 f(x_{i-1}(t))+af(x_i(t))\\ &+\beta_1 f(x_{i+1}(t))+\beta_2 f(x_{i+2}(t)).\end{aligned} \tag{3.3}$$

In the following table, we will present the simulation of these spreading speeds under some parameters and the results calculated under the formula (3.1) and (3.2).

Parameters	$c_-^*(S)$	$c_+^*(S)$	$c_-^*(\Phi)$	$c_+^*(\Phi)$	Sign of c_-^*	Sign of c_+^*
$\alpha_2 = 0.05, \alpha_1 = 0.1,$ $a = 0.5, \beta_1 = 0.4,$ $\beta_2 = 0.8$	4.70	0.36	4.83	0.38	positive	positive
$\alpha_2 = 0.05, \alpha_1 = 0.1,$ $a = 0.2, \beta_1 = 0.4,$ $\beta_2 = 0.8$	4.07	-0.01	4.16	0.00	positive	zero
$\alpha_2 = 0.05, \alpha_1 = 0.1,$ $a = 0.1, \beta_1 = 0.4,$ $\beta_2 = 0.8$	3.83	-0.16	3.91	-0.15	positive	negative
$\alpha_2 = \alpha_1 = 0,$ $a = 1, \beta_1 = 0.1,$ $\beta_2 = 0.2$	1.36	0.00	1.41	0.00	positive	zero
$\alpha_2 = \alpha_1 = 0,$ $a = 0.5, \beta_1 = 0.2,$ $\beta_2 = 0.8$	3.86	-0.32	3.95	-0.32	positive	negative

Table 1. $c_\pm^*(S)$ and $c_\pm^*(\Phi)$ represent the spreading speeds calculated by simulation(see, e.g., [11, Section 4.2]) and the spreading speeds calculated by the formula (3.1) and (3.2), respectively.

We now give more analyses for these cases.

