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Hopf Cyclicity of a Class of Liénard-Type Systems*

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Abstract The Hopf cyclicities of some smooth polynomial, rational polynomial and piecewise smooth Liénard systems are studied. For two Liénard systems with the same damping term and different restoring (or potential) terms, we provide sufficient conditions that the two systems have the same Hopf cyclicity. Then, some examples are given to illustrate the efficiency and applicability of our results.

Keywords Hopf cyclicity, polynomial and rational polynomial Liénard systems, smooth and piecewise smooth systems.

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1. Introduction

Consider the following Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

which appears in many models of classical Newtonian mechanics. Usually, we call $f(x)$ a damping term and $g(x)$ a restoring term (or potential term). Historically, Liénard equation is not only closely related to a large number of practical applications (see e.g., [8, 20]), but also plays an important role in the theoretical studies of qualitative theory (see e.g., [1, 6, 10, 14, 16, 19, 21, 23, 26–29]).

The above equation (1.1) is equivalent to the following planar differential system (called Liénard system)

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{1.2}$$

where $F(x) = \int_0^x f(x)dx$. Here, we assume that $g(0) = 0$, $g'(0) > 0$, so that the origin is a center or focus.

In the qualitative theory of differential systems, an important open problem is to determine the maximum number of limit cycles bifurcating from a center or a focus, which is related to the local version of Hilbert's 16th problem. One way to study the limit cycles of the system (1.2) is the Hopf bifurcation, that is, to study the small-amplitude limit cycles. Usually, the maximum number of small limit cycles obtained by Hopf bifurcation is called Hopf cyclicity. When $f(x)$ and $g(x)$ are polynomials of degree n and m respectively, we denote the Hopf cyclicity

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of system (1.1) by $H_L^s(m,n)$. Many results on $H_L^s(m,n)$ are obtained by computing Lyapunov coefficients. However, due to the difficulty of the problem, most of the results are obtained under the strict assumptions of $F(x)$ or $g(x)$, for example, the degrees and coefficients in $F(x)$ or $G(x)$ are fixed, etc. The results of $H_L^s(m,n)$ are rarely reported for arbitrary values of m or n.

For $m=2$, $g(x)$ is a quadratic polynomial. Without loss of generality, we can assume system (1.2) has the form

$$\dot{x} = y - F(x), \quad \dot{y} = -x(x+1), \tag{1.3}$$

where

$$F(x) = F_1(x) = \sum_{i=1}^{n+1} a_i x^i.$$

Han [11, 12], Christopher and Lynch [4] independently investigated system (1.3), and obtained respectively that the Hopf cyclicity $\mathbb{H}_L^s(2,n) = \left[\frac{2n+1}{3}\right]$ at the origin for $n \geq 1$. Moreover, it was verified in [4] that $\mathbb{H}_L^s(2,n) = \mathbb{H}_L^s(n,2)$. When the damping term

$$F(x) = F_2(x) = \begin{cases} \sum_{i=1}^{n+1} a_i^+ x^i, & x > 0, \\ \sum_{i=1}^{l+1} a_i^- x^i, & x \leq 0 \end{cases}$$

in system (1.3) is a piecewise smooth polynomial in x of degree l and n, Tian and Han [24] obtained the cyclicity $\left[\frac{3l+2n+4}{3}\right]$ (resp., $\left[\frac{3n+2l+4}{3}\right]$) for $n \leq l$ (resp., $n \geq l$) of the origin.

In the case of $m=3$, according to the conditions $g(0)=0$ and $g'(0)>0$, let us assume that $g(x) = x + b_1 x^2 + b_2 x^3$. Christopher and Lynch [4] also obtained the following results, when $F(x) = F_1(x)$:

(i) if $b_1 = 0$, the cyclicity $\mathbb{H}_L^s(3,n) = \mathbb{H}_L^s(n,3) = \left[\frac{n}{2}\right], n \geq 1$;

(ii) if $b_1 \neq 0$, by scaling x and y simultaneously, $g(x) = x + x^2 + bx^3$, the cyclicity $\mathbb{H}_L^s(3,n) = \mathbb{H}_L^s(n,3) = 2\left[\frac{3(n+2)}{8}\right], 1 \leq n \leq 50$.

Tian, Han and Xu [25] studied the system with a special cubic restoring term

$$\dot{x} = y - F(x), \quad \dot{y} = -\frac{1}{2}x(x+1)(x+2). \tag{1.4}$$

It was proved that the Hopf cyclicity is $\mathbb{H}_L^s(3,n) = \left[\frac{3n+2}{4}\right]$ (resp., $\left[\frac{2l+n+2}{2}\right]$ as $l \geq n$ or $\left[\frac{2n+l+2}{2}\right]$ as $l \leq n$), if $F(x) = F_1(x)$ (resp., $F_2(x)$), $n \geq 1$.

Recently, Sun and Yu [22] have considered a Liénard system with a quintic restoring term, which is equivalent to the following form

$$\dot{x} = y - F(x), \quad \dot{y} = x\left(x + \frac{1}{2}\right)(x-1)^3, \tag{1.5}$$

where $F(x) = F_1(x)$ or $F_2(x)$. The Hopf cyclicity of system (1.5) at the origin is $\mathbb{H}_L^s(5,n) = \left[\frac{2n+1}{3}\right]$ (resp., $\left[\frac{3l+2n+4}{3}\right]$, if $n \leq l$ or $\left[\frac{3n+2l+4}{3}\right]$, if $n \geq l$) for $F(x) = F_1(x)$ (resp., $= F_2(x)$), $n \geq 1$.

Notice that systems (1.3) and (1.5) have the same Hopf cyclicity at the origin for $F(x) = F_1(x)$ or $F_2(x)$. Is this a coincidence or does it contain some structural rules?

Inspired by the above observation, we investigate the following two Liénard systems

$$\dot{x} = y - F(x, \mathbf{a}), \quad \dot{y} = -g_1(x) \tag{1.6}$$

and

$$\dot{x} = y - F(x, \mathbf{a}), \quad \dot{y} = -g_2(x), \tag{1.7}$$

where $\mathbf{a}$ is a parameter vector, and F, g_1, g_2 are polynomials satisfying $F(x, \mathbf{a}) = F_1(x)$ or $F_2(x)$, $g_i(0) = 0$, and $g_i'(0) > 0$ for $i = 1, 2$. Thus, the origin is either an elementary center or an elementary focus point of the above two systems.

In addition, if the restoring terms $g_1(x)$ and $g_2(x)$ are piecewise smooth, such cases also deserve our attention. At this point, we consider the following two systems

$$\dot{x} = y - F(x, \mathbf{a}), \quad \dot{y} = -\widetilde{g}_1(x) \tag{1.8}$$

and

$$\dot{x} = y - F(x, \mathbf{a}), \quad \dot{y} = -\widetilde{g}_2(x), \tag{1.9}$$

where $F(x, \mathbf{a}) = F_1(x)$ or $F_2(x)$ with

$$F_1(x) = F_1^{\pm}(x) = \sum_{i=1}^{n+1} a_i x^i$$

or

$$F_2(x) = \begin{cases} F_2^+(x) = \sum_{i=1}^{n+1} a_i^+ x^i, & x > 0, \\ F_2^-(x) = \sum_{i=1}^{l+1} a_i^- x^i, & x \le 0. \end{cases}$$

$\widetilde{g}_i(x) = \begin{cases} g_i^+(x), & x > 0, \\ g_i^-(x), & x \le 0, \end{cases}$ for $i = 1, 2$. $g_1^{\pm}$ and $g_2^{\pm}$ are polynomials and satisfy $g_i^{\pm}(0) = 0$, $(g_i^{\pm})'(0) > 0$, $\left((F_i^{\pm})_x(0, \mathbf{a}_0)\right)^2 - 4\left(g_i^{\pm}\right)'(0) < 0$, for $i = 1, 2$. Then, the origins of systems (1.8) and (1.9) are FF type singularities, where "F" means "focus" (see [5, 9, 13]). That is, for any one of the systems, subsystem $x > 0$ and subsystem $x \le 0$ are the focus at the origin, and they have the same orientation.

Recently, there has been a growing interest in the cyclicity of rational Liénard systems (see e.g., [2, 15, 17, 24]). Jiang and Han [15] considered a rational Liénard system,

$$\dot{x} = y - \frac{q_n(x)}{p_m(x)}, \quad \dot{y} = -g(x), \tag{1.10}$$

where $p_m(x) = 1 + \sum_{i=1}^m b_i x^i$, $q_n(x) = \sum_{i=0}^n a_i x^i$, while $g(x)$ is an odd function with any degree. It was proved that the cyclicity is $\left[\frac{m+n-1}{2}\right]$ at the singular point $(0, a_0)$. Li and Han [17] obtained that the cyclicity of system (1.10) with an odd function $g(x)$ at the point $(0, a_0)$ is $\min\{n, \left[\frac{m+n-1}{2}\right]\}$ (resp., $\left[\frac{n+1}{2}\right]$ as $\alpha + \beta \ne 0$ or $\left[\frac{n-1}{2}\right]$ as $\alpha + \beta = 0$) if $p_m(x) = (1-x)^m$ (resp., $p_m(x) = (1-\alpha x)(1-\beta x)$, $\alpha\beta \ne 0$).

It is worth noting that as for the results of cyclicity of rational Liénard systems, there are always some conditions imposed on the restoring term $g(x)$ such as the odd function condition in [15, 17]. The question is why the cyclicity of the system does not change for any odd function $g(x)$. Therefrom, we consider the following two rational Liénard systems and provide the condition so that they have the same cyclicity at the singular point $(0, a_0)$

$$\dot{x} = y - \frac{q_n(x)}{p_m(x)}, \quad \dot{y} = -g_1(x) \tag{1.11}$$

and

$$\dot{x} = y - \frac{q_n(x)}{p_m(x)}, \quad \dot{y} = -g_2(x), \tag{1.12}$$

where $p_m(x) = 1 + \sum_{i=1}^{m} b_i x^i$ and $q_n(x) = \sum_{i=0}^{n} a_i x^i$. g_1 and g_2 are polynomials satisfying $g_i(0) = 0$ and $g_0'(0) > 0$.