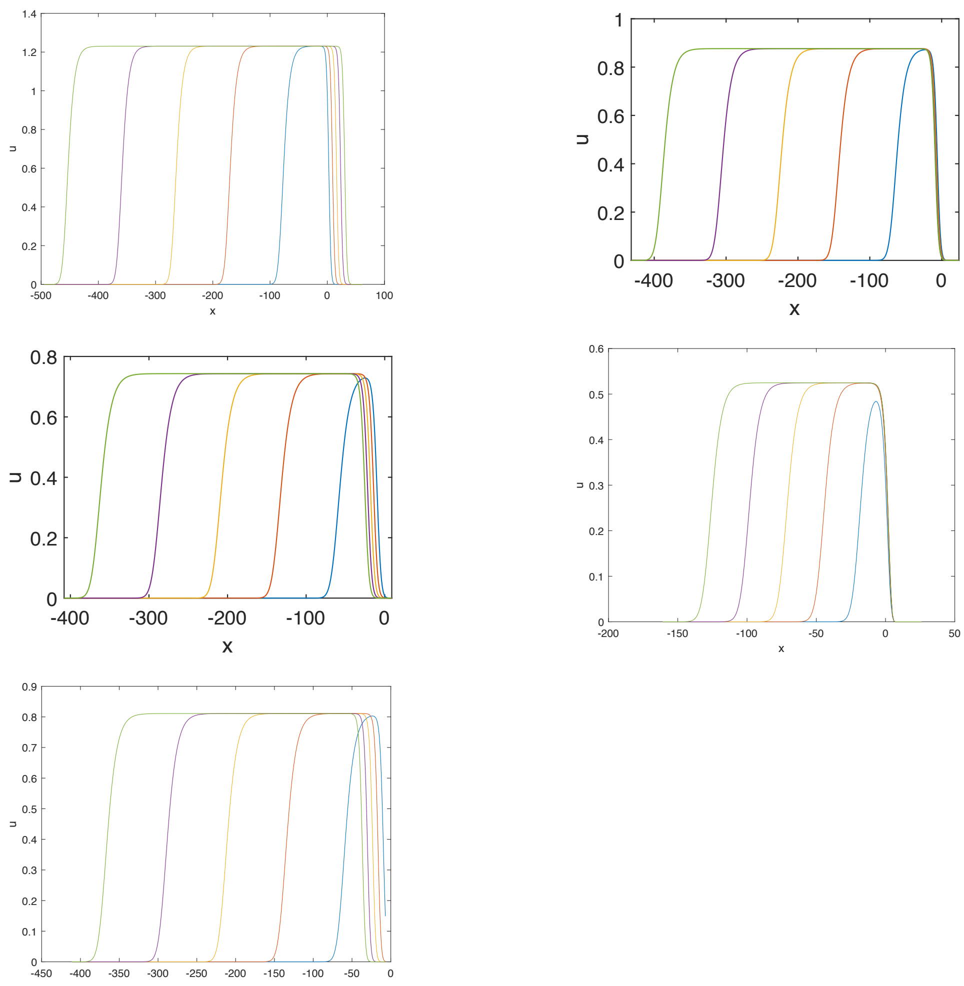

Figure 1. The three sets of parameters in the first row are $\alpha_2 = 0.05, \alpha_1 = 0.1, a = 0.5, \beta_1 = 0.4, \beta_2 = 0.8$, $\alpha_2 = 0.05, \alpha_1 = 0.1, a = 0.2, \beta_1 = 0.4, \beta_2 = 0.8$ and $\alpha_2 = 0.05, \alpha_1 = 0.1, a = 0.1, \beta_1 = 0.4, \beta_2 = 0.8$. The simulation results on the left show that the signal will transfer to both sides, the middle results show that the signal will transfer to the left side and stop on the right, and the results on the right show that the signal will transfer to the left and disappear on the right. The parameter to the left of the second row is $\alpha_2 = \alpha_1 = 0, a = 1, \beta_1 = 0.1, \beta_2 = 0.2$ and the simulation results indicate that the signal will transfer to the left side and stop on the right. The parameter to the right of the second row is $\alpha_2 = \alpha_1 = 0, a = 0.1, \beta_1 = 0.1, \beta_2 = 0.9$ and the simulation results imply that the signal will transfer to the left side and diminish on the right.

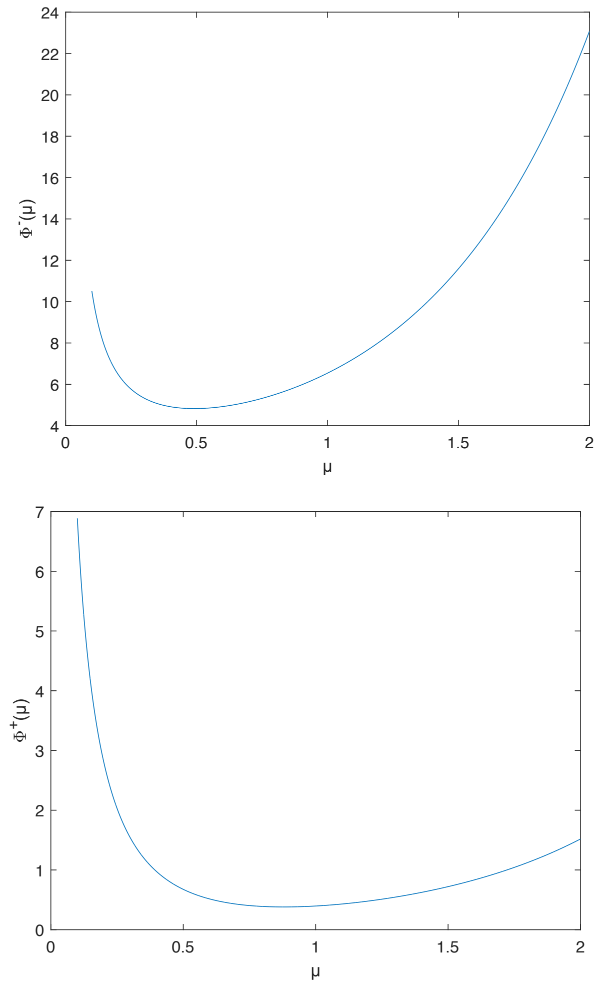

Figure 2. $\alpha_2 = 0.05, \alpha_1 = 0.1, a = 0.5, \beta_1 = 0.4, \beta_2 = 0.8$ and both of the infimum of $\Phi^-(\mu)$ and $\Phi^+(\mu)$ are positive and can be got at some finite value.