In this paper, we will provide sufficient conditions that two systems have the same Hopf cyclicity at the singular point. Our result is stated in the following theorem.

Theorem 1.1. *Assume that $g_i(0) = 0$ and $g_i'(0) > 0$ for $i = 1, 2$. For systems* (1.6) *and* (1.7) *with $F(x, \mathbf{a}) = F_1(x)$ or $F_2(x)$, if there exists some polynomial $A(x)$ with $A'(0) = 0$, $A''(0) \neq 0$ and polynomial functions $f_i(\cdot)$ (i=1,2) such that $g_i(x) = A'(x) f_i(A(x))$, then* (1.6) *and* (1.7) *have the same Hopf cyclicity at the origin.*

For the systems with discontinuous polynomial restoring terms, the result is as follows.

Theorem 1.2. *Suppose that $g_i^{\pm}(0) = 0$ and $(g_i^{\pm})'(0) > 0$ for $i = 1, 2$. Consider piecewise smooth systems* (1.8) *and* (1.9) *with $F(x, \mathbf{a}) = F_1(x)$ or $F_2(x)$. If there exist polynomials $A(x)$ and $B(x)$ with $A'(0) = B'(0) = 0$, $A''(0)B''(0) \neq 0$ and polynomial functions $\widetilde{f}_i(\cdot)$ such that $g_i^+(x) = A'(x)\widetilde{f}_i(A(x))$ and $g_i^-(x) = B'(x)\widetilde{f}_i(B(x))$ for $i = 1, 2$, then the two systems have the same Hopf cyclicity at the origin.*

For rational Liénard systems, we have the following theorem.

Theorem 1.3. *Suppose that $g_i(0) = 0$ and $g_i'(0) > 0$ for $i = 1, 2$. For rational Liénard systems* (1.11) *and* (1.12)*, if there exists some polynomial $A(x)$ with $A'(0) = 0$, $A''(0) \neq 0$ and polynomial functions $f_i(\cdot)$ (i=1,2) such that $g_i(x) = A'(x) f_i(A(x))$, then* (1.11) *and* (1.12) *have the same Hopf cyclicity at the singular point* $(0, a_0)$.

Let $G_i(x) = \int_0^x g_i(s)ds$ and $G_i^{\pm}(x) = \int_0^x g_i^{\pm}(s)ds$ for $i = 1, 2$.

Remark 1.1. The conclusion is also valid, if the conditions $g_i(x) = A'(x)f_i(A(x))$, $g_i^+(x) = A'(x)\widetilde{f}_i(A(x))$ and $g_i^-(x) = B'(x)\widetilde{f}_i(B(x))$ in Theorems 1.1-1.3 are replaced by the integral forms $G_i(x) = f_i(A(x))$, $G_i^+(x) = \widetilde{f}_i(A(x))$ and $G_i^-(x) = \widetilde{f}_i(B(x))$ respectively. In fact, there are the same displacement functions near the singular point.

This paper is organized as follows. In section 2, we will introduce some preliminaries and basic lemmas about the calculation of Lyapunov coefficients of smooth and piecewise smooth polynomial Liénard systems. We will prove our main results: Theorems 1.1-1.3 in section 3. The last section will provide two examples to illustrate the efficiency and sufficiency of our results.

2. Preliminaries and basic lemmas

One classical method to solve center-focus problems is based on computing Lyapunov constants. Due to the difficulty of calculating Lyapunov coefficients, many researchers try to explore new calculation methods. For simplicity, we only introduce some methods of studying Liénard systems involved in this paper.

Christopher [3] obtained a condition that the origin of system (1.2) is the center by the following algebraic method. Let us consider a field $\mathcal{F}$, which is the subfield of $\mathbb{R}(x)$ generated by the polynomials G_1 and G_2. The following lemma in [3] shows that $\mathcal{F}$ shares an important algebraic property with G_1 and G_2.

Lemma 2.1 ([3]). *Suppose that there exists an analytic function $z(x)$ with $z(0) = 0$, $z'(0) < 0$, such that both $G_1(z(x)) = G_1(x)$ and $G_2(z(x)) = G_2(x)$ are in a neighborhood of $x = 0$. Then, for all elements H of the field $\mathcal{F}$ generated by G_1 and G_2, we have $H(z(x)) = H(x)$, which is considered as meromorphic functions of x about $x = 0$. Furthermore, G_1 and G_2 are polynomials of some polynomial $A \in \mathcal{F}$.*

In the following lemma, we obtain an algebraic equivalent characterization that two systems (1.6) and (1.7) have the same analytic function at the origin.

Lemma 2.2. *Suppose that $g_i(0) = 0$ and $g_i'(0) > 0$ for $i = 1, 2$. Systems (1.6) and (1.7) have the same analytic function $z(x)$ defined by $G_i(z(x)) \equiv G_i(x)$ with $z(0) = 0$, $z'(0) < 0$, if and only if G_1 and G_2 are polynomials of some polynomial $A \in \mathcal{F}$ with $A'(0) = 0$, and $A''(0) \neq 0$.*

Proof. On one hand, if there is an analytic function $z(x)$ with $z(0) = 0$, $z'(0) < 0$ such that $G_1(z(x)) \equiv G_1(x)$ and $G_2(z(x)) \equiv G_2(x)$, then by Lemma 2.1, G_1 and G_2 are polynomials of some polynomial $A \in \mathcal{F}$ satisfying $A(z) = A(x)$. Since $g_i(0) = 0$ and $g_i'(0) > 0$, it is clear that $A'(0) = 0$ and $A''(0) \neq 0$ because of $G_i'(0) = 0$ and $G_i''(0) > 0$.

On the other hand, if we have $G_1(x) = \sum_{i=1}^{l_1} c_i A^i(x)$ and $G_2(x) = \sum_{i=1}^{l_2} \widetilde{c}_i A^i(x)$, where c_i and $\widetilde{c}_i$ are real constants, then from the conditions $g(0) = 0$ and $g'(0) > 0$, it follows that $c_1\widetilde{c}_1 \neq 0$. Hence,

$$G_i(z_i) - G_i(x) = (A(z_i) - A(x))\, Q_i(x, z_i), \quad i = 1, 2,$$

where $Q_i(x, z_i) = \sum_{r=1}^{l_i} \sum_{s=0}^{r-1} c_r A^{r-1-s}(x) A^s(z_i) \neq 0$, $i = 1, 2$ for $0 < x \ll 1$. It is apparent that $G_i(z) \equiv G_i(x)$. Since $A'(0) = 0$ and $A''(0) \neq 0$, there exists an analytic function $z(x)$ such that $A(z) = A(x)$ with $z(0) = 0$ and $z'(0) < 0$. □

Furthermore, we provide a sufficient condition, which can determine that systems (1.8) and (1.9) have the same analytic function at the origin, and let $G_i^{\pm}(x) = \int_0^x g_i^{\pm}(s)ds$.

Lemma 2.3. *Suppose that $g_i(0) = 0$ and $g_i'(0) > 0$ for $i = 1, 2$. If there exist polynomials $A(x)$ and $B(x)$ with $A'(0) = B'(0) = 0$, $A''(0)B''(0) \neq 0$ and polynomial functions $\widetilde{f}_i(\cdot)$ such that $g_i^+(x) = A'(x)\widetilde{f}_i(A(x))$ and $g_i^-(x) = B'(x)\widetilde{f}_i(B(x))$ for $i = 1, 2$, then one can find an analytic function $z(x)$ with $z(0) = 0$ and $z'(0) < 0$ satisfying $G_i^-(z) \equiv G_i^+(x)$ for $i = 1, 2$. However, the reverse does not hold.*

Proof. If there exist polynomials $A(x)$ and $B(x)$ satisfying $A'(0) = B'(0) = 0$, $A''(0)B''(0) \neq 0$, such that $G_1^+(x) = \sum_{i=1}^{l_1} d_i A^i(x)$, $G_1^-(x) = \sum_{i=1}^{l_1} d_i B^i(x)$,

$G_2^+(x) = \sum_{i=1}^{l_2} \widetilde{d}_i A^i(x)$ and $G_2^-(x) = \sum_{i=1}^{l_2} \widetilde{d}_i B^i(x)$, where d_i and $\widetilde{d}_i$ are real constants. Obviously, $d_1\widetilde{d}_1 \neq 0$. Note that

$$G_i^-(z_i) - G_i^+(x) = (B(z_i) - A(x))\,\widetilde{Q}_i(x, z_i), \quad i = 1, 2,$$

where $\widetilde{Q}_i(x, z_i) = \sum_{r=1}^{l_i} \sum_{s=0}^{r-1} d_r A^{r-1-s}(x) B^s(z_i) \neq 0$, $i = 1, 2$ for $0 < x \ll 1$. By $A'(0) = B'(0) = 0$ and $A''(0) = B''(0) \neq 0$, we can get an analytic function $z(x)$ satisfying $B(z) = A(x)$ with $z(0) = 0$ and $z'(0) < 0$. Hence, $G_i^-(z) \equiv G_i^+(x)$, for $i = 1, 2$.