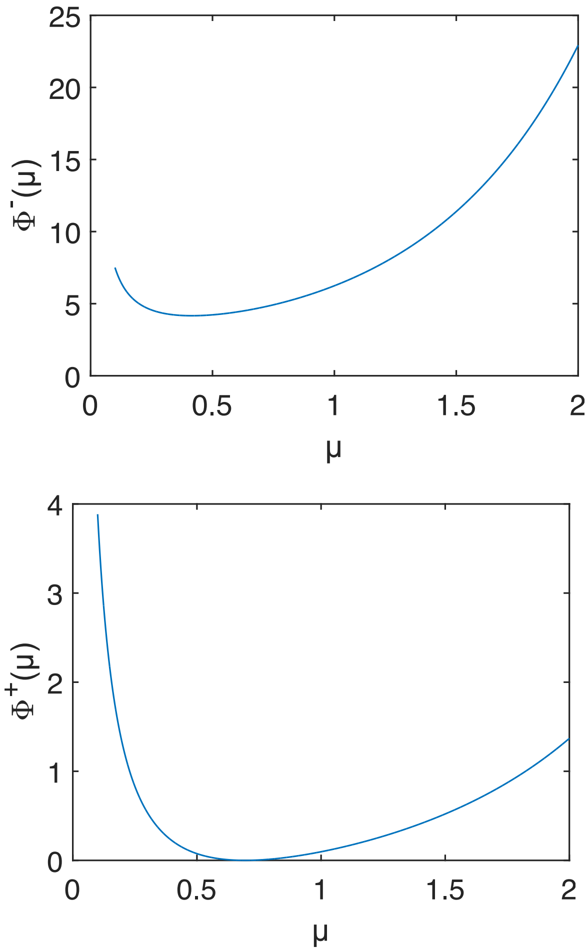

Figure 3. $\alpha_2 = 0.05, \alpha_1 = 0.1, a = 0.2, \beta_1 = 0.4, \beta_2 = 0.8$ and the infimum of $\Phi^-(\mu)$ is a positive number taken at some finite value, while the infimum of $\Phi^+(\mu)$ is zero at some finite value.

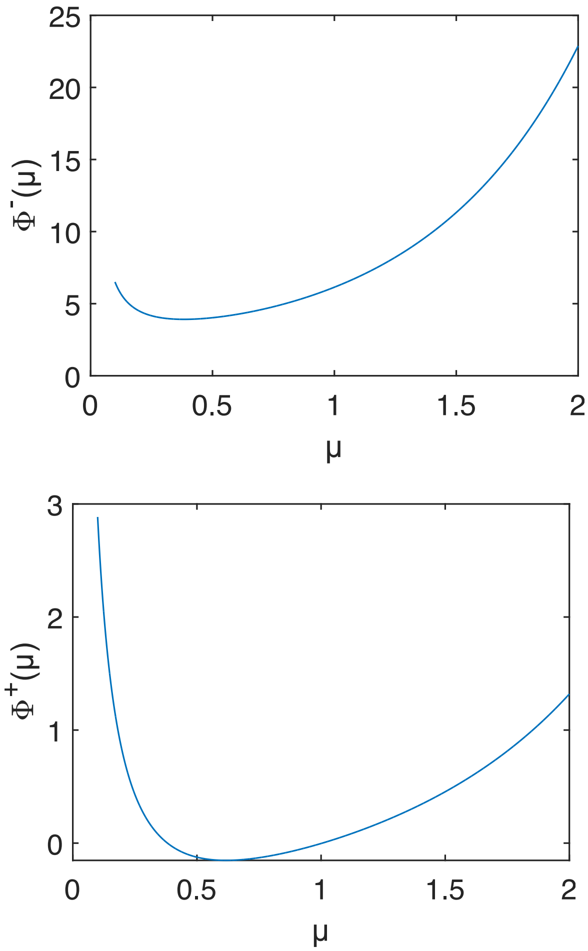

Figure 4. $\alpha_2 = 0.05, \alpha_1 = 0.1, a = 0.1, \beta_1 = 0.4, \beta_2 = 0.8$ and the infimum of $\Phi^-(\mu)$ is the positive number taken at some finite value, while the infimum of $\Phi^+(\mu)$ is the negative number taken at a finite value.