In order to show that the reverse is incorrect, $G_1^+(x) = G_1^-(x) = x^2$, $G_2^+(x) = x^2 - x^3$ and $G_2^-(x) = x^2 + x^3$ are set. It follows that there is an analytic function $z(x) = -x$ with $z(0) = 0$ and $z'(0) < 0$ satisfying $G_i^-(z) \equiv G_i^+(x)$, for $i = 1, 2$. However, there does not exist polynomials $A(x)$ or $B(x)$ such that $G_i^+(x)$ (resp. $G_i^-(x)$) is generated by $A(x)$ (resp. $B(x)$). □

For smooth Liénard type systems, one equivalent method to computing the Lyapunov coefficients was developed in [11, 12]. Consider the generalized Liénard system

$$\dot{x} = p(y) - F(x, \mu), \quad \dot{y} = -g(x), \tag{2.1}$$

where μ is a parameter vector. $p(y), g(x)$ and $F(x, \mu)$ are analytic functions. In order to ensure that the origin is a center or focus of system (2.1), it is necessary to assume that the following conditions hold,

$$p(0) = F(0, \mu) = g(0) = 0, p'(0) > 0, g'(0) > 0 \text{ and } F_x(0, \mu^*) = 0. \tag{2.2}$$

From [11,12], one can construct the displacement function of system (2.1) locally around the origin, which has the expansion as follows,

$$d(r, \mu) = \sum_{i=1}^{\infty} d_i(\mu) r^i, \text{ for } |r| \ll 1 \text{ and } |\mu - \mu^*| \ll 1,$$

where $d_i(\mu) \in C^\infty$. Let $G(x) = \int_0^x g(s)ds$ and suppose that

$$F(z(x), \mu) - F(x, \mu) = \sum_{i=1}^{\infty} B_i(\mu) x^i, \quad |x| \ll 1, \tag{2.3}$$

where $z(x) = -x + O(x^2)$ is the analytic function defined by $G(z(x)) \equiv G(x)$. It was proved in [11, 12] that the following result holds.

Lemma 2.4 ([11, 12]). *Let* (2.2) *hold. Assume that* (2.3) *holds formally. Then, we have*

$$d(r, \mu) = \sum_{i=1}^{\infty} d_i(\mu) r^i,$$

where

$$\begin{aligned} d_1 &= N_1(B_1) B_1, \\ d_{2j} &= O(|B_1, B_3, \cdots, B_{2j-1}|), \\ d_{2j+1} &= N_j(B_1) B_{2j+1} + O(|B_1, B_3, \cdots, B_{2j-1}|), \end{aligned}$$

with $N_j \in C^\infty$ *and* $N_j(0) > 0$.

The above expression relationship shows that we only need to calculate B_i and analyze its algebraic variety $\{B_i = 0\}$ $(i \geq 1)$ to obtain the cyclicity. The corresponding results can be seen in [11, 12].

For non-smooth Liénard system, Liu and Han [18] extended the method in [11, 12] to study the Hopf bifurcation of the following system

$$(\dot{x}, \dot{y}) = \begin{cases} \left(p(y) - F^+(x,\mu), -g^+(x)\right), \, x \geq 0, \\ \left(p(y) - F^-(x,\mu), -g^-(x)\right), \, x \leq 0, \end{cases} \tag{2.4}$$

where $\mu \in \mathbb{R}^n$ is an n-dimensional parameter vector. $p(y), g^{\pm}(x)$ and $F^{\pm}(x)$ are analytic functions satisfying

$$\begin{aligned} &p(0) = F^{\pm}(0,\mu) = g^{\pm}(0) = 0, p'(0) > 0, \\ &(g^{\pm})'(0) > 0 \text{ and } \left(F_x^{\pm}(0,\mu^*)\right)^2 - 4p'(0)\left(g^{\pm}\right)'(0) < 0. \end{aligned} \tag{2.5}$$

Let (2.4) satisfy (2.5). It is proved that the origin is a fine or weak focus for $\mu = \mu^*$, if and only if

$$\sqrt{(g^-)'(0)}F_x^+(0,\mu^*) + \sqrt{(g^+)'(0)}F_x^-(0,\mu^*) = 0. \tag{2.6}$$

It should be noted that the displacement function $d(r,\mu)$ is still analytic, which can be obtained as

$$d(r,\mu) = \sum_{i=1}^{\infty} \widetilde{d}_i(\mu) r^i,$$

for some $|r| \ll 1$ and $|\mu - \mu^*| \ll 1$. Let $G^{\pm}(x) = \int_0^x g^{\pm}(s)ds$. Assume that

$$F^-(z(x),\mu) - F^+(x,\mu) = \sum_{j=1}^{\infty} B_j(\mu) x^j, \quad 0 < x \ll 1, \tag{2.7}$$

where $z(x) = -\sqrt{\left(g^+\right)'(0)/\left(g^-\right)'(0)}x + O\left(x^2\right)$ is the analytic function defined by $G^{\pm}(x) = \int_0^x g^{\pm}(s)ds$ satisfying $G^-(z(x)) \equiv G^+(x)$.

The following lemma establish a relationship between the coefficients B_j and $\widetilde{d}_j$.