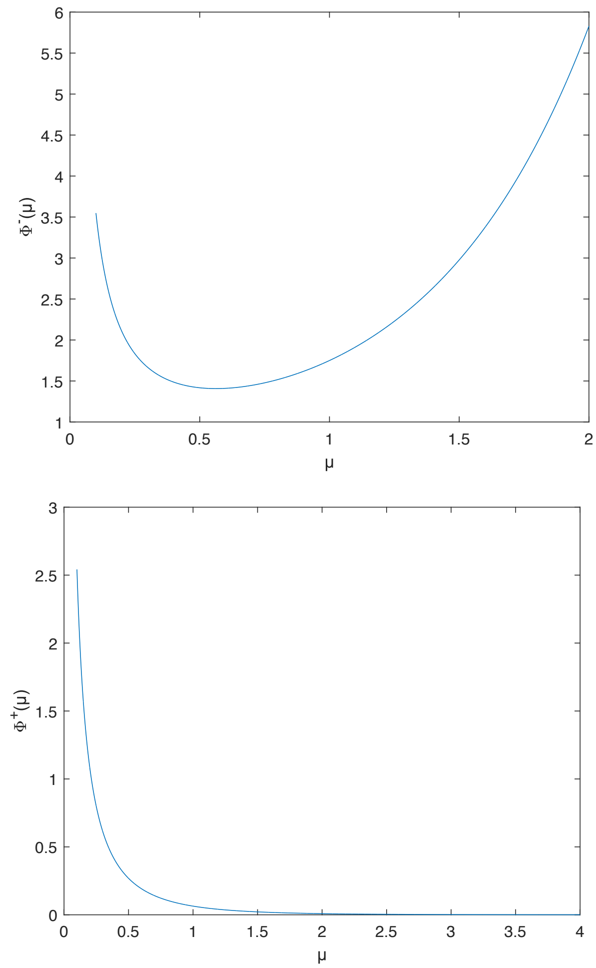

Figure 5. $\alpha_2 = \alpha_1 = 0, a = 1, \beta_1 = 0.1, \beta_2 = 0.2$ and the infimum of $\Phi^-(\mu)$ is positive at some finite value, while the infimum of $\Phi^+(\mu)$ is zero at infinity.

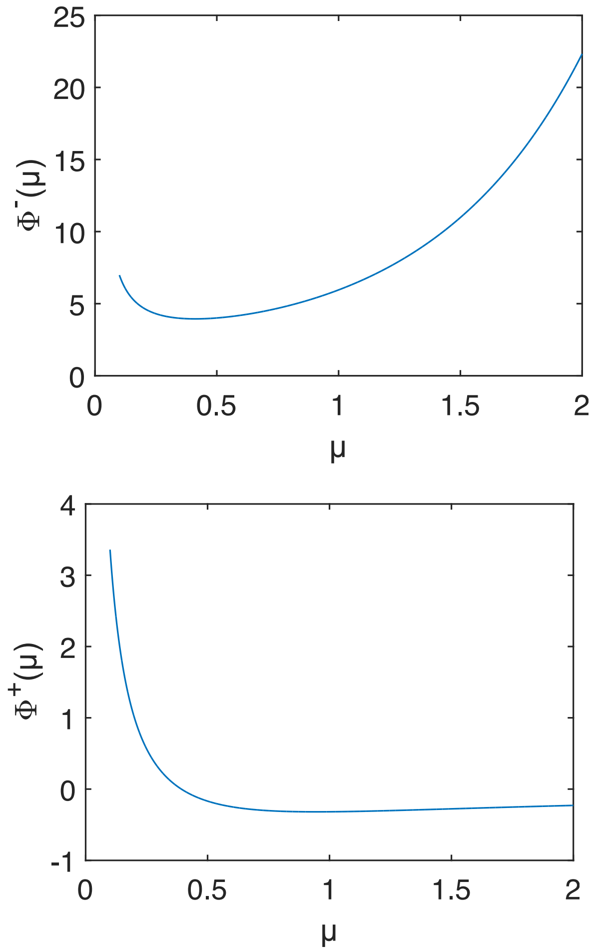

Figure 6. $\alpha_2 = \alpha_1 = 0, a = 0.1, \beta_1 = 0.1, \beta_2 = 0.2$ and the infimum of $\Phi^-(\mu)$ is positive at some finite value, while the infimum of $\Phi^+(\mu)$ is a negative number at some finite value.