Lemma 2.5 ([18]). *Suppose that* (2.5) *and* (2.6) *hold. Then, we have formally*

$$d(r,\mu) = \sum_{i=1}^{\infty} \widetilde{d}_i(\mu) r^i,$$

where

$$\begin{aligned} \widetilde{d}_1 &= N_1^*(\mu) B_1, \\ \widetilde{d}_j &= N_j^*(\mu) B_j + O\left(|B_1, B_2, \cdots, B_{j-1}|\right) \end{aligned}$$

with $N_j^* \in C^{\infty}$ *and* $N_j^*(0) > 0$, *for* $j \geq 1$.

3. Proof of the main results

Proof of Theorem 1.1. Following the ideas of [7], we define an invertible analytic transformation in a neighborhood of $x = 0$,

$$u_i = \sqrt{2G_i(x)}\,\mathrm{sgn}(x), \quad u_i(0) = 0, \quad u_i'(0) > 0, \tag{3.1}$$

where $G_i(x) = \int_0^x g_i(s)ds$ for $i = 1, 2$. Denote by $x = x(u_i)$ the inverse of the transformation (3.1). Then, we have

$$2G_i(x(u_i)) = u_i^2 = 2G_i(x(-u_i)), \quad x(u_i) = x, \quad i = 1, 2.$$

Let $z_i(x) = x(-u_i)$. Hence,

$$G_i(x) \equiv G_i(z_i(x)), \quad i = 1, 2.$$

It follows that

$$0 = G_i(x) - G_i(z_i) = (x - z_i)\left(\frac{1}{2}g_i'(0)(x + z_i) + o(x, z_i)\right). \tag{3.2}$$

Note that $z_i(0) = 0$ and $z_i'(0) < 0$. Combining the formula (3.2) and the conditions on $g_i(x)$, we have the following unique analytic expressions for $z_1(x)$ and $z_2(x)$ respectively

$$z_1(x) = -x + o(x) \quad \text{and} \quad z_2(x) = -x + o(x), \quad |x| \ll 1.$$

One can see [11, 12] for more details.

It can be seen that systems (1.6) and (1.7) have the same $F(x)$ term. According to Lemmas 2.4 and 2.5, if the two systems have the same analytical function satisfying (3.2), then they have the same Hopf cyclicity at the origin. In fact, according to the conditions in Theorem 1.1 and Lemma 2.2, we can obtain that systems (1.6) and (1.7) have the same analytical function z. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. It is similar to the previous proof of Theorem 1.1. Motivated by the idea of [18], we have the following transformation

$$u_i = \sqrt{2G_i^{\pm}(x)}\,\mathrm{sgn}(x), \tag{3.3}$$

where $G_i^{\pm}(x) = \int_0^x g_i^{\pm}(s)ds$ for $i = 1, 2$. Then,

$$2G_i^{+}(x(u_i)) = u_i^2 = 2G_i^{-}(x(-u_i)), \quad x(u_i) = x, \quad i = 1, 2.$$

Let $z_i(x) = x(-u_i)$, which implies that

$$G_i^{+}(x) \equiv G_i^{-}(z_i(x)), \quad i = 1, 2.$$

As a result,

$$z_1(x) = -\frac{\sqrt{(g_1^{+})'(0)}}{\sqrt{(g_1^{-})'(0)}}x + o(x) \quad \text{and} \quad z_2(x) = -\frac{\sqrt{(g_2^{+})'(0)}}{\sqrt{(g_2^{-})'(0)}}x + o(x), \quad |x| \ll 1.$$

If there exist polynomials $A(x)$ and $B(x)$ with $A'(0) = B'(0) = 0$, $A''(0)B''(0) \neq 0$ and polynomial functions $\widetilde{f}_i(\cdot)$ such that $g_i^+(x) = A'(x)\widetilde{f}_i(A(x))$ and $g_i^-(x) = B'(x)\widetilde{f}_i(B(x))$ for $i = 1, 2$, we obtain that systems (1.8) and (1.9) have the same analytical function $z = z_1 = z_2$, by applying Lemma 2.3 for $|x| \ll 1$. According to Lemma 2.5, if systems (1.8) and (1.9) have the same analytical function $z = z_1 = z_2$ for $|x| \ll 1$, they also have the same Hopf cyclicity at the origin. This completes the proof of Theorem 1.2.

Considering $F(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1+\sum_{i=1}^{m} b_i x^i} \in C^\infty$ at the origin, the proof method of Theorem 1.1 is also applicable for Theorem 1.3 (we omit the proof here).

4. Applications

In this section, we first analyze why the two systems (1.3) and (1.5) mentioned in the introduction have the same Hopf cyclicity. Secondly, we explain that the conditions of Theorems 1.1-1.3 are sufficient but not necessary.

From Lemma 2.2, one can see that if we want to find G_1 and G_2 with the same analytical function z, we can equivalently find the $A \in \mathcal{F}$ with $A'(0) = 0$, $A''(0) \neq 0$ such that G_1 and G_2 are polynomials of A.