Appendix

In this appendix, we will introduce some necessary notations, assumptions and theorems that will be used in the above research process. As mentioned in [18], we give the following hypotheses.

Let $\lambda(\mu)$ be a function in $C^2([0, +\infty))$ with the following properties.

(L1) $\lambda(\mu) > 0$ for any $\mu \in [0, +\infty)$.

(L2) $\lambda(0) > 1$.

(L3) $\ln \lambda(\mu)$ is convex with respect to $\mu \in [0, +\infty)$.

Then we can define $\Phi(\mu)$ and $\Psi(\mu)$ as follows:

$$\Phi(\mu) := \frac{\ln \lambda(\mu)}{\mu}, \ \mu \in (0, +\infty) \qquad \Psi(\mu) := \frac{\lambda'(\mu)}{\lambda(\mu)}, \ \mu \in [0, +\infty)$$

Let

$$c^* := \inf_{\mu>0} \Phi(\mu).$$

For the convenience, we denote

$$\Phi(+\infty) := \lim_{\mu\to+\infty} \Phi(\mu) \qquad \text{and} \qquad \Psi(+\infty) := \lim_{\mu\to+\infty} \Psi(\mu).$$

Without loss of generality, if $\Phi(+\infty)$ is a finite constant, we can assume that

(L4) $\Phi(+\infty) = 0$.

If $\lim_{\mu\to+\infty} \Phi(\mu)$ is infinite, it follows from Lemma 2.1 in [18] that $\lim_{\mu\to+\infty} \Phi(\mu) = +\infty$ and we set

(L4′) $\Phi(+\infty) = +\infty$.

Theorem A (Theorem 2.1 [18])**.** *Assume that* (L1)–(L3) *and* (L4) *hold. Then the following assertions hold:*

(i) If $\lambda(+\infty) \geq 1$, then $c^* = 0$.

(ii) If $\lambda(+\infty) < 1$, then $c^* < 0$.

For any $\mu \in [0, +\infty)$, let

$$h(\mu) = \ln \lambda(\mu), \qquad g(\mu) = h'(\mu)\mu - h(\mu).$$

Theorem B (Theorem 2.2 [18])**.** *Assume that* (L1)–(L3) *and* (L4) *hold. Then there is $\mu^* \in (0, +\infty)$ such that $g(\mu^*) = 0$. Moreover,*

(i) if $h(\mu^*) > 0$, then $c^* > 0$.

(ii) if $h(\mu^*) = 0$, then $c^* = 0$.

(iii) if $h(\mu^*) < 0$, then $c^* < 0$.

Let $\lambda_n \in C^2([0, +\infty))$ satisfies (L1)–(L3) and $\lambda_n(\mu)$ converges to $\lambda(\mu)$ as $n \to +\infty$ for any $\mu \in [0, +\infty)$ from above or below. So we have the following theorem.

Theorem C (Theorem 2.3 [18])**.** *Assume that $\lambda_n(\mu) \geq \lambda_{n+1}(\mu) \geq \lambda(\mu)$ or $\lambda_n(\mu) \leq \lambda_{n+1}(\mu) \leq \lambda(\mu)$ for all $n \geq 1$ and $\mu \in [0, +\infty)$. In addition, assume that $\lambda'_n(\mu)$ converges to $\lambda'(\mu)$ as $n \to +\infty$ for all $\mu \in [0, +\infty)$. Then $\lim_{n\to+\infty} c^*_n = c^*$.*

Corollary A (Corollary 2.1 [18])**.** *Assume that* (L1)–(L3) *and* (L4) *hold, and let $h_0 := \inf_{\mu>0} h(\mu) (= \min_{\mu\geq0} h(\mu))$. The following conclusions hold.*

(i) If $h_0 > 0$ then $c^* > 0$.

(ii) If $h_0 = 0$ then $c^* = 0$.

(iii) If $h_0 < 0$ then $c^* < 0$.

Acknowledgements

Thanks to the referee and editors for valuable suggestions and comments, which have improved our original manuscript.

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