For systems (1.3) and (1.5) at the origin, by a simultaneous scaling of the x- and y- axes respectively, it is easy to change them into the following system

$$\dot{x} = y - \sum_{i=1}^{n+1} a_i x^i, \quad \dot{y} = -x(1-x). \tag{4.1}$$

We note that such a scaling respects the weights of the Liapunov quantities. Therefore, it has no effect on the dynamics. Let

$$A(x) := G_1(x) = \int_0^x s(1-s)ds = \frac{1}{2}x^2 - \frac{1}{3}x^3. \tag{4.2}$$

Hence, $A'(0) = 0$ and $A''(0) \neq 0$.

Reviewing system (1.5) in [22] with $F = F_1$ or F_2, we obtain

$$\begin{aligned} G_2(x) &= \int_0^x s\left(s+\frac{1}{2}\right)(1-s)^3 ds = \frac{1}{4}x^2 - \frac{1}{6}x^3 - \frac{3}{8}x^4 + \frac{1}{2}x^5 - \frac{1}{6}x^6 \\ &= \frac{1}{2}A(x) - \frac{3}{2}A^2(x). \end{aligned} \tag{4.3}$$

It is clearly seen from (4.2) and (4.3) that $G_1(x)$ and $G_2(x)$ have the same polynomial factor $A(x)$ with $A(0) = 0$ and $A''(0) \neq 0$, which shows that systems (1.5) and (4.1) have the same analytical function z, and they have the same cyclicity using Theorem 1.1.

If $A'(0) = 0$, $A''(0) \neq 0$ and $\deg A(x) = 2$ hold, it follows that

$$g_i(-x) = -g_i(x) \Longleftrightarrow g_i = A'(x)f_i(A(x)), \text{ for } i = 1, 2,$$

which implies that two rational Liénard systems have the same cyclicity by using Theorem 1.1 or 1.3. When $\deg A(x) \geq 3$, for the convenience of readers to understand and apply Theorems 1.1-1.3, we present the following examples.

Example 4.1. Consider

$$\dot{x} = y - F(x), \quad \dot{y} = -A'(x)\sum_{i=0}^{k} b_i A^i(x), \tag{4.4}$$

where $A(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$, $F(x) = F_1(x)$ or $F_2(x)$. b_i are real constants, and $A''(0)b_0 > 0$. If $F(x) = F_1(x)$, the Hopf cyclicity of the origin is $\left[\frac{2n+1}{3}\right]$. If $F(x) = F_2(x)$, the Hopf cyclicity of the origin is $\left[\frac{3l+2n+4}{3}\right]$ for $l \geq n$, and $\left[\frac{3n+2l+4}{3}\right]$ for $n \geq l$.

In [24], Tian and Han investigated rational Liénard system (1.10) with a quadratic polynomial restoring $g(x) = x(x+1)$. It was obtained that an upper bound for the cyclicity of the point $(0, a_0)$ is $\left[\frac{4n+2m-4}{3}\right] - \left[\frac{n-m}{3}\right]$ (resp., $\left[\frac{4m+2n-4}{3}\right] - \left[\frac{m-n}{3}\right]$) if $n \geq m$ (resp., $m > n$) for arbitrary m and n.

Example 4.2. The system with restoring term of degree $3k+2$ is given by

$$\dot{x} = y - \frac{q_n(x)}{p_m(x)}, \quad \dot{y} = -A'(x)\sum_{i=0}^{k} b_i A^i(x), \tag{4.5}$$

where $q_n(x)$ and $p_m(x)$ satisfy condition (1.2), $A(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$, and b_i are real constants, $b_0 \neq 0$. An upper bound for the cyclicity of the singular point $(0, a_0)$ is $\left[\frac{4n+2m-4}{3}\right] - \left[\frac{n-m}{3}\right]$ (resp., $\left[\frac{4m+2n-4}{3}\right] - \left[\frac{m-n}{3}\right]$), if $n \geq m$ (resp., $m > n$) for arbitrary m and n.

In particular, taking $k = 1$, $b_0 = \frac{1}{2}$, $b_1 = -3$ and $b_{i\geq 2} = 0$, the estimation of cyclicity of the rational Liénard system with a quintic restoring term

$$g(x) = -x\left(x + \frac{1}{2}\right)(x-1)^3 \tag{4.6}$$

can be obtained immediately. The cyclicity of polynomial Liénard with this restoring term (4.6) has been studied in [22] mentioned above.

By a simultaneous scaling of the x- and y- axes respectively, it is easy to change $g(x) = x(x+1)$ into $g(x) = x(x-1)$, which has no effect on their dynamics. Let $g_1(x) = x(x-1)$ and $g_2(x) = A'(x)\sum_{i=0}^{k} b_i A^i(x)$. It follows that $G_1(x)$ and $G_2(x)$ have the same polynomial factor $A(x)$ with $A(0) = 0$ and $A''(0) \neq 0$. According to Theorem 1.3, we obtain that the rational Liénard system in [24] and system (4.5) have the same cyclicity.

Note that the the reverse of Lemma 2.3 is not true. Consequently, the conditions in Theorem 1.2 are sufficient rather than necessary. To illustrate that the conditions in Theorem 1.1 are sufficient rather than necessary, we consider the Hopf cyclicity of the following two Liénard systems at the origin, in which the degree of damping terms are fixed and restoring terms with degree 11 respectively.

Example 4.3. Consider the two systems below

$$\dot{x} = y - F(x), \quad \dot{y} = -\frac{1}{54}x(x-1)(8x^9 - 36x^8 + 54x^7 - 27x^6 - 54) \tag{4.7}$$

and

$$\dot{x} = y - F(x), \quad \dot{y} = -\frac{1}{384}x(x+1)(x+2)(3x^4 + 12x^3 + 12x^2 - 8)^2, \tag{4.8}$$

where $F(x) = F_1(x)$ or $F_2(x)$ with

$$F_1(x) = \sum_{i=1}^{5} \alpha_i x^i,$$
$$F_2(x) = \begin{cases} \sum_{i=1}^{3} \alpha_i^+ x^i, & x \geq 0, \\ \sum_{i=1}^{4} \alpha_i^- x^i, & x \leq 0. \end{cases}$$

We claim that at the origin that systems (4.7) and (4.8) have the same Hopf cyclicity 3 (resp., 5), if $F = F_1$ (resp., $F = F_2$).

In fact, the following relationship

$$-\frac{1}{54}x(x-1)(8x^9 - 36x^8 + 54x^7 - 27x^6 - 54) = -A'(x)\sum_{i=0}^{3} b_i A^i(x)$$

is satisfied, where $b_0 = 1, b_1 = b_2 = 0$, $b_3 = 4$ and $A(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$. According to Example 4.1, we get that the cyclicity of system (4.7) at the origin is 3 (resp., 5), if $F(x) = F_1(x)$ (resp., $F(x) = F_2(x)$).

Reviewing system (1.4) in the introduction, we obtain $G(x) = \int_0^x g(s)ds = \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4$. Let

$$\begin{aligned} \widetilde{g}(x) &= G'(x)\left(\frac{1}{3} - 2G(x) + 3G^2(x)\right) \\ &= \frac{1}{384}x(x+1)(x+2)(3x^4 + 12x^3 + 12x^2 - 8)^2. \end{aligned} \tag{4.9}$$

Furthermore, we get the form of system (4.8). Using Theorem 1.1, systems (1.4) and (4.8) have the same cyclicity with $F = F_1$ or F_2 at the origin. Next, it is only necessary to state that although systems (4.7) and (4.8) have the same cyclicity, we cannot find the same analytical function z with $z(0) = 0$ and $z'(0) < 0$, such that $G_1(x) \equiv G_1(z)$ and $G_2(x) \equiv G_2(z)$ hold, where

$$\begin{aligned} G_1(x) &= \int_0^x \frac{1}{384}s(s+1)(s+2)(3s^4 + 12s^3 + 12s^2 - 8)^2 ds \\ &= \frac{1}{1536}x^2(3x^8 + 24x^7 + 72x^6 + 96x^5 + 24x^4 - 96x^3 - 96x^2 + 64)(x+2)^2 \end{aligned}$$

and

$$\begin{aligned} G_2(x) &= \int_0^x \frac{1}{54}s(s-1)(8s^9 - 36s^8 + 54s^7 - 27s^6 - 54)ds \\ &= \frac{1}{1296}x^2(2x-3)(2x^3 - 3x^2 - 6)(4x^6 - 12x^5 + 9x^4 + 12x^3 - 18x^2 + 36). \end{aligned}$$

It follows that

$$0 = G_1(z_1) - G_1(x) = (z_1 - x)(x^2 + z_1^2 + 2x + 2z_1)h_1(x)$$

and

$$0 = G_2(z_2) - G_2(x) = (z_2 - x)(2x^2 + 2xz_2 + 2z_2^2 - 3x - 3z_2)h_2(x),$$

where $h_1(x)h_2(x) \neq 0$ for $0 < x \ll 1$. Hence,

$$
\begin{aligned}
&z_1 = -1 + \sqrt{1 - 2x - x^2} = -x - x^2 - x^3 + O(x^4)\\
&\neq\\
&z_2 = \frac{1}{4}(3 - 2x - \sqrt{3}\sqrt{3 + 4x - 4x^2}) = -x + \frac{2}{3}x^2 - \frac{4}{9}x^3 + O(x^4).
\end{aligned}
$$

Note that if $F(x)$ is replaced by $\frac{q_n(x)}{p_m(x)}$, the corresponding two systems still have the same cyclicity at the singular point $(0, a_0)$, which shows that the conditions of Theorem 1.3 are also sufficient.

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